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Existence of solutions to a class of quasilinear elliptic problems with nonlinear singular terms

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Abstract

The authors of this paper deal with the existence of weak solutions to the homogenous boundary value problem for the equation $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^\alpha}$ with $f \in L^m(\Omega)$ and $\alpha \geq 1$. The authors prove the existence of solutions in $W_0^{1,p}(\Omega)$ for suitable m and α.

MSC: 35J62; 35B25; 76D03

Keywords: quasilinear elliptic problem; nonlinear singular term; existence

1 Introduction

In this paper, we study the existence of solutions for the following quasi-linear elliptic problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \frac{f(x)}{u^\alpha}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^N $(N \geq 1)$ with smooth boundary $\partial\Omega$. $f \geq 0$, $f \not\equiv 0$, $p > 1$, $\alpha \geq 1$.

Model (1.1) may describe many physical phenomena such as chemical heterogeneous catalysts, nonlinear heat transfers, some biological experiments, *etc.* [1–3]. In the case when $p = 2$, Lazer and Mckenna in [4] studied the following problem:

$$\begin{cases} -\Delta u = \frac{1}{u^\alpha}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

They proved that the solution to problem (1.2) was in $W_0^{1,2}(\Omega)$ if and only if $\alpha < 3$, while it was not in $C^1(\overline{\Omega})$ if $\alpha > 1$. Later the authors of [5–7] dealt with the existence of solutions to

$$\begin{cases} -\Delta u = ag(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g(s)$ is singular at $s = 0$. They obtained similar results as that of [4]. Moreover, Boccardo and Orsina in [8] discussed how the summability of f and the values of α affected the existence, regularity and nonexistence of solutions. For more results, the interested readers may refer to [9, 10]. When $p \in (1,\infty)$, $p \neq 2$, Giacomoni, Schindler and Takáč in [11] applied lower and upper-solution method and the mountain pass theorem to prove that the problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda u^{-\delta} + u^q, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\delta \in (0,1)$, $q \in (p-1, p^* - 1)$, has multiple weak solutions. And then, the authors in [12] not only improved the results in [11] but also obtained that the solution was not in $W_0^{1,p}(\Omega)$ if $\alpha > \frac{2p-1}{p-1}$. However, we need to point out that all the papers mentioned discussed the existence of solutions by means of upper-lower solution techniques. In this paper, we apply the method of regularization and Schauder's fixed point theorem as well as a necessary compactness argument to overcome some difficulties arising from the nonlinearity of the differential operator, the singularity of nonlinear terms and the summability of the weighted function $f(x)$ and then prove the existence of positive solutions in $W_0^{1,p}(\Omega)$ for suitable m and α when $f(x) \in L^m(\Omega)$ and $\alpha \geq 1$, which implies that the summability of the weighted function $f(x)$ determines whether or not problem (1.1) has a solution in $W_0^{1,p}(\Omega)$.

2 Main results

In this section, we apply the method of regularization and Schauder's fixed point theorem to prove the existence of solutions. In order to prove the main results of this section, we consider the following auxiliary problem:

$$\begin{cases} -\operatorname{div}(|\nabla u_n|^{p-2}\nabla u_n) = \frac{f_n(x)}{(u_n+\frac{1}{n})^\alpha}, & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega, \end{cases} \tag{2.1}$$

where $f_n = \min\{f(x), n\}$.

Definition 2.1 A function $u \in W_0^{1,p}(\Omega)$ is called a solution of problem (1.1) if the following identity holds:

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx = \int_\Omega \frac{f}{u^\alpha}\varphi\,dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

Since the proof of the following lemmas are similar to that in [8], we only give a sketch of the proof.

Lemma 2.1 *Problem* (2.1) *has a unique nonnegative solution* $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *for any fixed* $n \in N^*$, $f \in L^1(\Omega)$.

