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Existence and continuous dependence of mild solutions for fractional neutral abstract evolution equations

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Abstract
This paper is mainly concerned with the existence, uniqueness and continuous dependence of mild solutions for fractional neutral functional differential equation with nonlocal initial conditions and infinite delay. The results are obtained by means of the classical fixed point theorems combined with theory of resolvent operators for integral equations.

Keywords: fractional neutral differential equations; nonlocal conditions; infinite delay; mild solutions; resolvent operators

1 Introduction

In this paper, we are concerned with the neutral fractional differential equation of the form

$$ {}^{C}D^{q}\big(x(t)-g(t,x_t)\big) = Ax(t) + f(t,x_t), \quad t \in (0,T], \tag{1.1}$$

$$ x_0(\vartheta) + \big(h(x_{t_1},x_{t_2},\ldots,x_{t_m})\big)(\vartheta) = \phi(\vartheta) \in \mathcal{B}_v, \quad \vartheta \le 0, \tag{1.2}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative of the order $0 < q \le 1$. $0 < t_i < T$, $i = 1,2,\ldots,m$. A is the infinitesimal generator of a C_0-semigroup of bounded linear operators, and densely defined on a Banach space $(X, \|\cdot\|)$. f, g, h, and ϕ are the given functions and satisfy some assumptions. The history $x_t : (-\infty,0] \to X$, $x_t(\theta) = x(t+\theta)$, $\theta \in (-\infty,0]$, belongs to an abstract phase space $\mathcal{B}_v$ which will be specified in Section 2. Moreover, the integral equation

$$ x(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{Ax(s)}{(t-s)^{1-q}}\,ds, \quad t \ge 0, \tag{1.3}$$

has an associated resolvent operator $(\mathcal{S}(t))_{t\ge 0}$ on X. By a resolvent operator, we mean the definition as follows.

Definition 1.1 [1, Definition 1.3] A one parameter family of bounded linear operators $(\mathcal{S}(t))_{t\ge 0}$ on X is called a resolvent operator for (1.3) if the following conditions hold:

(a) $\mathcal{S}(\cdot)x \in C([0,\infty),X)$ and $\mathcal{S}(0)x = x$ for all $x \in X$,

(b) $\mathcal{S}(t)D(A) \subset D(A)$ and $A\mathcal{S}(t)x = \mathcal{S}(t)Ax$ for all $x \in D(A)$ (the domain of A) and every $t \ge 0$,

(c) for every $x \in D(A)$ and $t \geq 0$,

$$\mathcal{S}(t)x = x + \frac{1}{\Gamma(q)} \int_0^t \frac{A\mathcal{S}(s)x}{(t-s)^{1-q}}\, ds.$$

To illustrate that our hypothesis on (1.3) is possible, we give the following example.

Example 1.1 We take $X = L^2([0,\pi];\mathbb{R})$ and let A be the operator given by $Ax = x''$ with domain $D(A) := \{x \in X : x'' \in X, x(0) = x(\pi) = 0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Furthermore, A has discrete spectrum with eigenvalues of the form $-n^2$, $n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by $z_n(\xi) := \frac{2}{\pi}\sin(n\xi)$. In addition, $\{z_n : n \in \mathbb{N}\}$ is an orthogonal basis for X; $T(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, z_n\rangle z_n$ for all $x \in X$ and every $t > 0$. From these expressions it follows that $(T(t))_{t\geq 0}$ is a uniformly bounded compact semigroup, so that $R(\lambda, A) = (\lambda - A)^{-1}$ is a compact operator for all $\lambda \in \rho(A)$.

From [1, Example 2.2.1] we know that the integral equation

$$x(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{Ax(s)}{(t-s)^{1-q}}\, ds, \quad t \geq 0,$$

has an associated analytic resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ on X. It is given by

$$\mathcal{S}(t) = \begin{cases} \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}(\lambda^q - A)^{-1}\, d\lambda, & t > 0; \\ I, & t = 0, \end{cases}$$

where $\Gamma_{r,\theta}$ denotes a contour consisting of the rays $\{re^{i\theta} : r \geq 0\}$ and $\{re^{-i\theta} : r \geq 0\}$ for some $\theta \in (\frac{\pi}{2}, \pi)$. Obviously, there exists a constant $\mathbb{S}_A$ such that $\|[\mathcal{S}'(t) - \mathcal{S}'(s)]x\| \leq \mathbb{S}_A|t - s|\|x\|_{[D(A)]}$ for all $t, s \geq 0$, where $\|\cdot\|_{[D(A)]}$ denote the graph norm.

Recently, fractional differential equations have been investigated extensively. The motivation for those works arises from both the development of the theory of fractional calculus itself and the applications of such constructions in various sciences such as physics, chemistry, aerodynamics, electrodynamics of the complex medium, and so on. For more details and examples see [1–5].

Especially, there are many papers treating the problem of the existence of a mild solution for abstract semilinear fractional differential equations; see [6–13]. But, as pointed out by Hernández *et al.* in [14], some concepts of mild solution are not realistic; see, for examples, [6, 9, 10]. Furthermore, in [14], the authors utilized an approach based on the well-developed theory of resolvent operators for integral equations to deal with the following abstract fractional equations

$$\begin{aligned} &D^\alpha\bigl(x(t) + g\bigl(t, x(t)\bigr)\bigr) = Ax(t) + f\bigl(t, x(t)\bigr), \quad t \in (0, a], \\ &x(0) = x_0, \end{aligned}$$

where D^α is the Riemann-Liouville fractional derivative of the order $0 < \alpha < 1$. For the resolvent operators, we refer the interested reader to [1, 14–19] and the references therein for more details.

Nonlocal conditions were initiated by Byszewski [20] when he proved the existence and uniqueness of mild and classical solutions of nonlocal Cauchy problems. As remarked by Byszewski and Lakshmikantham [21], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena. There are many papers concerned with the nonlocal conditions; see [9, 12, 13, 20, 21] and the references therein, for examples.

