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Some new fixed point theorems for the Meir-Keeler contractions on partial Hausdorff metric spaces

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Abstract
The purpose of this paper is to study fixed point theorems for a multi-valued mapping concerning with three classes of Meir-Keeler contractions with respect to the partial Hausdorff metric $\mathcal{H}$ in complete partial metric spaces. Our results generalize and improve many recent fixed point theorems for the partial Hausdorff metric in the literature.

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Keywords: fixed point; Meir-Keeler contraction; partial Hausdorff metric space

1 Introduction and preliminaries

Throughout this paper, by $\mathbb{R}^+$, we denote the set of all nonnegative real numbers, while $\mathbb{N}$ is the set of all natural numbers. Fixed point theory is one of the most crucial tools in nonlinear functional analysis and has application in distinct branches of mathematic. In 1922, Banach [1] introduced the most impressed fixed point result, and he concluded that each contraction has a unique fixed point in the complete metric space. Since then, this pioneer work has been generalized and extended in different abstract spaces. One of the interesting generalization of Banach fixed point theorem was given by Matthews [2] in 1994. In this paper, the author introduced the following notion of partial metric spaces and proved the Banach fixed point theorem in the context of complete partial metric space. We recall some basic definitions and fundamental results of partial metric spaces from the literature.

Definition 1 [2] A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

(p_1) $x = y$ if and only if $p(x,x) = p(x,y) = p(y,y)$;
(p_2) $p(x,x) \le p(x,y)$;
(p_3) $p(x,y) = p(y,x)$;
(p_4) $p(x,y) \le p(x,z) + p(z,y) - p(z,z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Remark 1 [2] It is clear that, if $p(x,y) = 0$, then from (p_1) and (p_2), we have $x = y$. But, if $x = y$, then the expression $p(x,y)$ may not be 0.

Each partial metric p on X generates a $\mathcal{T}_0$ topology τ_p on X which has as a base the family of open p-balls $\{B_p(x,\gamma) : x \in X, \gamma > 0\}$, where $B_p(x,\gamma) = \{y \in X : p(x,y) < p(x,x) + \gamma\}$ for all $x \in X$ and $\gamma > 0$. If p is a partial metric on X, then the function $d_p : X \times X \to \mathbb{R}^+$ given by

$$d_p(x,y) = 2p(x,y) - p(x,x) - p(y,y)$$

is a metric on X.

Definition 2 [2] Let (X,p) be a partial metric space. Then

(1) a sequence $\{x_n\}$ in a partial metric space (X,p) converges to $x \in X$ if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$;
(2) a sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if and only if $\lim_{m,n\to\infty} p(x_m,x_n)$ exists (and is finite);
(3) a partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x,x) = \lim_{m,n\to\infty} p(x_m,x_n)$;
(4) a subset A of a partial metric space (X,p) is closed if whenever $\{x_n\}$ is a sequence in A such that $\{x_n\}$ converges to some $x \in X$, then $x \in A$.

Remark 2 The limit in a partial metric space is not unique.

Lemma 1 [2, 3]

(1) $\{x_n\}$ *is a Cauchy sequence in a partial metric space* (X,p) *if and only if it is a Cauchy sequence in the metric space* (X,d_p);
(2) *a partial metric space* (X,p) *is complete if and only if the metric space* (X,d_p) *is complete. Furthermore*, $\lim_{n\to\infty} d_p(x_n,x) = 0$ *if and only if* $p(x,x) = \lim_{n\to\infty} p(x_n,x) = \lim_{n\to\infty} p(x_n,x_m)$.

Very recently Haghi *et al.* [4] proved that some fixed point results in partial metric space results are equivalent to results in the context of usual metric space. Recently, fixed point theory has developed rapidly on partial metric spaces; see *e.g.* [3, 5–12] and the references therein.

Let (X,d) be a metric space and $CB(X)$ denote the collection of all nonempty, closed and bounded subsets of X. For $A, B \in CB(X)$, we define

$$\mathcal{H}(A,B) := \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\},$$

where $d(x,B) := \inf\{d(x,b) : b \in B\}$, and it is well known that $\mathcal{H}$ is called the Hausdorff metric induced the metric d. A multi-valued mapping $T : X \to CB(X)$ is called a contraction if

$$\mathcal{H}(Tx,Ty) \leq kd(x,y),$$

for all $x, y \in X$ and $k \in [0,1)$. The study of fixed points for multi-valued contractions using the Hausdorff metric was introduced in Nadler [13].

Theorem 1 [13] *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued contraction. Then there exists $x \in X$ such that $x \in Tx$.*

Very recently, Aydi *et al.* [14] established the notion of partial Hausdorff metric $\mathcal{H}_p$ induced by the partial metric p. Let (X,p) be a partial metric space and $CB^p(X)$ be the collection of all nonempty, closed and bounded subset of the partial metric space (X,p). Note that closedness is taken from (X,τ_p) and boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \in \mathbb{R}$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(a,a) + M$. For $A,B \in CB^p(X)$ and $x \in X$, they define

$$\begin{aligned} &p(x,A) := \inf\{p(x,a) : a \in A\},\\ &\delta_p(A,B) := \sup\{p(a,B) : a \in A\},\\ &\delta_p(B,A) := \sup\{p(b,A) : b \in B\},\\ &\mathcal{H}_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}. \end{aligned}$$

It is immediate to see that if $p(x,A) = 0$, then $d_p(x,A) = 0$ where $d_p(x,A) = \inf\{d_p(x,a) : a \in A\}$.

Remark 3 [14] Let (X,p) be a partial metric space and A a nonempty subset of X. Then

$$a \in \overline{A} \quad \text{if and only if} \quad p(a,A) = p(a,a).$$

Aydi *et al.* [14] also introduced the following properties of mappings $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}$ and $\mathcal{H}_p : CB^p(X) \times CB^p(X) \to \mathbb{R}$.

Proposition 1 [14] *Let (X,p) be a partial metric space. For $A,B \in CB^p(X)$, the following properties hold*:

1. $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$;
2. $\delta_p(A,A) \le \delta_p(A,B)$;
3. $\delta_p(A,B) = 0$ *implies that* $A \subset B$;
4. $\delta_p(A,B) \le \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 2 [14] *Let (X,p) be a partial metric space. For $A,B \in CB^p(X)$, the following properties hold*:

1. $\mathcal{H}_p(A,A) \le \mathcal{H}_p(A,B)$;
2. $\mathcal{H}_p(A,B) = \mathcal{H}_p(B,A)$;
3. $\mathcal{H}_p(A,B) \le \mathcal{H}_p(A,C) + \mathcal{H}_p(C,B) - \inf_{c\in C} p(c,c)$;
4. $\mathcal{H}_p(A,B) = 0$ *implies that* $A = B$.