Proof Let $n \in N$ be fixed. For any $w \in L^p(\Omega)$, we get that the following problem has a unique solution $v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ by applying the variational method to

$$\begin{cases} -\operatorname{div}(|\nabla v|^{p-2}\nabla v) = \frac{f_n(x)}{(|w|+\frac{1}{n})^\alpha}, & x \in \Omega, \\ v = 0, & x \in \partial\Omega. \end{cases} \tag{2.2}$$

We may refer to [2, 10] for the existence and uniqueness of the solution for problem (2.2). So, for any $w \in L^p(\Omega)$, we may define the mapping $\Gamma : L^p(\Omega) \to L^p(\Omega)$ as $\Gamma(w) = v$. In fact, multiplying the first identity in (2.2) by v, and integrating over Ω, we have

$$\int_\Omega |\nabla v|^p\,dx = \int_\Omega \frac{f_n}{(|w| + \frac{1}{n})^\alpha} v\,dx \le n^{\alpha+1} \int_\Omega |v|\,dx.$$

Applying the embedding theorem $W^{1,p}(\Omega) \hookrightarrow L^1(\Omega)$, we obtain

$$\|v\|^p_{W^{1,p}} \le Cn^{\alpha+1} \|v\|_{W^{1,p}},$$

which implies that

$$\|v\|_{W^{1,p}} \le Cn^{\frac{\alpha+1}{p-1}}.$$

Due to the embedding $W^{1,p}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^p(\Omega)$, we get that Γ is a compact operator. Moreover, if $u = \lambda\Gamma u$ for some $0 < \lambda \le 1$, then $\Gamma u = \frac{u}{\lambda}$ and hence $\|u\|_{L^p(\Omega)} \le \|u\|_{W^{1,p}(\Omega)} \le C$ for a constant C independent of λ. Then by Schauder's fixed point theorem, we know that there exists $u_n \in W_0^{1,p}(\Omega)$ such that $u_n = \Gamma(u_n)$, *i.e.*, problem (2.1) has a solution. Noting that $\frac{f_n}{(|u_n|+\frac{1}{n})^\alpha} \ge 0$, the maximum principle in [13, 14] shows that $u_n \ge 0$, $u_n \in L^\infty(\Omega)$. □

Lemma 2.2 *The sequence $\{u_n\}$ is increasing with respect to n. $u_n > 0$ in Ω' for any $\Omega' \subset\subset \Omega$, and there exists a positive constant $C_{\Omega'}$ (independent of n) such that for all $n \in N^*$,*

$$u_n \ge C_{\Omega'} > 0 \quad \textit{for every } x \in \Omega'. \tag{2.3}$$

Proof Choosing $(u_n - u_{n+1})_+ = \max\{u_n - u_{n+1}, 0\}$ as a test function, observing that

$$\begin{aligned}
&\left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right)\nabla(u_n - u_{n+1})_+ \ge 0,\\
&\left[\left(u_{n+1} + \frac{1}{n+1}\right)^\alpha - \left(u_n + \frac{1}{n}\right)^\alpha\right](u_n - u_{n+1})_+ \le 0, \quad \text{for every } \alpha > 0,\\
&0 \le f_n \le f_{n+1},
\end{aligned}$$

we get

$$0 \le \int_\Omega \left(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right)\nabla(u_n - u_{n+1})_+\,dx \le 0.$$

This inequality yields $(u_n - u_{n+1})_+ = 0$ a.e. in Ω, that is, $u_n \le u_{n+1}$ for every $n \in N^*$. Since the sequence u_n is increasing with respect to n, we only need to prove that u_1 satisfies inequality (2.3). According to Lemma 2.1, we know that there exists a positive constant C (only depending on $|\Omega|, N, p$) such that $\|u_1\|_{L^\infty(\Omega)} \le C\|f_1\|_{L^\infty(\Omega)} \le C$, then

$$-\operatorname{div}\left(|\nabla u_1|^{p-2}\nabla u_1\right) = \frac{f_1}{(u_1+1)^\alpha} \ge \frac{f_1}{(C+1)^\alpha}.$$