In addition, Zhou and Jiao in [13] discuss a class of fractional neutral evolution equations with nonlocal conditions

$$\begin{aligned} &{}^{C}D^{q}\bigl[x(t)-h(t,x_t)\bigr]+Ax(t)=f(t,x_t), \quad t\in(0,a], \\ &x_0(\vartheta)+\bigl(g(x_{t_1},x_{t_2},\ldots,x_{t_m})\bigr)(\vartheta)=\varphi(\vartheta), \quad \vartheta\in[-r,0], \end{aligned}$$

where ${}^{C}D^{q}$ is the Caputo fractional derivative of the order $0 < q \leq 1$. $0 < t_i < a$, $i = 1,2,\ldots,m$. $-A$ is the infinitesimal generator of an analytic semigroup on a Banach space E. By considering an integral equation which is given in terms of probability density and semigroup, they establish criteria on the existence and uniqueness of mild solutions.

In this paper, we used the theory of resolvent operators coupled with fixed point theorem, to obtain some new results about the global uniqueness and existence of the problem (1.1)-(1.2) in Section 3. Moreover, in Section 4, the continuous dependence of the mild solution is also investigated.

2 Preliminaries

In this section, we shall introduce some basic definitions, notations, and lemmas which are used throughout this paper.

Definition 2.1 The fractional (arbitrary) order integral of the function $v(t)\in L^1([0,\infty],\mathbb{R})$ of $\mu\in\mathbb{R}^+$ is defined by

$$I^{\mu}v(t)=\frac{1}{\Gamma(\mu)}\int_0^t (t-s)^{\mu-1}v(s)\,ds, \quad t>0,$$

where Γ is the Euler gamma function.

Definition 2.2 The Caputo fractional derivative of order $\mu > 0$ for a function $v(t)$ given in the interval $[0,\infty)$ is defined by

$${}^{C}D^{\mu}v(t)=\frac{1}{\Gamma(n-\mu)}\int_a^t (t-s)^{n-\mu-1}v^{(n)}(s)\,ds$$

provided that the right-hand side is point-wise defined. Here $n = [\mu] + 1$ and $[\mu]$ means the integral part of the number μ, and Γ is the Euler gamma function.

More details on fractional derivatives and their properties can be found in [2, 3, 5].

Now, let $(Z;\|\cdot\|_Z)$ and $(W;\|\cdot\|_W)$ be Banach spaces. We denote by $\mathcal{L}(Z;W)$ the space of bounded linear operators from Z into W endowed with the operator norm denoted by $\|\cdot\|_{\mathcal{L}(Z;W)}$, and we can write simply $\mathcal{L}(Z)$ and $\|\cdot\|_{\mathcal{L}(Z)}$ when $Z = W$. The notation $[D(A)]$ stands for the domain of A endowed with the graph norm $\|x\|_{[D(A)]} = \|x\| + \|Ax\|$. In addition, $B_r(x;Z)$ represents the closed ball with center at x and radius r in Z.

In the sequel, we always assume that the resolvent operator $(\mathcal{S}(t))_{t\geq 0}$ of (1.3) is analytic and compact; see [1, Chapter 2] for details. In addition, $\|\mathcal{S}'(t)x\| \leq \varphi_A(t)\|x\|_{[D(A)]}$ for all $t \geq 0$ and $\|[\mathcal{S}'(t) - \mathcal{S}'(s)]x\| \leq \mathbb{S}_A|t-s|\|x\|_{[D(A)]}$ for all $t, s \geq 0$, where φ_A is a function in $L^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^+)$ and $\mathbb{S}_A$ is a constant.

Consider the abstract integral equation

$$x(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{Ax(s)}{(t-s)^{1-q}}\,ds + w(t), \quad t \in [0,T], \tag{2.1}$$

where $w \in C([0,T];X)$. From [1, Definition 1.1.1], we note the following concept of a mild solution.

Definition 2.3 A function $x \in C([0,T];X)$ is called a mild solution of the integral equation (2.1) on $[0,T]$ provided that $\int_0^t (t-s)^{q-1}x(s)\,ds \in D(A)$ for all $t \in [0,T]$ and

$$x(t) = \frac{1}{\Gamma(q)}A\int_0^t \frac{x(s)}{(t-s)^{1-q}}\,ds + w(t), \quad t \in [0,T]. \tag{2.2}$$

The next result plays an important part in our development, which follows from [1, Proposition I.1.2, Theorem II.2.4, Corollary II.2.6, Proposition I.1.3], and also can be found in [14, Lemma 1.1].

Lemma 2.1 *Assume that the resolvent operator* $(\mathcal{S}(t))_{t\geq 0}$ *of* (1.3) *is analytic and compact, and* $w \in C([0,T];D(A))$, *then the function* $x : [0,T] \to X$ *defined by*

$$x(t) = \int_0^t \mathcal{S}'(t-s)w(s)\,ds + w(t), \quad t \in [0,T],$$

is a mild solution of (2.1).