And, Aydi *et al.* [14] proved the following important result.

Lemma 2 *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a,b) \le h\mathcal{H}_p(A,B).$$

In this study, we also recall the notion of Meir-Keeler-type function (see [15]). A function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a Meir-Keeler-type function, if ξ satisfies the following condition:

$$\forall \eta > 0\ \exists \delta > 0\ \forall t \in \mathbb{R}^+ \quad \bigl(\eta \le t < \eta + \delta \Rightarrow \xi(t) < \eta\bigr).$$

Remark 4 It is clear that if ξ is a Meir-Keeler-type function, then we have

$$\xi(t) < t, \quad \text{for all } t \in \mathbb{R}^+.$$

We first introduce the notion of stronger Meir-Keeler-type function, as follows:

Definition 3 A function $\psi : \mathbb{R}^+ \to [0,1)$ is said to be a stronger Meir-Keeler-type function, if ψ satisfies the following condition:

$$\forall \eta > 0\ \exists \delta > 0\ \exists \gamma_\eta \in (0,1)\ \forall t \in \mathbb{R}^+ \quad \bigl(\eta \le t < \eta + \delta \Rightarrow \psi(t) < \gamma_\eta\bigr).$$

In 1972, Reich introduced the following important function.

Definition 4 [16] A function $\psi : \mathbb{R}^+ \to [0,1)$ is said to be a Reich function ($\mathcal{R}$-function, for short) if

$$\limsup_{s \to t^+} \psi(s) < 1, \quad \text{for all } t \in \mathbb{R}^+.$$

Remark 5 It is clear that, if the function $\psi : \mathbb{R}^+ \to [0,1)$ is a Reich function ($\mathcal{R}$-function), then ψ is also a stronger Meir-Keeler-type function.

We next introduce the notion of weaker Meir-Keeler function, as follows:

Definition 5 A function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a weaker Meir-Keeler-type function, if φ satisfies the following condition:

$$\forall \eta > 0\ \exists \delta > 0\ \forall t \in \mathbb{R}^+ \quad \bigl(\eta \le t < \eta + \delta \Rightarrow \exists n_0 \in \mathbb{N}, \varphi^{n_0}(t) < \eta\bigr).$$

The purpose of this paper is to study fixed point theorems for a multi-valued mapping concerning three classes of Meir-Keeler contractions with respect to the partial Hausdorff metric $\mathcal{H}$ in complete partial metric spaces. Our results generalize and improve many recent fixed point theorems for the partial Hausdorff metric in the literature.

2 Fixed point theorem (I)

In the sequel, we denote by Φ the class of functions $\phi : {\mathbb{R}^+}^4 \to \mathbb{R}^+$ satisfying the following conditions:

(1) ϕ is an increasing and continuous function in each coordinate;

(2) for $t \in \mathbb{R}^+ \backslash \{0\}$, $\phi(t,t,t,t) \le t$ and $\phi(t_1,t_2,t_3,t_4) = 0$ iff $t_1 = t_2 = t_3 = t_4 = 0$.

We now introduce the notion of (ψ,ϕ)-Meir-Keeler contraction on partial Hausdorff metric spaces.

Definition 6 Let (X,p) be a partial metric space, $\psi:\mathbb{R}^+\to[0,1)$ and $\phi\in\Phi$. We call $T:X\to CB^p(X)$ a (ψ,ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, if the following conditions hold:

(c_1) ψ is a stronger Meir-Keeler-type function;
(c_2) for all $x,y\in X$, we have

$$\mathcal{H}_p(Tx,Ty)\le\psi\bigl(p(x,y)\bigr)\phi\biggl(p(x,y),p(x,Tx),p(y,Ty),\frac{p(x,Ty)+p(y,Tx)}{2}\biggr).$$

We state and prove the main fixed point result for the (ψ,ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$.

Theorem 2 *Let (X,p) be a complete partial metric space. Suppose $T:X\to CB^p(X)$ is a (ψ,ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$. Then T has a fixed point in X, that is, there exists $x^*\in X$ such that $x^*\in Tx^*$.*

Proof Let $x_0\in X$ be given and let $x_1\in Tx_0$. Since $T:X\to CB^p(X)$ is a (ψ,ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned}&\mathcal{H}_p(Tx_0,Tx_1)\\&\quad\le\psi\bigl(p(x_0,x_1)\bigr)\phi\biggl(p(x_0,x_1),p(x_0,Tx_0),p(x_1,Tx_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\biggr).\end{aligned}\tag{2.1}$$

Put $p(x_0,x_1)=\eta_0>0$. Since ψ is a stronger Meir-Keeler-type function, there exists $\gamma_{\eta_0}\in(0,1)$ such that

$$\psi\bigl(p(x_0,x_1)\bigr)<\gamma_{\eta_0}.\tag{2.2}$$

From (2.1) and (2.2), we have

$$\mathcal{H}_p(Tx_0,Tx_1)<\gamma_{\eta_0}\cdot\phi\biggl(p(x_0,x_1),p(x_0,Tx_0),p(x_1,Tx_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\biggr).\tag{2.3}$$

From Lemma 2 with $h=\frac{1}{\sqrt{\gamma_{\eta_0}}}>1$, there exists $x_2\in Tx_1$ such that

$$p(x_1,x_2)\le\frac{1}{\sqrt{\gamma_{\eta_0}}}\mathcal{H}_p(Tx_0,Tx_1).\tag{2.4}$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned}p(x_1,x_2)&<\sqrt{\gamma_{\eta_0}}\cdot\phi\biggl(p(x_0,x_1),p(x_0,Tx_0),p(x_1,Tx_1),\frac{p(x_0,Tx_1)+p(x_1,Tx_0)}{2}\biggr)\\&\le\sqrt{\gamma_{\eta_0}}\cdot\phi\biggl(p(x_0,x_1),p(x_0,x_1),p(x_1,x_2),\frac{p(x_0,x_2)+p(x_1,x_1)}{2}\biggr)\\&\le\sqrt{\gamma_{\eta_0}}\cdot\phi\biggl(p(x_0,x_1),p(x_0,x_1),p(x_1,x_2),\frac{p(x_0,x_1)+p(x_1,x_2)}{2}\biggr).\end{aligned}\tag{2.5}$$