Noting that $\frac{f_1}{(C+1)^\alpha} \geq 0$, $\frac{f_1}{(C+1)^\alpha} \not\equiv 0$, the strong maximum principle implies that $u_1 > 0$ in Ω, *i.e.*, inequality (2.3) holds. □

Theorem 2.1 *Suppose that f is a nonnegative function in $L^1(\Omega)$ and $\alpha = 1$, then problem* (1.1) *has a solution in $W_0^{1,p}(\Omega)$.*

Proof We consider the existence of solutions in the case when $f(x) \in L^1(\Omega)$. Multiplying the first identity in problem (2.1) by u_n and integrating over Ω, we get

$$\int_\Omega |\nabla u_n|^p \, dx = \int_\Omega \frac{f_n u_n}{u_n + \frac{1}{n}} \, dx \leq \int_\Omega |f_n| \, dx \leq \int_\Omega |f| \, dx,$$

i.e., $\|u_n\|_{W_0^{1,p}(\Omega)} \leq \|f\|_{L^1(\Omega)}^{\frac{1}{p}}$.

Then we know that there exist $u \in W^{1,p}(\Omega)$ and $\vec{V} \in L^{\frac{p}{p-1}}(\Omega, R^N)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega), \\ u_n \to u & \text{a.e. in } \Omega, \\ |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \vec{V} & \text{weakly in } L^{\frac{p}{p-1}}(\Omega, R^N). \end{cases} \tag{2.4}$$

For every $\varphi \in C_0^\infty(\Omega)$, we get from inequality (2.3) that

$$0 \leq \left| \frac{f_n \varphi}{u_n + \frac{1}{n}} \right| \leq \frac{\|\varphi\|_{L^\infty}}{C_{\Omega'}} f(x),$$

where $\Omega' = \{x : \varphi \neq 0\}$. Then applying Lebesgue's dominated convergence theorem, one has that

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n \varphi}{u_n + \frac{1}{n}} \, dx = \int_\Omega \frac{f\varphi}{u} \, dx \tag{2.5}$$

as u_n satisfies the following identity:

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\varphi \, dx = \int_\Omega \frac{f_n \varphi}{u_n + \frac{1}{n}} \, dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.6}$$

Combining with (2.4)-(2.6), we have that

$$\int_\Omega \vec{V} \nabla\varphi \, dx = \int_\Omega \frac{f\varphi}{u} \, dx, \quad \forall \varphi \in C_0^\infty(\Omega). \tag{2.7}$$

Next, we shall prove that $\vec{V} = |\nabla u|^{p-2}\nabla u$ a.e. in Ω. It is easy to see that both (2.6) and (2.7) hold for all $\varphi \in W^{1,p}(\Omega)$ with compact support. Thus in (2.6) we choose $\varphi = (u_n - \xi)\zeta$, where $\zeta \in C_0^\infty(\Omega)$, $\zeta \geq 0$, and $\xi \in W^{1,p}(\Omega)$, to obtain

$$\begin{aligned} &\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla\zeta (u_n - \xi) \, dx + \int_\Omega \zeta |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - \xi) \, dx \\ &\quad = \int_\Omega \frac{\zeta f_n (u_n - \xi)}{u_n + \frac{1}{n}} \, dx. \end{aligned} \tag{2.8}$$

Noting that $\zeta(|\nabla u_n|^{p-2}\nabla u_n - |\nabla\xi|^{p-2}\nabla\xi)(\nabla u_n - \nabla\xi) \geq 0$, we obtain that

$$\begin{aligned}
&\int_\Omega \zeta|\nabla\xi|^{p-2}\nabla\xi\nabla(u_n-\xi)\,dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\zeta(u_n-\xi)\,dx \\
&\quad\leq \int_\Omega \zeta|\nabla u_n|^{p-2}\nabla u_n\nabla(u_n-\xi)\,dx + \int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\zeta(u_n-\xi)\,dx \\
&\quad= \int_\Omega \frac{\zeta f_n(u_n-\xi)}{u_n+\frac{1}{n}}\,dx.
\end{aligned} \tag{2.9}$$