Lemma 2.2 [14, Lemma 2.2] *Assume that* $\mathcal{S}(t)$ *is compact for all* $t > 0$. *Then* $\mathcal{S}'(t)$ *is compact for all* $t > 0$ *and the inclusion map* $i_c : [D(A)] \to X$ *is compact.*

Next, we present the abstract phase space $\mathcal{B}_\nu$, which has been used in [22]. Assume that $\nu : (-\infty, 0] \to (0, +\infty)$ is a continuous function with $l = \int_{-\infty}^0 \nu(s)\,ds < +\infty$. For any $a > 0$, we define

$$\mathcal{B} = \big\{\xi : [-a,0] \to X \text{ such that } \xi(t) \text{ is bounded and measurable}\big\},$$

and provide the space $\mathcal{B}$ with the norm $\|\xi\|_{[-a,0]} = \sup_{t\in[-a,0]}|\xi(t)|$. Then we define

$$\mathcal{B}_\nu = \left\{\xi : [-\infty,0] \to X \text{ such that, for any } c > 0, \xi|_{[-c,0]} \in \mathcal{B} \right.$$
$$\left. \text{and } \int_{-\infty}^0 \nu(s)\|\xi\|_{[s,0]}\,ds < +\infty\right\}.$$

Obviously, the phase space $\mathcal{B}_\nu$ endowed with the norm $\|\xi\|_{\mathcal{B}_\nu} = \int_{-\infty}^0 \nu(s)\|\xi\|_{[s,0]}\,ds$ is a Banach space and $|\xi(0)| \leq l^{-1}\|\xi\|_{\mathcal{B}_\nu}$. Now we consider the space

$$\mathcal{B}_{\nu T} = \big\{\xi : [-\infty,T] \to X \text{ such that, } \xi|_{[0,T]} \in C\big([0,T],X\big) \cap D(A) \text{ and } \xi_0 \in \mathcal{B}_\nu\big\},$$

with the seminorm $\|\cdot\|_*$ defined by

$$\|x\|_* = \|x_0\|_{\mathcal{B}_\nu} + l\|x\|_{[0,T]}, \quad x \in \mathcal{B}_{\nu T}.$$

Lemma 2.3 [22] *Assume $x \in \mathcal{B}_{\nu T}$, then for $0 \le t \le T$, $x_t \in \mathcal{B}_\nu$. Moreover,*

$$l\bigl|x(t)\bigr| \le \|x_t\|_{\mathcal{B}_\nu} \le \|x_0\|_{\mathcal{B}_\nu} + l\|x\|_{[0,T]}.$$

We will now introduce the concept of a mild solution of (1.1)-(1.2). To this end, note that if $x \in \mathcal{B}_{\nu T}$ is a solution of (1.1)-(1.2), we can expect that

$$\begin{cases} x(t) = \phi(0) - (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(0) - g(0, x_0) + g(t, x_t) \\ \qquad + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,x_s)}{(t-s)^{1-q}}\,ds + \frac{1}{\Gamma(q)}\int_0^t \frac{Ax(s)}{(t-s)^{1-q}}\,ds, \quad 0 < t \le T; \\ x_0(\vartheta) + (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(\vartheta) = \phi(\vartheta), \quad \vartheta \le 0. \end{cases}$$

Furthermore, in view of $x_0(t) = x(t)$, $t \le 0$, we can give the following definition.

Definition 2.4 A function $x \in \mathcal{B}_{\nu T}$ is called a mild solution of (1.1)-(1.2) provided that $\int_0^t (t-s)^{q-1} x(s)\,ds \in D(A)$ for all $t \in [0,T]$ and

$$x(t) = \begin{cases} \phi(0) - (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(0) - g(0, x_0) + g(t, x_t) \\ \quad + \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,x_s)}{(t-s)^{1-q}}\,ds + \frac{1}{\Gamma(q)}A\int_0^t \frac{x(s)}{(t-s)^{1-q}}\,ds, & 0 < t \le T; \\ \phi(t) - (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(t), & t \le 0. \end{cases} \tag{2.3}$$

3 Existence and uniqueness of mild solutions

To simplify our development, in the rest of this work, for a function $x \in \mathcal{B}_{\nu T}$ we use the notations $G_x, F_x : [0,T] \to X$ given by

$$G_x(t) = \phi(0) - \bigl(h(x_{t_1}, x_{t_2}, \ldots, x_{t_m})\bigr)(0) - g(0, x_0) + g(t, x_t)$$

and

$$F_x(t) = \frac{1}{\Gamma(q)}\int_0^t \frac{f(s,x_s)}{(t-s)^{1-q}}\,ds.$$

By Lemma 2.1, one sees that

$$x(t) = \begin{cases} \int_0^t \mathcal{S}'(t-s)[G_x(s) + F_x(s)]\,ds + G_x(t) + F_x(t), & 0 < t \le T; \\ \phi(t) - (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(t), & t \le 0, \end{cases} \tag{3.1}$$

is a mild solution of (1.1)-(1.2).

From now on, we denote the set $\{x : x \in \mathcal{B}_{\nu T}, \|x\|_* \le r\}$ by B_r. Then B_r, for each $r > 0$, is a bounded, closed, nonempty, and convex subset in X.

Now, we introduce the map $\Psi : \mathcal{B}_{\nu T} \to \mathcal{B}_{\nu T}$ by

$$\Psi x(t) = \begin{cases} \int_0^t \mathcal{S}'(t-s)[G_x(s) + F_x(s)]\,ds + G_x(t) + F_x(t), & 0 < t \le T; \\ \phi(t) - (h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}))(t), & t \le 0. \end{cases}$$

By Lemma 2.1 and the argument above, it is easy to see that the operator Ψ having a fixed point $x_* \in \mathcal{B}_{\nu T}$ is equivalent to the problem (1.1)-(1.2) having a mild solution $x_*(t)$, $-\infty < t \leq T$.

To prove our results, we introduce the following conditions.

(H1) $f, g \in C([0,T] \times \mathcal{B}_\nu, X)$.

(H2) There exists a constant L_f, such that, for all $x, y \in \mathcal{B}_\nu$,

$$\big\| f(t,x) - f(t,y) \big\|_{[D(A)]} \leq L_f \|x - y\|_{\mathcal{B}_\nu}, \quad t \in [0,T]. \tag{3.2}$$

(H3) There exist constants L_f, L_g, L_h, such that, for all $x, y \in \mathcal{B}_\nu$,

$$\begin{aligned} &\big\| g(t,x) - g(t,y) \big\|_{[D(A)]} \leq L_g \|x - y\|_{\mathcal{B}_\nu}, \\ &\big\| h(y_{t_1}, y_{t_2}, \ldots, y_{t_m}) - h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \big\|_{[D(A)]} \leq L_h \|x_t - y_t\|_{\mathcal{B}_\nu}. \end{aligned} \tag{3.3}$$

(H4) $l(L_h + 2L_g)(\|\varphi_A\|_{L^1} + 1) + lL_h < 1$.