If $p(x_0,x_1) \leq p(x_1,x_2)$, then by the definition of the function ϕ, we have

$$\begin{aligned} p(x_1,x_2) &< \sqrt{\gamma_{\eta_0}} \cdot \phi\left(p(x_0,x_1), p(x_0,x_1), p(x_1,x_2), \frac{p(x_0,x_1)+p(x_1,x_2)}{2}\right) \\ &\leq \sqrt{\gamma_{\eta_0}} \cdot \phi\big(p(x_1,x_2), p(x_1,x_2), p(x_1,x_2), p(x_1,x_2)\big) \\ &\leq \sqrt{\gamma_{\eta_0}} \cdot (x_1,x_2), \end{aligned}$$

which implies a contradiction, and hence $p(x_0,x_1) > p(x_1,x_2)$. Therefore, we have

$$\begin{aligned} p(x_1,x_2) &< \sqrt{\gamma_{\eta_0}} \cdot \phi\left(p(x_0,x_1), p(x_0,x_1), p(x_1,x_2), \frac{p(x_0,x_1)+p(x_1,x_2)}{2}\right) \\ &\leq \sqrt{\gamma_{\eta_0}} \cdot \phi\big(p(x_0,x_1), p(x_0,x_1), p(x_0,x_1), p(x_0,x_1)\big) \\ &\leq \sqrt{\gamma_{\eta_0}} \cdot p(x_0,x_1). \end{aligned} \tag{2.6}$$

Put $p(x_1,x_2) = \eta_1 > 0$. Since ψ is a stronger Meir-Keeler-type function, there exists $\gamma_{\eta_1} \in (0,1)$ such that

$$\psi\big(p(x_1,x_2)\big) < \gamma_{\eta_1}. \tag{2.7}$$

From Lemma 2 with $h = \frac{1}{\sqrt{\gamma_{\eta_1}}} > 1$, we have

$$p(x_2,x_3) \leq \frac{1}{\sqrt{\gamma_{\eta_1}}} \mathcal{H}_p(Tx_1, Tx_2). \tag{2.8}$$

Since $T : X \to CB^p(X)$ is a (ψ,ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned} &\mathcal{H}_p(Tx_1, Tx_2) \\ &\quad \leq \psi\big(p(x_1,x_2)\big)\phi\left(p(x_1,x_2), p(x_1,Tx_1), p(x_2,Tx_2), \frac{p(x_1,Tx_2)+p(x_2,Tx_1)}{2}\right) \\ &\quad < \gamma_{\eta_1} \cdot \phi\left(p(x_1,x_2), p(x_1,x_2), p(x_2,x_3), \frac{p(x_1,x_3)+p(x_2,x_2)}{2}\right) \\ &\quad \leq \gamma_{\eta_1} \cdot \phi\left(p(x_1,x_2), p(x_1,x_2), p(x_2,x_3), \frac{p(x_1,x_2)+p(x_2,x_3)}{2}\right). \end{aligned} \tag{2.9}$$

Using (2.8) and (2.9), we obtain

$$p(x_2,x_3) < \sqrt{\gamma_{\eta_1}} \cdot \phi\left(p(x_1,x_2), p(x_1,x_2), p(x_2,x_3), \frac{p(x_1,x_2)+p(x_2,x_3)}{2}\right).$$

If $p(x_1,x_2) \leq p(x_2,x_3)$, then

$$\begin{aligned} p(x_2,x_3) &< \sqrt{\gamma_{\eta_1}} \cdot \phi\left(p(x_1,x_2), p(x_1,x_2), p(x_2,x_3), \frac{p(x_1,x_2)+p(x_2,x_3)}{2}\right) \\ &\leq \sqrt{\gamma_{\eta_1}} \cdot \phi\big(p(x_2,x_3), p(x_2,3), p(x_2,x_3), p(x_2,x_3)\big) \\ &\leq \sqrt{\gamma_{\eta_1}} \cdot p(x_2,x_3), \end{aligned}$$

which implies a contradiction, and hence $p(x_1, x_2) > p(x_2, x_3)$. Therefore

$$p(x_2, x_3) \le \sqrt{\gamma_{\eta_0}}\sqrt{\gamma_{\eta_1}} \cdot p(x_0, x_1). \tag{2.10}$$

Continuing this process, we can obtain a sequence $\{x_n\}$ of X recursively as follows:

$$x_n \in Tx_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Put $p(x_n, x_{n+1}) = \eta_n > 0$. Since ψ is a stronger Meir-Keeler-type function, there exists $\gamma_{\eta_n} \in (0, 1)$ such that

$$(x_n, x_{n+1}) < \gamma_{\eta_n}. \tag{2.11}$$

Since $T : X \to CB^p(X)$ is a (ψ, ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have for all $n \in \mathbb{N} \cup \{0\}$

$$\begin{aligned}
&\mathcal{H}_p(Tx_n, Tx_{n+1}) \\
&\quad \le \psi\big(p(x_n, x_{n+1})\big)\phi\bigg(p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \\
&\quad\quad \frac{1}{2}\big[p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)\big]\bigg) \\
&\quad < \gamma_{\eta_n}\phi\bigg(p(x_n, x_{n+1}), p(x_n, Tx_n), p(x_{n+1}, Tx_{n+1}), \frac{1}{2}\big[p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n)\big]\bigg) \\
&\quad < \gamma_{\eta_n}\phi\bigg(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}\big[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})\big]\bigg).
\end{aligned} \tag{2.12}$$

From Lemma 2 with $h = \frac{1}{\sqrt{\gamma_{\eta_n}}} > 1$, we have

$$p(x_{n+1}, x_{n+2}) \le \frac{1}{\sqrt{\gamma_{\eta_n}}}\mathcal{H}_p(Tx_n, Tx_{n+1}), \quad n \in \mathbb{N} \cup \{0\}. \tag{2.13}$$

Using (2.12) and (2.13), we obtain

$$\begin{aligned}
&p(x_{n+1}, x_{n+2}) \\
&\quad < \sqrt{\gamma_{\eta_n}}\phi\bigg(p(x_n, x_{n+1}), p(x_n, x_{n+1}), p(x_{n+1}, x_{n+2}), \frac{1}{2}\big[p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2})\big]\bigg).
\end{aligned}$$

