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Existence of anti-periodic solutions with symmetry for some high-order ordinary differential equations

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Abstract
The existence of anti-periodic solutions with symmetry for high-order Duffing equations and a high-order Duffing type p-Laplacian equation has been studied by using degree theory. The results obtained enrich some known works to some extent.
MSC: 34B15; 34C25

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1 Introduction

Anti-periodic problems arise naturally from the mathematical models of various physical processes (see [1, 2]) and also appear in the study of partial differential equations and abstract differential equations (see [3–5]). For instance, electron beam focusing system in traveling-wave tube theories is an anti-periodic problem (see [6]).

In mechanics, the simplest model of oscillation equation is a single pendulum equation

$$x'' + \omega^2 \sin x = e(t) \ \bigl(\equiv e(t + 2\pi)\bigr),$$

whose anti-periodic solutions satisfy

$$x(t + \pi) = -x(t), \quad \forall t \in \mathbb{R}.$$

During the past twenty years, anti-periodic problems have been studied extensively by numerous scholars. For example, for first-order ordinary differential equations, a Massera's type criterion was presented in [7] and the validity of the monotone iterative technique was shown in [8]. Moreover, for higher-order ordinary differential equations, the existence of anti-periodic solutions was considered in [9–12]. Recently, existence results were extended to anti-periodic boundary value problems for impulsive differential equations (see [13]), and anti-periodic wavelets were discussed in [14].

It is well known that higher-order p-Laplacian equations are derived from many fields such as fluid mechanics and nonlinear elastic mechanics. In the past few decades, many important results on higher-order p-Laplacian equations with certain boundary conditions have been obtained. We refer the readers to [15–19] and the references cited therein.

In [10], the authors considered the existence of anti-periodic solutions for the high-order Duffing equation as follows:

$$x^{(n)} + \sum_{i=1}^{n-1} a_i x^{(i)} + g(t,x) = e(t). \tag{1.1}$$

Moreover, in [15] the authors discussed the existence of anti-periodic solutions for the following higher-order Liénard type p-Laplacian equation:

$$\big(\phi_p\big(x^{(n)}\big)\big)^{(n)} + f(x)x' + g(t,x) = e(t). \tag{1.2}$$

However, to the best of our knowledge, there exist relatively few results on the existence of anti-periodic solutions with symmetry for (1.1) and (1.2). Thus, it is worthwhile to continue to investigate the existence of anti-periodic solutions with symmetry for (1.1) and (1.2).

Motivated by the works mentioned previously, in this paper, we study the existence of anti-periodic solutions with symmetry for high-order Duffing equations of the forms:

$$x^{(2m+1)} + \sum_{i=1}^{m} a_i x^{(2i-1)} + g(t,x) = e(t), \tag{1.3}$$

$$x^{(2m+2)} + \sum_{i=1}^{m} a_i x^{(2i)} + g(t,x) = e(t) \tag{1.4}$$

and high-order Duffing type p-Laplacian equation of the form:

$$\big(\phi_p\big(x^{(m+1)}\big)\big)^{(m+1)} + g(t,x) = e(t), \tag{1.5}$$

where $p > 1$ is a constant, $m \geq 1$ is an integer, $\phi_p(s) = |s|^{p-2}s$; $a_i \in \mathbb{R}$, $g \in C(\mathbb{R}^2, \mathbb{R})$, $e \in C(\mathbb{R}, \mathbb{R})$ with $g(t+\pi, -x) \equiv -g(t,x)$, $e(t+\pi) \equiv -e(t)$. Obviously, the inverse operator of ϕ_p is ϕ_q, where $q > 1$ is a constant such that $\frac{1}{p} + \frac{1}{q} = 1$.

Notice that, when $p = 2$, the nonlinear operator $(\phi_p(x^{(m+1)}))^{(m+1)}$ reduces to the linear operator $x^{(2m+2)}$. On the other hand, $x(t)$ is also a 2π-periodic solution if $x(t)$ is a π-anti-periodic solution. Hence, from the arguments in this paper, we can also obtain the existence results on periodic solutions for the above equations.