(H5)

$$\lim_{\|y_t\|_{\mathcal{B}_\nu} \to \infty} \sup_{t \in [0,T]} \frac{\|f(t, y_t)\|_{[D(A)]}}{\|y_t\|_{\mathcal{B}_\nu}} = \lambda_3, \qquad \lim_{\|y_t\|_{\mathcal{B}_\nu} \to \infty} \sup_{t \in [0,T]} \frac{\|g(t, y_t)\|_{[D(A)]}}{\|y_t\|_{\mathcal{B}_\nu}} = \lambda_4.$$

First, we present a uniqueness result.

Theorem 3.1 *Suppose that* (H1)-(H3) *hold, then system* (1.1)-(1.2) *has a unique mild solution, provided that*

$$\wedge = l\left[L_h + 2L_g + \frac{T^q L_f}{q\Gamma(q)}\right]\big(\|\varphi_A\|_{L^1} + 1\big) + lL_h < 1.$$

Proof For $y \in \mathcal{B}_{\nu T}$, From the assumption on f, g we see that

$$\begin{aligned} &\int_0^t \big\| \mathcal{S}'(t-s)\big[G_y(s) + F_y(s)\big] \big\| \, ds \\ &\quad \leq \int_0^t \varphi_A(t-s) \big\| G_y(s) + F_y(s) \big\|_{[D(A)]} \, ds \\ &\quad \leq \big\| G_y(s) + F_y(s) \big\|_{C([0,T],[D(A)])} \int_0^t \varphi_A(t-s) \, ds \\ &\quad \leq \big(\big\| G_y(s) \big\|_{C([0,T],[D(A)])} + \big\| F_y(s) \big\|_{C([0,T],[D(A)])} \big) \int_0^t \varphi_A(t-s) \, ds \\ &\quad \leq \left(\big\| G_y(s) \big\|_{C([0,T],[D(A)])} + \|f\|_{C([0,T],[D(A)])} \frac{T^q}{q\Gamma(q)} \right) \|\varphi_A\|_{L^1}, \end{aligned}$$

which implies that the function $s \to \mathcal{S}'(t-s)[G_y(s) + F_y(s)]$ is integrable on $[0,t]$ for all $t \in [0,T]$, since $f, g \in C([0,T] \times \mathcal{B}_\nu, X)$. Then Ψ is well defined.

From the definition of the G_y, F_y, we can see that, for $x, y \in \mathcal{B}_{\nu T}$,

$$\begin{aligned} \|G_x - G_y\|_{[D(A)]} \leq{} & \big\| \big(h(x_{t_1}, x_{t_2}, \ldots, x_{t_m})\big)(0) \\ & - \big(h(y_{t_1}, y_{t_2}, \ldots, y_{t_m})\big)(0) \big\|_{[D(A)]} \end{aligned}$$

$$\begin{aligned}
&\quad + \big\| g(0,x_0) - g(0,y_0) \big\|_{[D(A)]} \\
&\quad + \big\| g(t,x_t) - g(t,y_t) \big\|_{[D(A)]} \\
&\leq L_h \|x_t - y_t\|_{\mathcal{B}_\nu} + 2L_g \|x_t - y_t\|_{\mathcal{B}_\nu} \\
&\leq (L_h + 2L_g) \|x_t - y_t\|_{\mathcal{B}_\nu}
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\|F_x - F_y\|_{[D(A)]} &\leq \frac{T^q \|f(s,x_s) - f(s,y_s)\|_{[D(A)]}}{q\Gamma(q)} \\
&\leq \frac{T^q L_f}{q\Gamma(q)} \|x_t - y_t\|_{\mathcal{B}_\nu}.
\end{aligned} \tag{3.5}$$

Then, for $x, y \in \mathcal{B}_{\nu T}$, and $t \in [0,T]$, we have

$$\begin{aligned}
l\big\| \Psi x(t) - \Psi y(t) \big\| &\leq l\bigg\{ \int_0^t \big\| \mathcal{S}'(t-s)\big[F_x(s) - F_y(s)\big] \big\| \, ds \\
&\quad + \int_0^t \big\| \mathcal{S}'(t-s)\big[G_x(s) - G_y(s)\big] \big\| \, ds \\
&\quad + \|G_x - G_y\|_{[D(A)]} + \|F_x - F_y\|_{[D(A)]} \bigg\} \\
&\leq l\bigg\{ \int_0^t \varphi_A(t-s) \big\| F_x(s) - F_y(s) \big\|_{[D(A)]} \, ds \\
&\quad + \int_0^t \varphi_A(t-s) \big\| G_x(s) - G_y(s) \big\|_{[D(A)]} \, ds \\
&\quad + \|G_x - G_y\|_{[D(A)]} + \|F_x - F_y\|_{[D(A)]} \bigg\} \\
&\leq l\bigg\{ \bigg[L_h + 2L_g + \frac{T^q L_f}{q\Gamma(q)} \bigg] \big(\|\varphi_A\|_{L^1} + 1 \big) \bigg\} \|x_t - y_t\|_{\mathcal{B}_\nu}.
\end{aligned}$$

On the other hand, for $x, y \in \mathcal{B}_{\nu T}$, and $t < 0$, we have

$$\begin{aligned}
\big\| (\Psi x)_0 - (\Psi y)_0 \big\|_{\mathcal{B}_\nu} &\leq \int_{-\infty}^0 \nu(s) \big\| h(y_{t_1}, y_{t_2}, \ldots, y_{t_m}) - h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \big\|_{[D(A)]} \, ds \\
&\leq l \big\| h(y_{t_1}, y_{t_2}, \ldots, y_{t_m}) - h(x_{t_1}, x_{t_2}, \ldots, x_{t_m}) \big\|_{[D(A)]} \\
&\leq l L_h \|x_t - y_t\|_{\mathcal{B}_\nu}.
\end{aligned} \tag{3.6}$$

Hence,

$$\|\Psi x - \Psi y\|_* \leq \wedge \|x - y\|_*,$$

which implies that Ψ is a contraction and there exists a unique fixed point $x_* \in \mathcal{B}_{\nu T}$ of Ψ. $x_*(t)$, $-\infty < t \leq T$ is a mild solution of system (1.1)-(1.2). The proof is complete. □

Next, we give some generally existence results. For this purpose, we present a fixed point theorem due to Krasnoselskii which can be found in [23, p.370] or Smart [24, p.31] and also can be found in [25, Theorem K].