If $p(x_n, x_{n+1}) \le p(x_{n+1}, x_{n+2})$, then

$$\begin{aligned}
p(x_{n+1}, x_{n+2}) &< \sqrt{\gamma_{\eta_n}}\phi\big(p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2}), p(x_{n+1}, x_{n+2})\big) \\
&\le \sqrt{\gamma_{\eta_n}} p(x_{n+1}, x_{n+2}),
\end{aligned}$$

which implies a contradiction, and hence $p(x_0, x_1) > p(x_1, x_2)$. Therefore, we have

$$p(x_{n+1}, x_{n+2}) \le \sqrt{\gamma_{\eta_n}} p(x_n, x_{n+1}). \tag{2.14}$$

By the mathematical induction, we obtain

$$\begin{aligned} p(x_{n+1},x_{n+2}) &\le \sqrt{\gamma_{\eta_n}}\,p(x_n,x_{n+1}) \\ &\le \sqrt{\gamma_{\eta_n}}\sqrt{\gamma_{\eta_{n-1}}}\,p(x_{n-1},x_n) \\ &\le \cdots \\ &\le \sqrt{\gamma_{\eta_n}}\sqrt{\gamma_{\eta_{n-1}}}\cdots\sqrt{\gamma_{\eta_0}}\,p(x_0,x_1). \end{aligned} \tag{2.15}$$

Put

$$\overline{k} = \max\bigl\{\sqrt{\gamma_{\eta_n}} : n \in \mathbb{N}\cup\{0\}\bigr\}. \tag{2.16}$$

Using (2.15) and (2.16), we obtain

$$p(x_{n+1},x_{n+2}) \le (\overline{k})^{n+1} p(x_0,x_1), \quad \text{for all } n \in \mathbb{N}\cup\{0\}. \tag{2.17}$$

Let $n \to \infty$ in (2.17). Then

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{2.18}$$

By the property (p_2) of a partial metric and using (2.18), we have

$$\lim_{n\to\infty} p(x_n,x_n) = 0. \tag{2.19}$$

Using (2.17) and the property (p_4) of a partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n,x_{n+m}) &\le \sum_{i=1}^{m} p(x_{n+i-1},x_{n+i}) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}) \\ &\le \sum_{i=1}^{m} (\overline{k})^{n+i-1} p(x_0,x_1) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}) \\ &\le \frac{(\overline{k})^n}{(1-\overline{k})} p(x_0,x_1) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}). \end{aligned} \tag{2.20}$$

Using (2.19) and (2.20), we get

$$\lim_{n\to\infty} p(x_n,x_{n+m}) = 0.$$

By the definition of d_p, we see that, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d_p(x_n,x_{n+m}) \le \lim_{n\to\infty} 2p(x_n,x_{n+m}) = 0. \tag{2.21}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X,d_p). Since (X,p) is complete, from Lemma 1, (X,d_p) is a complete metric space. Therefore, $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric d_p, and we also have

$$p\bigl(x^*,x^*\bigr) = \lim_{n\to\infty} p\bigl(x_n,x^*\bigr) = \lim_{n\to\infty} p(x_n,x_m) = 0. \tag{2.22}$$

Since $T : X \to CB^p(X)$ is a (ψ, ϕ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned}
&\mathcal{H}_p(Tx_n, Tx^*) \\
&\quad \le \psi\big(p(x_n, x^*)\big)\phi\left(p(x_n, x^*), p(x_n, Tx_n), p(x^*, Tx^*), \frac{p(x_n, Tx^*) + p(x^*, Tx_n)}{2}\right) \\
&\quad \le \psi\big(p(x_n, x^*)\big)\phi\left(p(x_n, x^*), p(x_n, x_{n+1}), p(x^*, Tx^*), \frac{p(x_n, Tx^*) + p(x^*, x_{n+1})}{2}\right).
\end{aligned}$$

By the definition of the mapping ψ and using (2.22), we get

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx^*) &< \phi\left(0, 0, p(x^*, Tx^*), \frac{1}{2}p(x^*, Tx^*)\right) \\
&\le p(x^*, Tx^*). \qquad (2.23)
\end{aligned}$$

Now $x_{n+1} \in Tx_n$ shows

$$p(x_{n+1}, Tx^*) \le \delta_p(Tx_n, Tx^*) \le \mathcal{H}_p(Tx_n, Tx^*).$$

Using (2.23), we get

$$p(x^*, Tx^*) = \lim_{n\to\infty} p(x_{n+1}, Tx^*) < p(x^*, Tx^*),$$

a contradiction. So, we have

$$p(x^*, Tx^*) = 0.$$

Therefore, from (2.22), $p(x^*, x^*) = 0$, we obtain

$$p(x^*, x^*) = p(x^*, Tx^*),$$

which implies $x^* \in Tx^*$ by Remark 3. □

Using the Reich function and stronger Meir-Keeler function, we establish the following notion of (ψ, ϕ)-Reich's contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$.

Definition 7 Let (X, p) be a partial metric space, $\psi : \mathbb{R}^+ \to [0, 1)$, and $\phi \in \Phi$. We call $T : X \to CB^p(X)$ a (ψ, ϕ)-Reich's contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$ if the following conditions hold:

(1) ψ is a Reich function ($\mathcal{R}$-function);

(2) for all $x, y \in X$, we have

$$\mathcal{H}_p(Tx, Ty) \le \psi\big(p(x, y)\big)\phi\left(p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right).$$

Apply above Remark 5, Definition 7, and Theorem 2, we are easy to get the following theorem.

Theorem 3 *Let (X,p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a (ψ,ϕ)-Reich's contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$. Then T has a fixed point in X, that is, there exists $x^* \in X$ such that $x^* \in Tx^*$.*

3 Fixed point theorem (II)

In this section, we let Ξ be the class of all non-decreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(1) φ is a weaker Meir-Keeler-type function;
(2) for all $t \in (0,\infty)$, $\{\varphi^n(t)\}_{n\in\mathbb{N}}$ is decreasing;
(3) $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) = 0$,
(4) for $t > 0$, if $\lim_{n\to\infty} \varphi^n(t) = 0$, then $\lim_{n\to\infty} \sum_{i=n}^{m} \varphi^i(t) = 0$, where $m > n$;
(5) for $t_n \in \mathbb{R}^+$, if $\lim_{n\to\infty} t_n = 0$, then $\lim_{n\to\infty} \varphi(t_n) = 0$.

We recall the notion of α-admissible function that was introduced in [17].

Definition 8 [17] Let $f : X \to X$ be a self-mapping of a set X and $\alpha : X \times X \to \mathbb{R}^+$. Then f is called a α-admissible if

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(fx,fy) \geq 1.$$

In [18], the authors introduced the following notion of strictly α-admissible.

Definition 9 Let (X,p) be a partial metric space, $T : X \to CB^p(X)$ and $\alpha : X \times X \to \mathbb{R}^+\setminus\{0\}$. We say that T is strictly α-admissible if

$$\alpha(x,y) > 1 \quad \text{implies} \quad \alpha(y,z) > 1, \quad x \in X, y \in Tx, z \in Ty.$$

We now introduce the notion of (α,φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, as follows:

Definition 10 Let (X,p) be a partial metric space, $\varphi \in \Xi$, and $\alpha : X \times X \to \mathbb{R}^+\setminus\{0\}$. We call $T : X \to CB^p(X)$ a (α,φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$ if the following conditions hold:

(c_1) T is strictly α-admissible;
(c_2) for each $x,y \in X$,

$$\alpha(x,y)\mathcal{H}_p(Tx,Ty) \leq \varphi\bigl(p(x,y)\bigr).$$

We now state and prove our main result for the (α,φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$.