Letting $n\to\infty$ in (2.9) and using identity (2.7), we get

$$\begin{aligned}
&\int_\Omega \zeta|\nabla\xi|^{p-2}\nabla\xi\nabla(u-\xi)\,dx + \int_\Omega \vec{V}\nabla\zeta(u-\xi)\,dx \\
&\quad\leq \int_\Omega \frac{\zeta f(u-\xi)}{u}\,dx = \int_\Omega \vec{V}\nabla\big(\zeta(u-\xi)\big)\,dx = \int_\Omega \zeta\vec{V}\nabla(u-\xi)\,dx + \int_\Omega \vec{V}\nabla\zeta(u-\xi)\,dx,
\end{aligned}$$

which implies that

$$\int_\Omega \zeta\big(|\nabla\xi|^{p-2}\nabla\xi - \vec{V}\big)\nabla(u-\xi)\,dx \leq 0. \tag{2.10}$$

Let $u-\xi = \varepsilon\psi$ in (2.10), where ψ is an arbitrary function in $W^{1,p}(\Omega)$ and $\varepsilon > 0$ is a constant, we get that

$$\int_\Omega \zeta\varepsilon\big(|\nabla u - \varepsilon\nabla\psi|^{p-2}(\nabla u - \varepsilon\nabla\psi) - \vec{V}\big)\nabla\psi\,dx \leq 0,$$

i.e.,

$$\int_\Omega \zeta\big(|\nabla u - \varepsilon\nabla\psi|^{p-2}(\nabla u - \varepsilon\nabla\psi) - \vec{V}\big)\nabla\psi\,dx \leq 0.$$

Let $\varepsilon\to 0^+$, we have that

$$\int_\Omega \zeta\big(|\nabla u|^{p-2}\nabla u - \vec{V}\big)\nabla\psi\,dx \leq 0. \tag{2.11}$$

Since ψ is an arbitrary function, we obtain that

$$\int_\Omega \zeta\big(|\nabla u|^{p-2}\nabla u - \vec{V}\big)\nabla\psi\,dx = 0.$$

We choose $\psi = px$, where $p \in R^N$ is a constant, and we have that

$$\int_\Omega \zeta p\big(|\nabla u|^{p-2}\nabla u - \vec{V}\big)\,dx = 0,$$

which yields that $\vec{V} = |\nabla u|^{p-2}\nabla u$ a.e. in Ω. This proves that u is a weak solution of problem (1.1) when $f(x) \in L^1(\Omega)$. □

The first question is what happens to the solution if the inhomogeneous function $f(x)$ is not in $L^1(\Omega)$ but a nonnegative bounded Radon measure μ. Since a nonnegative Radon measure μ may always be approximated by a sequence f_n of $L^\infty(\Omega)$ functions, we want to know whether the approximate solutions may converge to a nontrivial function in $W_0^{1,p}(\Omega)$ or whether the approximate solutions converge. The existence of solutions in this case is still unknown, but we have the following result.

Theorem 2.2 *Suppose that μ is a nonnegative Radon measure concentrated on a Borel set E of zero p-capacity, and that g_n is a bounded sequence of nonnegative $L^1(\Omega)$ functions which converges to μ in the narrow topology of measures. Let u_n be the solution of problem* (2.1) *with the non-homogeneous function $f_n = g_n(x)$. Then*

$$\lim_{n\to\infty}\int_\Omega |u_n|^p\,dx = 0.$$

Proof By the conclusion of Theorem 2.1, we get that the solution u_n of problem (2.1) with $f_n = g_n$ is bounded in $W_0^{1,p}(\Omega)$. Since the set E has zero p-capacity, by [1, Lemma 5.1], for any real number $\sigma > 0$, there exists a function $\Psi_\sigma \in C_0^\infty(\Omega)$ satisfying

$$0 \le \Psi_\sigma \le 1, \qquad 0 \le \int_\Omega (1-\Psi_\sigma)\,d\mu \le \sigma, \qquad \int_\Omega |\nabla\Psi_\sigma|^p\,dx \le \sigma. \tag{2.12}$$