Theorem 3.2 *Let M be a closed, convex, and nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that A and B map M into X such that*

(i) $x, y \in M \Rightarrow Ax + By \in M$,

(ii) *A is compact and continuous,*

(iii) *B is a contraction mapping.*

Then $\exists y \in M$ with $y = Ay + By$.

In order to apply the fixed point theorem above, we introduce the decomposition $\Psi = \Psi_1 + \Psi_2 + \Psi_3$, where

$$\Psi_1 y(t) = \begin{cases} F_y(t), & 0 \leq t \leq T; \\ 0, & t \leq 0, \end{cases}$$

$$\Psi_2 y(t) = \begin{cases} \int_0^t \mathcal{S}'(t-s)F_y(s)\,ds, & 0 \leq t \leq T; \\ 0, & t \leq 0, \end{cases}$$

and

$$\Psi_3 y(t) = \begin{cases} \int_0^t \mathcal{S}'(t-s)G_y(s)\,ds + G_y(t), & 0 \leq t \leq T; \\ \phi(t) - (h(y_{t_1}, y_{t_2}, \dots, y_{t_m}))(t), & t \leq 0. \end{cases}$$

Then we prove the following lemma.

Lemma 3.1 *Suppose that* (H1), (H3), *and* (H4) *hold, then the operator Ψ_1, Ψ_2 are completely continuous and Ψ_3 is a contraction mapping on B_r, $r > 0$.*

Proof We divide this proof into three parts.

Part 1. Ψ_1 is completely continuous.

Step 1: Ψ_1 is continuous.

Let $\{y^n\}$ be a sequence in $\mathcal{B}_{vT}$ such that $y^n \to y$ as $n \to \infty$. Then for $0 \leq t \leq T$ we have

$$\begin{aligned} \|\Psi_1 y^n(t) - \Psi_1 y(t)\| &\leq \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left[f(s, y_s^n) - f(s, y_s) \right] ds \right\| \\ &\leq \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \left\| f(s, y_s^n) - f(s, y_s) \right\| ds \\ &\leq \frac{T^q}{q\Gamma(q)} \left\| f(t, y_t^n) - f(t, y_t) \right\|_{[0,T]}, \end{aligned} \tag{3.7}$$

which proves that $\Psi_1 y^n \to \Psi_1 y$ as $n \to \infty$ and Ψ_1 is continuous, by the continuity of f.

Step 2: Ψ_1 is a compact operator.

Let $0 < \epsilon < t < T$. From the mean value theorem for the Bochner integral (see [26, Lemma 2.1.3]), for $y \in \mathcal{B}_{vT}$ we see that

$$\begin{aligned} \Psi_1 y(t) &= \frac{1}{\Gamma(q)} \int_0^\epsilon \frac{f(s, y_s)}{(t-s)^{1-q}}\,ds + \frac{1}{\Gamma(q)} \int_\epsilon^t \frac{f(s, y_s)}{(t-s)^{1-q}}\,ds \\ &\in B_{\frac{M_f \epsilon^q}{q\Gamma(q)}}(\theta; X) \\ &\quad + (t-\epsilon)\overline{co\left(\left\{ \frac{f(s, y_s)}{(t-s)^{1-q}} : s \in [\epsilon, t], y \in B_r \right\} \right)}, \end{aligned} \tag{3.8}$$

where $M_f = \sup\{f(t,y) : t \in [0,T], y \in B_r\}$ and $co(S)$ denotes the convex hull of a set S. Since the map i_c is compact and f is continuous, we know that Diam of the first part of the right-hand side of (3.8) tends to 0 as $\epsilon \to 0$ and the second part is compact. This proves that the set $\Psi_1 B_r(t)$ is relatively compact in X for all $t \in [0,T]$.

On the other hand, for $0 \leq t_1 < t_2 \leq T$, we have

$$\begin{aligned} &\|\Psi_1 y(t_2) - \Psi_1 y(t_1)\| \\ &\quad\leq \frac{1}{\Gamma(q)} \left\| \int_0^{t_2} (t_2 - s)^{q-1} f(s, y_s)\, ds - \int_0^{t_1} (t_1 - s)^{q-1} f(s, y_s)\, ds \right\| \\ &\quad\leq \frac{1}{\Gamma(q)} \left\{ \left\| \int_{t_1}^{t_2} (t_2 - s)^{q-1} f(s, y_s)\, ds \right\| \right. \\ &\qquad \left. + \int_0^{t_1} \left[(t_2 - s)^{q-1} - (t_1 - s)^{q-1} \right] ds \left\| f(t, y_t) \right\|_{[0,T]} \right\} \\ &\quad\leq \frac{1}{q\Gamma(q)} \left(t_2^q - t_1^q \right) \left\| f(t, y_t) \right\|_{[0,T]} \end{aligned} \tag{3.9}$$

and $\|f(t,y_t)\|_{[0,T]} < +\infty$ when $y \in B_r$ since $f \in C([0,T] \times \mathcal{B}_v, X)$. Hence, (3.9) implies the set of functions $\Psi_1 B_r$ is equicontinuous. This proves that Ψ_1 is a compact operator.