Theorem 4 *Let (X,p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a (α,φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$. Suppose also that*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,y) > 1$ for all $y \in Tx_0$;*
(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n,x) \geq 1$ for all n.*

Then T has a fixed point in X (that is, there exists $x^* \in X$ such that $x^* \in Tx^*$).

Proof Let $x_1 \in Tx_0$. Since $T : X \to CB^p(X)$ is a (α, φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\alpha(x_0, x_1)\mathcal{H}_p(Tx_0, Tx_1) \leq \varphi\big(p(x_0, x_1)\big). \tag{3.1}$$

Put $\alpha(x_0, x_1) = h_0 > 1$. From Lemma 2 with $h = h_0$, there exists $x_2 \in Tx_1$ such that

$$p(x_1, x_2) \leq h_0\mathcal{H}_p(Tx_0, Tx_1). \tag{3.2}$$

Using (3.1) and (3.2), we obtain

$$p(x_1, x_2) \leq \varphi\big(p(x_0, x_1)\big). \tag{3.3}$$

Continuing this process, we can obtain a sequence $\{x_n\}$ of X recursively as follows:

$$x_n \in Tx_{n-1}, \quad \text{for all } n \in \mathbb{N}.$$

Since T is strictly α-admissible, we deduce that $\alpha(x_1, x_2) = h_1 > 1$. Continuing this process, we have

$$\alpha(x_n, x_{n+1}) = h_n > 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.4}$$

And, since $T : X \to CB^p(X)$ is a (α, φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\alpha(x_n, x_{n+1})\mathcal{H}_p(Tx_n, Tx_{n+1}) \leq \varphi\big(p(x_n, x_{n+1})\big), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.5}$$

From Lemma 2 with h_n, we have

$$p(x_{n+1}, x_{n+2}) \leq h_n\mathcal{H}_p(Tx_n, Tx_{n+1}), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.6}$$

Using (3.5) and (3.6), we obtain

$$p(x_{n+1}, x_{n+2}) \leq \varphi\big(p(x_n, x_{n+1})\big), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{3.7}$$

Therefore, we conclude that

$$\begin{aligned} p(x_{n+1}, x_{n+2}) &\leq \varphi\big(p(x_n, x_{n+1})\big) \\ &\leq \varphi^2\big(p(x_{n-1}, x_n)\big) \\ &\leq \cdots \\ &\leq \varphi^{n+1}\big(p(x_0, x_1)\big). \end{aligned} \tag{3.8}$$

By the condition (φ_2), $\{\varphi^n(p(x_0, x_1))\}_{n\in\mathbb{N}}$ is decreasing, it must converges to some $\eta \geq 0$. We claim that $\eta = 0$. On the contrary, assume that $\eta > 0$. Then by the definition of the weaker Meir-Keeler-type function, there exists $\delta > 0$ such that for $x_0, x_1 \in X$ with $\eta \leq p(x_0, x_1) <$

$\delta + \eta$, there exists $n_0 \in \mathbb{N}$ such that $\varphi^{n_0}(p(x_0, x_1)) < \eta$. Since $\lim_{n\to\infty} \varphi^n(p(x_0, x_1)) = \eta$, there exists $m_0 \in \mathbb{N}$ such that $\eta \le \varphi^m(p(x_0, x_1)) < \delta + \eta$, for all $m \ge m_0$. Thus, we conclude that $\varphi^{m_0+n_0}(p(x_0, x_1)) < \eta$. So we get a contradiction. So $\lim_{n\to\infty} \varphi^n(p(x_0, x_1)) = 0$, and so

$$\lim_{n\to\infty} p(x_n, x_{n+1}) = 0. \tag{3.9}$$

By the property (p_2) of a partial metric and using (3.9), we have

$$\lim_{n\to\infty} p(x_n, x_n) = 0. \tag{3.10}$$

We will prove that the sequence $\{x_n\}$ is a Cauchy sequence. Using (3.8), we have

$$\begin{aligned} p(x_n, x_{n+m}) &= p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+m}) - p(x_{n+1}, x_{n+1}) \\ &\le p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+m}) \\ &\quad - p(x_{n+1}, x_{n+1}) - p(x_{n+2}, x_{n+2}) \\ &\le \sum_{i=1}^{k} p(x_{n+i-1}, x_{n+i}) - \sum_{i=1}^{k-1} p(x_{n+i}, x_{n+i}) \\ &\le \sum_{i=1}^{m} \varphi^{n+i-1} p(x_0, x_1) - \sum_{i=1}^{m-1} p(x_{n+i}, x_{n+i}) \\ &\le \sum_{i=1}^{m} \varphi^{n+i-1} p(x_0, x_1). \end{aligned}$$

Letting $n \to \infty$, then, by using the condition (φ_4), we have

$$\lim_{n\to\infty} p(x_n, x_{n+m}) = 0. \tag{3.11}$$

By the definition of d_p, we see that, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d_p(x_n, x_{n+m}) \le \lim_{n\to\infty} 2p(x_n, x_{n+m}) = 0. \tag{3.12}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, from Lemma 1, (X, d_p) is a complete metric space. Therefore, $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric d_p, and we also have

$$p\bigl(x^*, x^*\bigr) = \lim_{n\to\infty} p\bigl(x_n, x^*\bigr) = \lim_{n\to\infty} p(x_n, x_m) = 0. \tag{3.13}$$

Since $T : X \to CB^p(X)$ is a (α, φ)-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\alpha\bigl(x_n, x^*\bigr)\mathcal{H}_p\bigl(Tx_n, Tx^*\bigr) \le \varphi\bigl(p\bigl(x_n, x^*\bigr)\bigr).$$