Noting that g_n converges to μ in the narrow topology of measure, one has from (2.12) that

$$\lim_{n\to\infty}\int_\Omega g_n(1-\Psi_\sigma)\,dx = \int_\Omega (1-\Psi_\sigma)\,d\mu \le \sigma. \tag{2.13}$$

Define $T_1(u_n) = \min\{u_n, 1\}$. Choosing $T_1(u_n)(1-\Psi_\sigma)$ as a test function in (2.1) with a non-homogeneous function g_n, we obtain that

$$\begin{aligned}&\int_\Omega \left|\nabla T_1(u_n)\right|^p(1-\Psi_\sigma)\,dx - \int_\Omega T_1(u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\Psi_\sigma\,dx\\ &\quad= \int_\Omega \frac{g_n T_1(u_n)(1-\Psi_\sigma)}{u_n+\frac{1}{n}}\,dx \le \int_\Omega \frac{g_n(1-\Psi_\sigma)u_n}{u_n}\,dx = \int_\Omega g_n(1-\Psi_\sigma)\,dx.\end{aligned} \tag{2.14}$$

Using $\|u_n\|_{W_0^{1,p}(\Omega)} \le C$, we assume that u_n is any subsequence such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$ and $u_n \to u$ in $L^p(\Omega)$. We show that the two limits in the theorem hold for any such subsequence. This completes the proof. Note that

$$\begin{aligned}&\left|\int_\Omega T_1(u_n)|\nabla u_n|^{p-2}\nabla u_n\nabla\Psi_\sigma\,dx\right|\\ &\quad\le \left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\Psi_\sigma\,dx\right| \le \|\nabla u_n\|_{L^p}^{p-1}\|\nabla\Psi_\sigma\|_{L^p} \le C\|\nabla\Psi_\sigma\|_{L^p}.\end{aligned} \tag{2.15}$$

By (2.12)-(2.15) and weak lower semi-continuity, we have

$$\int_\Omega \left|\nabla T_1(u)\right|^p(1-\Psi_\sigma)\,dx \le \sigma + \sigma^{\frac{1}{p}}. \tag{2.16}$$

Letting $\sigma \to 0^+$, we have

$$0 \le \int_\Omega \left|\nabla T_1(u)\right|^p\,dx \le 0,$$

which implies $u = 0$ a.e. in Ω. This completes the proof of the theorem. □

The above theorem shows that problem (1.1) has a solution in $W_0^{1,p}(\Omega)$ when $f \in L^1(\Omega)$ and $\alpha = 1$. But if f is only a Radon measure, the solution may not exist. At least, the solution can not be approximated by the solution of problem (2.1). The second question we are interested in is whether this problem has a solution in $W_0^{1,p}(\Omega)$ when $f \in L^m(\Omega)$ ($m > 1$) and $\alpha > 1$. We have the following.

Theorem 2.3 *Let f be a nonnegative function in $L^m(\Omega)$ ($f \not\equiv 0$) ($m > 1$). If $1 < \alpha < 2 - \frac{1}{m}$, then problem* (1.1) *has a solution $u \in W_0^{1,p}(\Omega)$ satisfying*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi\,dx = \int_\Omega \frac{f\varphi}{u^\alpha}\,dx, \quad \forall\varphi \in C_0^\infty(\Omega).$$

In order to prove this theorem, we need the following lemma.