Part 2. Ψ_2 is completely continuous.

It is easy to show that Ψ_2 is continuous. We will prove that the set $\Psi_2 B_r(t)$ is relatively compact in X for all $t \in [0,T]$ and the set of functions $\Psi_2 B_r$ is equicontinuous on $[0,T]$.

Firstly, let $0 < t \leq T$ and $\epsilon < \min\{t, 1\}$. Since $\mathcal{S}'(t) \in C((0,T]; \mathcal{L}(X))$, there are numbers $0 = s_0 < s_1 < s_2 < \cdots < s_n < s_{n+1} = t$ such that $|s_i - s_{i+1}| < \epsilon$ for all $i = 0, 1, 2, \ldots, n$, and $\|\mathcal{S}'(s) - \mathcal{S}'(s_j)\|_{\mathcal{L}(X)} < \epsilon$ for each $s \in [s_j, s_{j+1}]$ and every $j = 0, 1, 2, \ldots, n$. Under these conditions, for $y \in B_r$ we get

$$\begin{aligned} \Psi_2 y(t) &= \int_0^{s_1} \mathcal{S}'(s) F_y(t-s)\, ds + \sum_{i=1}^n \int_{s_i}^{s_{i+1}} \left[\mathcal{S}'(s) - \mathcal{S}'(s_i) \right] F_y(t-s)\, ds \\ &\quad + \sum_{i=1}^n \mathcal{S}'(s_i) \int_{s_i}^{s_{i+1}} F_y(t-s)\, ds. \end{aligned} \tag{3.10}$$

Note now that

$$\begin{aligned} &\left\| \int_0^{s_1} \mathcal{S}'(s) F_y(t-s)\, ds \right\| \leq \|\varphi_A\|_{L^1([0,\epsilon],\mathbb{R}^+)} \frac{M_f \epsilon^q}{q\Gamma(q)}, \\ &\left\| \sum_{i=1}^n \int_{s_i}^{s_{i+1}} \left[\mathcal{S}'(s) - \mathcal{S}'(s_i) \right] F_y(t-s)\, ds \right\| \leq \epsilon \frac{M_f T^{q+1}}{q(q+1)\Gamma(q)}, \end{aligned}$$

where M_f is defined as in Part 1.

From Lemma 2.2 and similarly to the proof of Step 1 in Part 1, we infer that the set $\Psi_2 B_r(t)$ is relatively compact in X for all $t \in [0,T]$.

Secondly, let $0 \leq \tau_1 < \tau_2 \leq T$, we have

$$\begin{aligned} &\|\Psi_2 y(\tau_2) - \Psi_2 y(\tau_1)\| \\ &\quad\leq \left\| \int_0^{\tau_2} \mathcal{S}'(\tau_2 - s) F_y(s)\, ds - \int_0^{\tau_1} \mathcal{S}'(\tau_1 - s) F_y(s)\, ds \right\| \end{aligned}$$

$$
\begin{aligned}
&\leq \left\| \int_{\tau_1}^{\tau_2} \mathcal{S}'(\tau_2 - s)F_y(s)\,ds \right\| + \left\| \int_0^{\tau_1} \mathcal{S}'(\tau_2 - s)F_y(s) - \mathcal{S}'(\tau_1 - s)F_y(s)\,ds \right\| \\
&\leq \left\| \int_{\tau_1}^{\tau_2} \mathcal{S}'(\tau_2 - s)F_y(s)\,ds \right\| + \mathbb{S}_A|\tau_2 - \tau_1|^2 \|F_y\|_{[D(A)]} \\
&\leq |\tau_2 - \tau_1| \|\varphi_A\|_{L^1} \|F_y\|_{[D(A)]} + \mathbb{S}_A|\tau_2 - \tau_1|^2 \|F_y\|_{[D(A)]}. \qquad (3.11)
\end{aligned}
$$

The equicontinuity of $\Psi_2 B_r$ is proved. Hence, Ψ_2 is completely continuous.

Part 3. Ψ_3 is a contraction on B_r.

Proceeding as in the proof of Theorem 3.1, we can show that, for $x, y \in \mathcal{B}_{vT}$,

$$
\begin{aligned}
&\|\Psi_3 x - \Psi_3 y\|_* \\
&\quad \leq l\big[(L_h + 2L_g)\big(\|\varphi_A\|_{L^1} + 1\big) + L_h\big]\|x - y\|_*.
\end{aligned}
$$

This proves Ψ_3 is a contraction on B_r, because of (H4). □

Theorem 3.3 *Suppose that* (H1), (H3)-(H5) *hold, then system* (1.1)-(1.2) *has at least a mild solution, provided that*

$$
l\left(L_h + 2\lambda_4 + \frac{T^q \lambda_3}{q\Gamma(q)}\right)\big(1 + \|\varphi_A\|_{L^1}\big) + lL_h < 1.
$$

Proof Obviously, $x(t)$ is a mild solution of (1.1)-(1.2) if and only if the operator equation $\Psi x = \Psi_1 x + \Psi_2 x + \Psi_3 x$ has a solution $x \in B_r$.

First, we prove that there exists a sufficiently large constant R, such that $\Psi_1 x + \Psi_2 x + \Psi_3 y \in B_R$, for $\forall x, y \in B_R$.