The rest of this paper is organized as follows. Section 2 contains some necessary preliminaries. In Section 3 and Section 4, basing on the Leray-Schauder principle, we establish some existence theorems on anti-periodic solutions with symmetry of (1.3), (1.4) and (1.5). Our results are different from those of bibliographies listed in the previous texts.

2 Preliminaries

For the sake of convenience, we set

$$C^{k,\pi} = \big\{x \in C^k(\mathbb{R}, \mathbb{R}) : x(t+\pi) \equiv -x(t)\big\}, \quad k \in \{0, 1, \ldots\}$$

with the norm

$$\|x\|_{C^k} = \max_{i \in \{0,1,\ldots,k\}} \big\{\big\|x^{(i)}\big\|_0\big\},$$

where $\|x\|_0 = \max_{t\in[0,2\pi]} |x(t)|$, and

$$C_0^{k,\pi} = \{x \in C^{k,\pi} : x(-t) \equiv x(t)\},$$
$$C_1^{k,\pi} = \{x \in C^{k,\pi} : x(-t) \equiv -x(t)\}$$

with the norm $\|\cdot\|_{C^k}$.

Notice that, $x \in C_0^{0,\pi}$ may be written as Fourier series as follows:

$$x(t) = \sum_{i=0}^{\infty} a_{2i+1}\cos(2i+1)t,$$

and $x \in C_1^{0,\pi}$ may be written as the following Fourier series:

$$x(t) = \sum_{i=0}^{\infty} b_{2i+1}\sin(2i+1)t,$$

where $a_{2i+1}, b_{2i+1} \in \mathbb{R}$. We define the mapping $J_0 : C_0^{0,\pi} \longrightarrow C_1^{1,\pi}$ by

$$(J_0x)(t) = \int_0^t x(s)\,ds = \sum_{i=0}^{\infty} \frac{a_{2i+1}}{2i+1}\sin(2i+1)t, \quad \forall t \in \mathbb{R}$$

and the mapping $J_1 : C_1^{0,\pi} \longrightarrow C_0^{1,\pi}$ by

$$\begin{aligned}(J_1x)(t) &= \int_0^t x(s)\,ds - \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1} \\ &= -\sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1}\cos(2i+1)t, \quad \forall t \in \mathbb{R}.\end{aligned}$$

It is easy to prove that the mappings J_0, J_1 are completely continuous by using the Arzelà-Ascoli theorem.

Next, we introduce a continuation theorem (see [20]) as follows.

Lemma 2.1 (Continuation theorem) *Let Ω be open bounded in a linear normal space X. Suppose that f is a completely continuous field on $\overline{\Omega}$. Moreover, assume that the Leray-Schauder degree*

$$\deg(f, \Omega, p) \neq 0, \quad \textit{for } p \in X \backslash f(\partial\Omega).$$

Then the equation $f(x) = p$ has at least one solution in Ω.

3 Anti-periodic solutions with symmetry of (1.3) and (1.4)

In this section, some existence results on anti-periodic solutions with symmetry of (1.3) and (1.4) will be given.

Theorem 3.1 *Assume that*

(H_1) *the functions* $g(t,x)$ *and* $e(t)$ *are odd in* t, *i.e.*,

$$g(-t,\cdot) = -g(t,\cdot), \qquad e(-t) = -e(t), \quad \forall t \in \mathbb{R};$$

(H_2) *there exist non-negative functions* $\alpha_1, \beta_1 \in C(\mathbb{R}, \mathbb{R}^+)$ *such that*

$$\left|g(t,x)\right| \le \alpha_1(t)|x| + \beta_1(t), \quad \forall t, x \in \mathbb{R};$$

(H_3) $\sum_{i=1}^{m} |a_i| + \|\alpha_1\|_0 - 1 < 0$.