Lemma 2.3 *The solution u_1 to problem* (2.1) *with $n = 1$ satisfies*

$$\int_\Omega u_1^{-r}\,dx < \infty, \quad \forall r < 1. \tag{2.17}$$

Proof By $\frac{\min\{f(x),1\}}{(u_1+1)^\alpha} \le 1$, and Lemma 2.2 in [14], we know that there exists $0 < \beta < 1$ such that $u_1 \in C^{1,\beta}(\overline{\Omega})$ and $\|u_1\|_{C^{1,\beta}} \le C$, which implies that the gradient of u_1 exists everywhere, then the Hopf lemma in [15] shows that $\frac{\partial u_1(x)}{\partial\nu} > 0$, in $\overline{\Omega}$, where ν is the outward unit normal vector of $\partial\Omega$ at x. Moreover, following the lines of proof of the lemma in [4], we get

$$\int_\Omega u_1^r\,dx < \infty \quad \text{if and only if} \quad r > -1. \qquad \square$$

Proof of Theorem 2.3 Multiplying the first identity in problem (2.1) by u_n, integrating over Ω, and applying Hölder's inequality and Lemma 2.3, we get

$$\begin{aligned}\int_\Omega |\nabla u_n|^p\,dx &= \int_\Omega \frac{f_n u_n}{(u_n+\frac{1}{n})^\alpha}\,dx \le \int_\Omega \frac{f_n}{u_n^{\alpha-1}}\,dx \le \int_\Omega \frac{f(x)}{u_1^{\alpha-1}}\,dx \\ &\le \|f\|_{L^m}\left\|u_1^{1-\alpha}\right\|_{L^{m'}} \le c\|f\|_{L^m},\end{aligned} \tag{2.18}$$

as $(1-\alpha)m' = (1-\alpha)\frac{m}{m-1} > -1$ by the assumption $1 < \alpha < 2 - \frac{1}{m}$; hence

$$\|u_n\|_{W_0^{1,p}} \le c\|f\|_{L^m(\Omega)}^{\frac{1}{p}}.$$

From (2.18), we know that there exist $u \in W_0^{1,p}(\Omega)$ and $\vec{V} \in L^{\frac{p}{p-1}}(\Omega, R^N)$ such that

$$\begin{cases} u_n \rightharpoonup u & \text{weakly in } W_0^{1,p}(\Omega) \text{ and strongly in } L^p(\Omega), \\ u_n \to u & \text{a.e. in } \Omega, \\ |\nabla u_n|^{p-2}\nabla u_n \rightharpoonup \vec{V} & \text{weakly in } L^{\frac{p}{p-1}}(\Omega, R^N). \end{cases} \tag{2.19}$$

For every $\varphi \in C_0^\infty(\Omega)$, from Lemma 2.2, we get that

$$0 \le \left|\frac{f_n\varphi}{(u_n+\frac{1}{n})^\alpha}\right| \le \frac{\|\varphi\|_{L^\infty}}{C'_\Omega} f(x).$$

Then applying Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\alpha}\,dx=\int_\Omega \frac{f\varphi}{u^\alpha}\,dx \tag{2.20}$$

since u_n satisfies the following identity:

$$\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\,dx=\int_\Omega \frac{f_n\varphi}{(u_n+\frac{1}{n})^\alpha}\,dx,\quad \forall\varphi\in C_0^\infty(\Omega). \tag{2.21}$$

In (2.21), letting $n\to\infty$ and using (2.19) and (2.20), we have

$$\int_\Omega \vec{V}\nabla\varphi\,dx=\int_\Omega \frac{f\varphi}{u^\alpha}\,dx,\quad \forall\varphi\in C_0^\infty(\Omega).$$

Following the lines of proof of Theorem 2.1, we get that problem (1.1) has a solution in $W_0^{1,p}(\Omega)$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
Both authors collaborated in all the steps concerning the research and achievements presented in the final manuscript.

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