From the condition (H5), we know that, for a sufficiently small ϵ, such that

$$
M_\epsilon = l\left[\left(L_h + 2\epsilon + 2\lambda_4 + \frac{T^q(\epsilon + \lambda_3)}{q\Gamma(q)}\right)\big(1 + \|\varphi_A\|_{L^1}\big) + L_h\right] < 1,
$$

there exists a constant $R_0 > 0$, for $y_t \in \mathcal{B}_v$, $\|y_t\|_{\mathcal{B}_v} > R_0$, and we have

$$
\begin{aligned}
&\big\|f(t, y_t)\big\|_{[D(A)]} \leq (\epsilon + \lambda_3)\|y_t\|_{\mathcal{B}_v}, \\
&\big\|g(t, y_t)\big\|_{[D(A)]} \leq (\epsilon + \lambda_4)\|y_t\|_{\mathcal{B}_v}.
\end{aligned}
$$

On the other hand, by the continuity of f, g, we know

$$
\begin{aligned}
&\big\|f(t, y_t)\big\|_{[D(A)]} \leq M_f, \\
&\big\|g(t, y_t)\big\|_{[D(A)]} \leq M_g, \qquad \|y_t\|_{\mathcal{B}_v} \leq R_0,
\end{aligned}
$$

where M_f, M_g are constants. Hence,

$$
\begin{aligned}
&\big\|f(t, y_t)\big\|_{[D(A)]} \leq M_f + (\epsilon + \lambda_3)\|y_t\|_{\mathcal{B}_v}, \\
&\big\|g(t, y_t)\big\|_{[D(A)]} \leq M_g + (\epsilon + \lambda_4)\|y_t\|_{\mathcal{B}_v}, \quad \text{for all } y_t \in \mathcal{B}_v.
\end{aligned}
$$

Define the function $\upsilon \in C((-\infty, T], X)$ such that $\|\upsilon\|_* = 0$. Immediately, for all $y \in \mathcal{B}_{vT}$, by Lemma 2.3, we obtain

$$\begin{aligned}
\|G_y\|_{[D(A)]} &\le \big\|\phi(0) - \big(h(y_{t_1}, y_{t_2}, \dots, y_{t_m})\big)(0)\big\|_{[D(A)]} \\
&\quad + \big\|g(0, y_0)\big\|_{[D(A)]} + \big\|g(t, y_t)\big\|_{[D(A)]} \\
&\le L_h\|y_t\|_{\mathcal{B}_v} + 2(\epsilon + \lambda_4)\|y_t\|_{\mathcal{B}_v} \\
&\quad + \big\|\phi(0)\big\|_{[D(A)]} + \big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} + 2M_g \\
&\le (L_h + 2\epsilon + 2\lambda_4)\|y_t\|_{\mathcal{B}_v} \\
&\quad + \big\|\phi(0)\big\|_{[D(A)]} + \big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} + 2M_g \\
&\triangleq (L_h + 2\epsilon + 2\lambda_4)\|y_t\|_{\mathcal{B}_v} + M_G
\end{aligned}$$

and

$$\|F_y\|_{[D(A)]} \le \frac{T^q(M_f + (\epsilon + \lambda_3)\|y_t\|_{\mathcal{B}_v})}{q\Gamma(q)} \triangleq \frac{T^q(\epsilon + \lambda_3)\|y_t\|_{\mathcal{B}_v}}{q\Gamma(q)} + M_F.$$

In view of $M_\epsilon < 1$, we can select a sufficiently large constant $R > R_0$ such that

$$\begin{aligned}
&M_\epsilon R + l\frac{T^q M_f}{q\Gamma(q)} + l\big\|\phi(0)\big\|_{[D(A)]} + l\big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} \\
&\quad + l(M_G + M_F)\big(1 + \|\varphi_A\|_{L^1}\big) \\
&\quad < R.
\end{aligned}$$

Hence, for $y, x \in B_R$, we get

$$\begin{aligned}
&\|\Psi_1 y + \Psi_2 y + \Psi_3 x\|_* \\
&\quad \le l\left((L_h + 2\epsilon + 2\lambda_4)\|x_t\|_{\mathcal{B}_v} + M_G + \frac{T^q(\epsilon + \lambda_3)\|y_t\|_{\mathcal{B}_v}}{q\Gamma(q)} + M_F\right)\big(1 + \|\varphi_A\|_{L^1}\big) \\
&\qquad + lL_h\|x_t\|_{\mathcal{B}_v} + l\big\|\phi(0)\big\|_{[D(A)]} + l\big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} \\
&\quad \le l\left[\left(L_h + 2\epsilon + 2\lambda_4 + \frac{T^q(\epsilon + \lambda_3)}{q\Gamma(q)}\right)\big(1 + \|\varphi_A\|_{L^1}\big) + L_h\right]R \\
&\qquad + l\big\|\phi(0)\big\|_{[D(A)]} + l\big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} \\
&\qquad + (M_G + M_F)\big(1 + \|\varphi_A\|_{L^1}\big) \\
&\quad \le M_\epsilon R + l\big\|\phi(0)\big\|_{[D(A)]} + l\big\|\big(h(\upsilon_{t_1}, \upsilon_{t_2}, \dots, \upsilon_{t_m})\big)(0)\big\|_{[D(A)]} \\
&\qquad + l(M_G + M_F)\big(1 + \|\varphi_A\|_{L^1}\big) \\
&\quad < R,
\end{aligned}$$

which implies $\Psi_1 y + \Psi_2 y + \Psi_3 x \in B_R$.

By Lemma 3.1 and Theorem 3.2, the operator Ψ has at least a fixed point $y_* \in B_R$. $y_*(t)$, $-\infty < t \le T$, is a mild solution of system (1.1)-(1.2). The proof is complete. □

4 Continuous dependence of mild solution

Now, we give the continuous dependence of mild solution of system (1.1)-(1.2) on the initial condition. As a matter of convenience, and in view of the characteristic of A, we have $\|\phi_1 - \phi_2\|_{[D(A)]} \le L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu}$, L_ϕ is a constant.