Then (1.3) *has at least one even anti-periodic solution* $x(t)$, *i.e.*, $x(t)$ *satisfies*

$$x(t+\pi) = -x(t), \qquad x(-t) = x(t), \quad \forall t \in \mathbb{R}.$$

Proof For making use of the Leray-Schauder degree theory to prove the existence of even anti-periodic solutions for (1.3), we consider the following homotopic equation of (1.3):

$$x^{(2m+1)} = -\lambda \sum_{i=1}^{m} a_i x^{(2i-1)} - \lambda g(t,x) + \lambda e(t), \quad \lambda \in [0,1]. \tag{3.1}$$

Define the operator $D_{01} : C_0^{2m+1,\pi} \longrightarrow C_1^{0,\pi}$ by

$$(D_{01}x)(t) = x^{(2m+1)}(t), \quad \forall t \in \mathbb{R}.$$

Obviously, the operator D_{01} is invertible. Let $N_{01} : C_0^{2m-1,\pi} \longrightarrow C^0$ be the Nemytskii operator

$$(N_{01}x)(t) = -\sum_{i=1}^{m} a_i x^{(2i-1)}(t) - g\bigl(t, x(t)\bigr) + e(t), \quad \forall t \in \mathbb{R}.$$

By hypothesis (H_1), it is easy to see that

$$(N_{01}x)(t+\pi) \equiv -(N_{01}x)(t), \qquad (N_{01}x)(-t) \equiv -(N_{01}x)(t), \quad \forall x \in C_0^{2m-1,\pi}.$$

Thus, the operator N_{01} sends $C_0^{2m-1,\pi}$ into $C_1^{0,\pi}$. Hence, the problem of even anti-periodic solutions for (3.1) is equivalent to the operator equation

$$D_{01}x = \lambda N_{01}x, \quad x \in C_0^{2m+1,\pi}.$$

From hypotheses (H_2), (H_3) and (5) in [10], for the possible even anti-periodic solution $x(t)$ of (3.1), there exists *a prior bounds* in $C_0^{2m+1,\pi}$, *i.e.*, $x(t)$ satisfies

$$\|x\|_{C^{2m+1}} \le T_1, \tag{3.2}$$

where T_1 is a positive constant independent of λ. So, our problem is reduced to construct one completely continuous operator F_λ, which sends $C_0^{2m+1,\pi}$ into $C_0^{2m+1,\pi}$, such that the fixed points of operator F_1 in some open bounded set are the even anti-periodic solutions of (1.3).

With this in mind, let us define the set as follows:

$$\Omega_{01} = \left\{x \in C_0^{2m+1,\pi} : \|x\|_{C^{2m+1}} < T_1 + 1\right\}.$$

Obviously, the set Ω_{01} is a open bounded set in $C_0^{2m+1,\pi}$ and zero element $\theta \in \Omega_{01}$. Define the completely continuous operator $F_\lambda : \overline{\Omega_{01}} \longrightarrow C_0^{2m+1,\pi}$ by

$$F_\lambda x = \underbrace{J_1 J_0 \cdots J_0 J_1}_{2m+1} \lambda N_{01} x = \lambda D_{01}^{-1} N_{01} x, \quad \lambda \in [0,1].$$

Let us define the completely continuous field $h_\lambda(x) : \overline{\Omega_{01}} \times [0,1] \longrightarrow C_0^{2m+1,\pi}$ by

$$h_\lambda(x) = x - F_\lambda x.$$

By (3.2), we get that zero element $\theta \notin h_\lambda(\partial\Omega)$ for all $\lambda \in [0,1]$. So, the following Leray-Schauder degrees are well defined and

$$\begin{aligned} \deg(id - F_1, \Omega, \theta) &= \deg(h_1, \Omega, \theta) \\ &= \deg(h_0, \Omega, \theta) = \deg(id, \Omega, \theta) = 1 \neq 0. \end{aligned}$$