By the definition of the mapping α, we have $\alpha(x_n, x^*) > 0$. By the condition (φ_5) and using (3.13), we get

$$\lim_{n\to\infty} \mathcal{H}_p\bigl(Tx_n, Tx^*\bigr) = 0. \tag{3.14}$$

Now $x_{n+1} \in Tx_n$ shows

$$p(x_{n+1}, Tx^*) \leq \delta_p(Tx_n, Tx^*) \leq \mathcal{H}_p(Tx_n, Tx^*).$$

Using (3.14), we get

$$p(x^*, Tx^*) = \lim_{n\to\infty} p(x_{n+1}, Tx^*) = 0.$$

Therefore, from (3.13), $p(x^*, x^*) = 0$, we obtain

$$p(x^*, x^*) = p(x^*, Tx^*),$$

which implies $x^* \in Tx^*$ by Remark 3. □

4 Fixed point theorem (III)

In this section, we consider the family

$$\Omega = \{(\xi_1, \xi_2, \xi_3, \xi_4) | \xi_i : \mathbb{R}^+ \to \mathbb{R}^+, i = 1, 2, 3, 4\}$$

such that:

(1) $\xi_1(t), \xi_2(t), \xi_3(t) \leq \xi_4(t)$ for all $t > 0$;
(2) $\xi_1(t)$, $\xi_2(t)$, $\xi_3(t)$, $\xi_4(t)$ are continuous;
(3) $\xi_1(t_1) = \xi_2(t_2) = \xi_3(t_3) = \xi_4(t_4) = 0$ if and only if $t_1 = t_2 = t_3 = t_4 = 0$;
(4) ξ_4 is a Meir-Keeler-type function;
(5) $\xi_4(t_1 + t_2) \leq \xi_4(t_1) + \xi_4(t_2)$ for all $t_1, t_2 > 0$.

We now introduce the notion of $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction on partial Hausdorff metric spaces.

Definition 11 Let (X, p) be a partial metric space, $\phi \in \Phi$, $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega$, and $\alpha : X \times X \to \mathbb{R}^+ \setminus \{0\}$. We call $T : X \to CB^p(X)$ a $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$ if the following conditions hold:

(1) T is strictly α-admissible;
(2) for all $x, y \in X$, we have

$$\alpha(x,y)\mathcal{H}_p(Tx, Ty) \leq \phi\left(\xi_1\big(p(x,y)\big), \xi_2\big(p(x, Tx)\big), \xi_3\big(p(y, Ty)\big), \frac{\xi_4(p(x, Ty) + p(y, Tx))}{2}\right).$$

We now state and prove our main result for the $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$.

Theorem 5 *Let (X, p) be a complete partial metric space. Suppose $T : X \to CB^p(X)$ is a $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$. Suppose also that*

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, y) > 1$ for all $y \in Tx_0$;*
(ii) *if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then $\alpha(x_n, x) \geq 1$ for all n.*

Then T has a fixed point in X (that is, there exists $x^* \in X$ such that $x^* \in Tx^*$).

Proof Let $x_1 \in Tx_0$. Since $T : X \to CB^p(X)$ is a $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned}&\alpha(x_0,x_1)\mathcal{H}_p(Tx_0,Tx_1)\\&\quad\le\phi\left(\xi_1\bigl(p(x_0,x_1)\bigr),\xi_2\bigl(p(x_0,Tx_0)\bigr),\xi_3\bigl(p(x_1,Tx_1)\bigr),\frac{\xi_4(p(x_0,Tx_1)+p(x_1,Tx_0))}{2}\right).\end{aligned}\tag{4.1}$$

By the condition (i), we have $\alpha(x_0, x_1) > 1$. Put $\alpha(x_0, x_1) = h_0$. Note $h_0 > 1$. From Lemma 2 with $h = \sqrt{h_0}$, there exists $x_2 \in Tx_1$ such that

$$p(x_1,x_2)\le\sqrt{h_0}\mathcal{H}_p(Tx_0,Tx_1).\tag{4.2}$$

Using (4.1) and (4.2), we have

$$\begin{aligned}0&\le p(x_1,x_2)\\&\le\frac{1}{\sqrt{h_0}}\phi\left(\xi_1\bigl(p(x_0,x_1)\bigr),\xi_2\bigl(p(x_0,Tx_0)\bigr),\xi_3\bigl(p(x_1,Tx_1)\bigr),\frac{\xi_4(p(x_0,Tx_1)+p(x_1,Tx_0))}{2}\right)\\&\le\frac{1}{\sqrt{h_0}}\phi\left(\xi_1\bigl(p(x_0,x_1)\bigr),\xi_2\bigl(p(x_0,x_1)\bigr),\xi_3\bigl(p(x_1,x_2)\bigr),\frac{\xi_4(p(x_0,x_1)+p(x_1,x_2))}{2}\right)\\&\le\frac{1}{\sqrt{h_0}}\phi\left(\xi_1\bigl(p(x_0,x_1)\bigr),\xi_2\bigl(p(x_0,x_1)\bigr),\xi_3\bigl(p(x_1,x_2)\bigr),\frac{\xi_4(p(x_0,x_1))+\xi_4(p(x_1,x_2))}{2}\right).\end{aligned}\tag{4.3}$$

Now, if $\xi_4(p(x_0,x_1)) \le \xi_4(p(x_1,x_2))$, then by using (4.3) and since $\phi \in \Phi$, $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega$, we have

$$\begin{aligned}0&\le p(x_1,x_2)\\&\le\frac{1}{\sqrt{h_0}}\phi\left(\xi_4\bigl(p(x_0,x_1)\bigr),\xi_4\bigl(p(x_0,x_1)\bigr),\xi_4\bigl(p(x_1,x_2)\bigr),\frac{\xi_4(p(x_0,x_1)+p(x_1,x_2))}{2}\right)\\&\le\frac{1}{\sqrt{h_0}}\phi\bigl(\xi_4\bigl(p(x_1,x_2)\bigr),\xi_4\bigl(p(x_1,x_2)\bigr),\xi_4\bigl(p(x_1,x_2)\bigr),\xi_4\bigl(p(x_1,x_2)\bigr)\bigr)\\&\le\frac{1}{\sqrt{h_0}}\xi_4\bigl(p(x_1,x_2)\bigr).\end{aligned}$$

By Remark 4, we also have

$$p(x_1,x_2)\le\frac{1}{\sqrt{h_0}}\xi_4\bigl(p(x_1,x_2)\bigr)<p(x_1,x_2),$$

which implies a contradiction. Therefore, we have

$$p(x_1,x_2)\le\frac{1}{\sqrt{h_0}}\xi_4\bigl(p(x_0,x_1)\bigr)\le\frac{1}{\sqrt{h_0}}p(x_0,x_1).$$

Since $T : X \to CB^p(X)$ is a $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned}&\alpha(x_1,x_2)\mathcal{H}_p(Tx_1,Tx_2)\\&\quad\le\phi\biggl(\xi_1\bigl(p(x_1,x_2)\bigr),\xi_2\bigl(p(x_1,Tx_1)\bigr),\xi_3\bigl(p(x_2,Tx_2)\bigr),\frac{\xi_4(p(x_1,Tx_2)+p(x_2,Tx_1))}{2}\biggr).\end{aligned}\tag{4.4}$$