Theorem 4.1 *Suppose that* (H1)-(H3) *hold. Then for* $\phi_1, \phi_2 \in \mathcal{B}_\nu$ *and the corresponding mild solutions* y_1, y_2 *of the problems* (1.1)-(1.2), *the following inequality holds*:

$$\|y_1 - y_2\|_* \le \wedge_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} + \wedge \|y_1 - y_2\|_*, \tag{4.1}$$

where $\wedge$ *is defined in Theorem* 3.1 *and*

$$\wedge_\phi = l\big(\|\varphi_A\|_{L^1} + 2\big)L_\phi.$$

Furthermore if $\wedge < 1$, *we have*

$$\|y_1 - y_2\|_* \le \frac{\wedge_\phi}{1 - \wedge} \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu}. \tag{4.2}$$

Proof Suppose that $\phi_i \in \mathcal{B}_\nu$ $(i = 1, 2)$ are arbitrary functions and that y_1, y_2 are the corresponding mild solutions of the problem (1.1)-(1.2). Let

$$\begin{aligned} G_{y\phi_i} &= \phi_i(0) - \big(h\big((y_i)_{t_1}, (y_i)_{t_2}, \ldots, (y_i)_{t_m}\big)\big)(0) \\ &\quad - g\big(0, (y_i)_0\big) + g\big(t, (y_i)_t\big), \quad i = 1, 2, \\ F_{y\phi_i} &= \frac{1}{\Gamma(q)} \int_0^t \frac{f(s, (y_i)_s)}{(t-s)^{1-q}}\, ds, \quad i = 1, 2. \end{aligned}$$

Then, for $0 < t \le T$, we have

$$\begin{aligned} y_1(t) - y_2(t) &= \int_0^t \mathcal{S}'(t-s)\big[G_{y\phi_1}(s) - G_{y\phi_2}(s)\big]\, ds + G_{y\phi_1}(t) - G_{y\phi_2}(t) \\ &\quad + \int_0^t \mathcal{S}'(t-s)\big[F_{y\phi_1}(s) - F_{y\phi_2}(s)\big]\, ds \\ &\quad + F_{y\phi_1}(t) - F_{y\phi_2}(t), \end{aligned} \tag{4.3}$$

for $t \le 0$,

$$\begin{aligned} y_1(t) - y_2(t) &= \phi_1(t) - \big(h\big((y_1)_{t_1}, (y_1)_{t_2}, \ldots, (y_1)_{t_m}\big)\big)(t) \\ &\quad - \phi_2(t) + \big(h\big((y_2)_{t_1}, (y_2)_{t_2}, \ldots, (y_2)_{t_m}\big)\big)(t). \end{aligned} \tag{4.4}$$

Analogously to (3.4) and (3.5), we have

$$\begin{aligned} &\|G_{y\phi_1} - G_{y\phi_2}\|_{[D(A)]} \\ &\quad \le \big\|\phi_1 - \phi_2 - h\big((y_1)_{t_1}, (y_1)_{t_2}, \ldots, (y_1)_{t_m}\big) \\ &\qquad + h\big((y_2)_{t_1}, (y_2)_{t_2}, \ldots, (y_2)_{t_m}\big)\big\|_{[D(A)]} \end{aligned}$$

$$\begin{aligned}
&+ \left\| g\big(0,(y_1)_0\big) - g\big(0,(y_2)_0\big) \right\|_{[D(A)]} \\
&+ \left\| g\big(t,(y_1)_t\big) - g\big(t,(y_2)_t\big) \right\|_{[D(A)]} \\
&\le L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} + (L_h + 2L_g) \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu},
\end{aligned} \tag{4.5}$$

and

$$\|F_{y\phi_1} - F_{y\phi_2}\|_{[D(A)]} \le \frac{L_f T^q}{q\Gamma(q)} \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu}. \tag{4.6}$$

From (4.3), (4.5), (4.5), and (4.6), we get, for $0 < t \le T$

$$\begin{aligned}
l\left\| y_1(t) - y_2(t) \right\| \le l\bigg[&L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} + (L_h + 2L_g)\|y_1 - y_2\|_{\mathcal{B}_\nu} \\
&+ \frac{L_f T^q}{q\Gamma(q)} \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu} \bigg] \big(\|\varphi_A\|_{L^1} + 1 \big).
\end{aligned}$$

Immediately,

$$\begin{aligned}
l\|y_1 - y_2\|_{[0,T]} \le{}& l\big(\|\varphi_A\|_{L^1} + 1 \big) L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} \\
&+ l\left(L_h + 2L_g + \frac{L_f T^q}{q\Gamma(q)} \right) \big(\|\varphi_A\|_{L^1} + 1 \big) \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu}.
\end{aligned} \tag{4.7}$$

Similarly to (3.6), from (4.4), we have

$$\left\| (y_1)_0 - (y_2)_0 \right\|_{\mathcal{B}_\nu} \le lL_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} + lL_h \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu}. \tag{4.8}$$

Equations (4.7) and (4.8) imply that

$$\begin{aligned}
&\|y_1 - y_2\|_* \\
&\quad\le l\big(\|\varphi_A\|_{L^1} + 2 \big) L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} \\
&\qquad + \left[lL_h + l\left(L_h + 2L_g + \frac{L_f T^q}{q\Gamma(q)} \right) \big(\|\varphi_A\|_{L^1} + 1 \big) \right] \left\| (y_1)_t - (y_2)_t \right\|_{\mathcal{B}_\nu} \\
&\quad\le l\big(\|\varphi_A\|_{L^1} + 2 \big) L_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} \\
&\qquad + \left[lL_h + l\left(L_h + 2L_g + \frac{L_f T^q}{q\Gamma(q)} \right) \big(\|\varphi_A\|_{L^1} + 1 \big) \right] \|y_1 - y_2\|_* \\
&\quad\le \wedge_\phi \|\phi_1 - \phi_2\|_{\mathcal{B}_\nu} + \wedge \|y_1 - y_2\|_*.
\end{aligned}$$

Therefore, (4.1) holds. Inequality (4.2) is a consequence of (4.1). This completes the proof. □

Competing interests
The authors declare that they have no competing interests.

Authors? contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final version of the manuscript.

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