Consequently, the operator F_1 has at least one fixed point in Ω_{01} by using Lemma 2.1. Namely, (1.3) has at least one even anti-periodic solution. The proof is complete. □

Theorem 3.2 *Assume that*

(H$_4$) *the function $g(t,x)$ is even in t, x and $e(t)$ is even in t, i.e.,*

$$g(-t,-x) = g(t,x), \qquad e(-t) = e(t), \quad \forall t \in \mathbb{R}$$

and the assumptions (H$_2$), (H$_3$) *are true.*

Then (1.3) *has at least one odd anti-periodic solution $x(t)$, i.e., $x(t)$ satisfies*

$$x(t+\pi) = -x(t), \qquad x(-t) = -x(t), \quad \forall t \in \mathbb{R}.$$

Proof We consider the homotopic equation (3.1) of (1.3). Define the operator $D_{11} : C_1^{2m+1,\pi} \longrightarrow C_0^{0,\pi}$ by

$$(D_{11}x)(t) = x^{(2m+1)}(t), \quad \forall t \in \mathbb{R}.$$

Let $N_{11} : C_1^{2m-1,\pi} \longrightarrow C^{0,\pi}$ be the Nemytskii operator

$$(N_{11}x)(t) = -\sum_{i=1}^{m} a_i x^{(2i-1)}(t) - g\big(t, x(t)\big) + e(t), \quad \forall t \in \mathbb{R}.$$

By hypothesis (H_4), it is easy to see that

$$(N_{11}x)(-t) \equiv (N_{11}x)(t), \quad \forall x \in C_1^{2m-1,\pi}.$$

Thus, the operator N_{11} sends $C_1^{2m-1,\pi}$ into $C_0^{0,\pi}$. Hence, the problem of odd anti-periodic solutions for (3.1) is equivalent to the operator equation

$$D_{11}x = \lambda N_{11}x, \quad x \in C_1^{2m+1,\pi}.$$

Our problem is reduced to construct one completely continuous operator G_λ, which sends $C_1^{2m+1,\pi}$ into $C_1^{2m+1,\pi}$, such that the fixed points of operator G_1 in some open bounded set are the odd anti-periodic solutions of (1.3). With this in mind, let us define the following set:

$$\Omega_{11} = \left\{x \in C_1^{2m+1,\pi} : \|x\|_{C^{2m+1}} < T_1 + 1\right\}.$$

Define the completely continuous operator $G_\lambda : \overline{\Omega_{11}} \longrightarrow C_1^{2m+1,\pi}$ by

$$G_\lambda x = \underbrace{J_0 J_1 \cdots J_1 J_0}_{2m+1} \lambda N_{11} x = \lambda D_{11}^{-1} N_{11} x, \quad \lambda \in [0,1].$$

The remainder of the proof work is quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

Theorem 3.3 *Assume that*

(H_5) *the functions $g(t,x)$ and $e(t)$ are even in t, i.e.,*

$$g(-t,\cdot) = g(t,\cdot), \qquad e(-t) = e(t), \quad \forall t \in \mathbb{R}$$

and the assumptions (H_2), (H_3) *are true.*

Then (1.4) *has at least one even anti-periodic solution.*

Proof We consider the homotopic equation of (1.4) as follows:

$$x^{(2m+2)} = -\lambda \sum_{i=1}^{m} a_i x^{(2i)} - \lambda g(t,x) + \lambda e(t), \quad \lambda \in [0,1]. \tag{3.3}$$

Define the operator $D_{02} : C_0^{2m+2,\pi} \longrightarrow C_0^{0,\pi}$ by

$$(D_{02}x)(t) = x^{(2m+2)}(t), \quad \forall t \in \mathbb{R}.$$

Let $N_{02} : C_0^{2m,\pi} \longrightarrow C^{0,\pi}$ be the Nemytskii operator

$$(N_{02}x)(t) = -\sum_{i=1}^{m} a_i x^{(2i)}(t) - g\big(t, x(t)\big) + e(t), \quad \forall t \in \mathbb{R}.$$