Since T is strictly α-admissible, we have $\alpha(x_1,x_2)>1$. Put $\alpha(x_1,x_2)=h_1$. Note $h_1>1$. From Lemma 2 with $h=\sqrt{h_1}$, there exists $x_3\in Tx_2$ such that

$$p(x_2,x_3)\le\sqrt{h_1}\mathcal{H}_p(Tx_1,Tx_2).\tag{4.5}$$

Using (4.4) and (4.5), we have

$$\begin{aligned}0&\le p(x_2,x_3)\\&\le\frac{1}{\sqrt{h_1}}\phi\biggl(\xi_1\bigl(p(x_1,x_2)\bigr),\xi_2\bigl(p(x_1,Tx_1)\bigr),\xi_3\bigl(p(x_2,Tx_2)\bigr),\frac{\xi_4(p(x_1,Tx_2)+p(x_2,Tx_1))}{2}\biggr)\\&\le\frac{1}{\sqrt{h_1}}\phi\biggl(\xi_1\bigl(p(x_1,x_2)\bigr),\xi_2\bigl(p(x_1,x_2)\bigr),\xi_3\bigl(p(x_2,x_3)\bigr),\frac{\xi_4(p(x_1,x_2)+p(x_2,x_3))}{2}\biggr)\\&\le\frac{1}{\sqrt{h_1}}\phi\biggl(\xi_1\bigl(p(x_1,x_2)\bigr),\xi_2\bigl(p(x_1,x_2)\bigr),\xi_3\bigl(p(x_2,x_3)\bigr),\frac{\xi_4(p(x_1,x_2))+\xi_4(p(x_2,x_3))}{2}\biggr).\end{aligned}\tag{4.6}$$

Now, if $\xi_4(p(x_1,x_2))\le\xi_4(p(x_2,x_3))$, then by using (4.6) and since $\phi\in\Phi$, $(\xi_1,\xi_2,\xi_3,\xi_4)\in\Omega$, we have

$$\begin{aligned}0&\le p(x_2,x_3)\\&\le\frac{1}{\sqrt{h_1}}\phi\biggl(\xi_4\bigl(p(x_1,x_2)\bigr),\xi_4\bigl(p(x_1,x_2)\bigr),\xi_4\bigl(p(x_2,x_3)\bigr),\frac{\xi_4(p(x_1,x_2)+p(x_2,x_3))}{2}\biggr)\\&\le\frac{1}{\sqrt{h_1}}\phi\bigl(\xi_4\bigl(p(x_2,x_3)\bigr),\xi_4\bigl(p(x_2,x_3)\bigr),\xi_4\bigl(p(x_2,x_3)\bigr),\xi_4\bigl(p(x_2,x_3)\bigr)\bigr)\\&\le\frac{1}{\sqrt{h_1}}\xi_4\bigl(p(x_2,x_3)\bigr).\end{aligned}$$

By Remark 4,we also have

$$p(x_2,x_3)\le\frac{1}{\sqrt{h_1}}\xi_4\bigl(p(x_2,x_3)\bigr)<p(x_2,x_3),$$

which implies a contradiction. Therefore, we have

$$p(x_2,x_3)\le\frac{1}{\sqrt{h_1}}\xi_4\bigl(p(x_1,x_2)\bigr)\le\frac{1}{\sqrt{h_1}}p(x_1,x_2)\le\frac{1}{\sqrt{h_0}}\frac{1}{\sqrt{h_1}}p(x_0,x_1).$$

Continuing this process, we can obtain a sequence $\{x_n\}$ of X recursively as follows:

$$x_n\in Tx_{n-1},\quad\text{for all } n\in\mathbb{N}.$$

Since $T : X \to CB^p(X)$ is a $(\alpha, \phi, \xi_1, \xi_2, \xi_3, \xi_4)$-Meir-Keeler contraction with respect to the partial Hausdorff metric $\mathcal{H}_p$, we have

$$\begin{aligned}
&\alpha(x_n, x_{n+1})\mathcal{H}_p(Tx_n, Tx_{n+1}) \\
&\quad \le \phi\biggl(\xi_1\bigl(p(x_n, x_{n+1})\bigr), \xi_2\bigl(p(x_n, Tx_n)\bigr), \xi_3\bigl(p(x_{n+1}, Tx_{n+1})\bigr), \\
&\qquad \frac{\xi_4(p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n))}{2}\biggr).
\end{aligned} \tag{4.7}$$

Since T is strictly α-admissible, we have $\alpha(x_n, x_{n+1}) > 1$ for all $n \in \mathbb{N}$. Put $\alpha(x_n, x_{n+1}) = h_n$. From Lemma 2 with $h = \sqrt{h_n}$, there exists $x_{n+2} \in Tx_{n+1}$ such that

$$p(x_{n+1}, x_{n+2}) \le \sqrt{h_n}\mathcal{H}_p(Tx_n, Tx_{n+1}). \tag{4.8}$$

Using (4.7) and (4.8), we have

$$\begin{aligned}
0 &\le p(x_{n+1}, x_{n+2}) \\
&\le \frac{1}{\sqrt{h_n}}\phi\biggl(\xi_1\bigl(p(x_n, x_{n+1})\bigr), \xi_2\bigl(p(x_n, Tx_n)\bigr), \xi_3\bigl(p(x_{n+1}, Tx_{n+1})\bigr), \\
&\qquad \frac{\xi_4(p(x_n, Tx_{n+1}) + p(x_{n+1}, Tx_n))}{2}\biggr) \\
&\le \frac{1}{\sqrt{h_n}}\phi\biggl(\xi_1\bigl(p(x_n, x_{n+1})\bigr), \xi_2\bigl(p(x_n, x_{n+1})\bigr), \xi_3\bigl(p(x_{n+1}, x_{n+2})\bigr), \\
&\qquad \frac{\xi_4(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}))}{2}\biggr) \\
&\le \frac{1}{\sqrt{h_n}}\phi\biggl(\xi_1\bigl(p(x_n, x_{n+1})\bigr), \xi_2\bigl(p(x_n, x_{n+1})\bigr), \xi_3\bigl(p(x_{n+1}, x_{n+2})\bigr), \\
&\qquad \frac{\xi_4(p(x_n, x_{n+1})) + \xi_4(p(x_{n+1}, x_{n+2}))}{2}\biggr).
\end{aligned} \tag{4.9}$$

Now, if $\xi_4(p(x_n, x_{n+1})) \le \xi_4(p(x_{n+1}, x_{n+2}))$, then by using (4.9) and since $\phi \in \Phi$, $(\xi_1, \xi_2, \xi_3, \xi_4) \in \Omega$, we have

$$\begin{aligned}
0 &\le p(x_{n+1}, x_{n+2}) \\
&\le \frac{1}{\sqrt{h_n}}\phi\biggl(\xi_4\bigl(p(x_n, x_{n+1})\bigr), \xi_4\bigl(p(x_n, x_{n+1})\bigr), \xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr), \\
&\qquad \frac{\xi_4(p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}))}{2}\biggr) \\
&\le \frac{1}{\sqrt{h_n}}\phi\bigl(\xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr), \xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr), \xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr), \xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr)\bigr) \\
&\le \frac{1}{\sqrt{h_n}}\xi_4\bigl(p(x_{n+1}, x_{n+2})\bigr).
\end{aligned}$$

By Remark 4,we also have

$$p(x_{n+1},x_{n+2}) \leq \frac{1}{\sqrt{h_n}}\xi_4\big(p(x_{n+1},x_{n+2})\big) < p(x_{n+1},x_{n+2}),$$

which implies a contradiction. Therefore, we have

$$\begin{aligned} p(x_{n+1},x_{n+2}) &\leq \frac{1}{\sqrt{h_n}}\xi_4\big(p(x_n,x_{n+1})\big) \\ &\leq \frac{1}{\sqrt{h_n}}p(x_n,x_{n+1}) \\ &\leq \cdots \\ &\leq \frac{1}{\sqrt{h_n}}\frac{1}{\sqrt{h_{n-1}}}\cdots\frac{1}{\sqrt{h_0}}p(x_0,x_1). \end{aligned} \tag{4.10}$$

Since $h_n > 1$ for all $n \in \mathbb{N} \cup \{0\}$, we get

$$\frac{1}{\sqrt{h_n}} < 1, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

Put

$$\overline{k} = \max\left\{\frac{1}{\sqrt{h_n}} : n \in \mathbb{N} \cup \{0\}\right\}. \tag{4.11}$$

Using (4.10) and (4.11), we obtain

$$p(x_{n+1},x_{n+2}) \leq (\overline{k})^{n+1}p(x_0,x_1), \quad \text{for all } n \in \mathbb{N} \cup \{0\}. \tag{4.12}$$

Let $n \to \infty$ in (4.12). Then

$$\lim_{n\to\infty} p(x_n,x_{n+1}) = 0. \tag{4.13}$$

By the property (p_2) of a partial metric and using (4.13), we have

$$\lim_{n\to\infty} p(x_n,x_n) = 0. \tag{4.14}$$

Using (4.12) and the property (p_4) of a partial metric, for any $m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n,x_{n+m}) &\leq \sum_{i=1}^{m} p(x_{n+i-1},x_{n+i}) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}) \\ &\leq \sum_{i=1}^{m} (\overline{k})^{n+i-1}p(x_0,x_1) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}) \\ &\leq \frac{(\overline{k})^n}{(1-\overline{k})}p(x_0,x_1) - \sum_{i=1}^{m-1} p(x_{n+i},x_{n+i}). \end{aligned} \tag{4.15}$$

Using (4.14) and (4.15), we get

$$\lim_{n\to\infty} p(x_n, x_{n+m}) = 0.$$

By the definition of d_p, we see that, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} d_p(x_n, x_{n+m}) \le \lim_{n\to\infty} 2p(x_n, x_{n+m}) = 0. \tag{4.16}$$

This shows that $\{x_n\}$ is a Cauchy sequence in (X, d_p). Since (X, p) is complete, from Lemma 1, (X, d_p) is a complete metric space. Therefore, $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric d_p, and we also have

$$p(x^*, x^*) = \lim_{n\to\infty} p(x_n, x^*) = \lim_{n\to\infty} p(x_n, x_m) = 0. \tag{4.17}$$

By the definition of the mapping α, we have $\alpha(x_n, x^*) > 0$. By using (4.17) together with the properties of the auxiliary functions ϕ, ξ_1, ξ_2, ξ_3, ξ_4, and the condition (ii), we get

$$\begin{aligned}
&\mathcal{H}_p(Tx_n, Tx^*) \\
&\quad\le \alpha(x_n, x^*)\mathcal{H}_p(Tx_n, Tx^*) \\
&\quad\le \phi\left(\xi_1(p(x_n, x^*)), \xi_2(p(x_n, Tx_n)), \xi_3(p(x^*, Tx^*)), \frac{\xi_4(p(x_n, Tx^*) + p(x^*, Tx_n))}{2}\right) \\
&\quad\le \phi\Bigg(\xi_1(p(x_n, x^*)), \xi_2(p(x_n, x_{n+1})), \xi_3(p(x^*, Tx^*)), \\
&\qquad \frac{\xi_4(p(x_n, x^*) + p(x^*, Tx^*) - p(x^*, x^*) + p(x^*, x_{n+1}))}{2}\Bigg) \\
&\quad\le \phi\Bigg(\xi_4(p(x_n, x^*)), \xi_4(p(x_n, x_{n+1})), \xi_4(p(x^*, Tx^*)), \\
&\qquad \frac{\xi_4(p(x_n, x^*) + p(x^*, Tx^*) - p(x^*, x^*) + p(x^*, x_{n+1}))}{2}\Bigg).
\end{aligned} \tag{4.18}$$

Let $n \to \infty$ in (4.18). By Remark 4,

$$\begin{aligned}
&\lim_{n\to\infty} \mathcal{H}_p(Tx_n, Tx^*) \\
&\quad\le \phi(0, 0, \xi_4(p(x^*, Tx^*)), \frac{1}{2}\xi_4(p(x^*, Tx^*)) \\
&\quad\le \xi_4(p(x^*, Tx^*)) \\
&\quad< p(x^*, Tx^*).
\end{aligned} \tag{4.19}$$

Now $x_{n+1} \in Tx_n$ shows

$$p(x_{n+1}, Tx^*) \le \delta_p(Tx_n, Tx^*) \le \mathcal{H}_p(Tx_n, Tx^*). \tag{4.20}$$

Using (4.18), (4.19), and (4.20), we get

$$p(x^*, Tx^*) = \lim_{n\to\infty} p(x_{n+1}, Tx^*) < p(x^*, Tx^*),$$

a contradiction. So, we have

$$p(x^*, Tx^*) = 0.$$

Therefore, from (4.17), $p(x^*, x^*) = 0$, we obtain

$$p(x^*, x^*) = p(x^*, Tx^*),$$

which implies $x^* \in Tx^*$ by Remark 3. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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