By hypothesis (H_5), it is easy to see that

$$(N_{02}x)(-t) \equiv (N_{02}x)(t), \quad \forall x \in C_0^{2m,\pi}.$$

Thus, the operator N_{02} sends $C_0^{2m,\pi}$ into $C_0^{0,\pi}$. Hence, the problem of even anti-periodic solutions for (3.3) is equivalent to the operator equation

$$D_{02}x = \lambda N_{02}x, \quad x \in C_0^{2m+2,\pi}.$$

Our problem is reduced to construct one completely continuous operator L_λ, which sends $C_0^{2m+2,\pi}$ into $C_0^{2m+2,\pi}$, such that the fixed points of operator L_1 in some open bounded set are the even anti-periodic solutions of (1.4). With this in mind, let us define the following set:

$$\Omega_{02} = \{x \in C_0^{2m+2,\pi} : \|x\|_{C^{2m+2}} < T_2 + 1\},$$

where T_2 is a positive constant independent of λ. Define the completely continuous operator $L_\lambda : \overline{\Omega_{02}} \longrightarrow C_0^{2m+2,\pi}$ by

$$L_\lambda x = \underbrace{J_1 J_0 \cdots J_1 J_0}_{2m+2} \lambda N_{02}x = \lambda D_{02}^{-1} N_{02}x, \quad \lambda \in [0,1].$$

The remainder of the proof work is quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

Theorem 3.4 *Assume that*

(H_6) *the function $g(t,x)$ is odd in t, x and $e(t)$ is odd in t, i.e.,*

$$g(-t,-x) = -g(t,x), \qquad e(-t) = -e(t), \quad \forall t \in \mathbb{R}$$

and the assumptions (H_2), (H_3) *are true.*

Then (1.4) *has at least one odd anti-periodic solution.*

Proof We consider the homotopic equation (3.3) of (1.4). Define the operator $D_{12} : C_1^{2m+2,\pi} \longrightarrow C_1^{0,\pi}$ by

$$(D_{12}x)(t) = x^{(2m+2)}(t), \quad \forall t \in \mathbb{R}.$$

Let $N_{12} : C_1^{2m,\pi} \longrightarrow C^{0,\pi}$ be the Nemytskii operator

$$(N_{12}x)(t) = -\sum_{i=1}^{m} a_i x^{(2i)}(t) - g(t,x(t)) + e(t), \quad \forall t \in \mathbb{R}.$$

By hypothesis (H_6), it is easy to see that

$$(N_{12}x)(-t) \equiv -(N_{12}x)(t), \quad \forall x \in C_1^{2m,\pi}.$$

Thus, the operator N_{12} sends $C_1^{2m,\pi}$ into $C_1^{0,\pi}$. Hence, the problem of odd anti-periodic solutions for (3.3) is equivalent to the operator equation

$$D_{12}x = \lambda N_{12}x, \quad x \in C_1^{2m+2,\pi}.$$

Our problem is reduced to construct one completely continuous operator P_λ which sends $C_1^{2m+2,\pi}$ into $C_1^{2m+2,\pi}$, such that the fixed points of operator P_1 in some open bounded set are the odd anti-periodic solutions of (1.4). With this in mind, let us define the set as follows:

$$\Omega_{12} = \left\{x \in C_1^{2m+2,\pi} : \|x\|_{C^{2m+2}} < T_2 + 1\right\}.$$

Define the completely continuous operator $P_\lambda : \overline{\Omega_{12}} \longrightarrow C_1^{2m+2,\pi}$ by

$$P_\lambda x = \underbrace{J_0 J_1 \cdots J_0 J_1}_{2m+2} \lambda N_{12}x = \lambda D_{12}^{-1} N_{12}x, \quad \lambda \in [0,1].$$

The remainder of the proof work is quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

When $g(t,x) = g(x)$, we can remove the assumption (H_2) in Theorem 3.1, Theorem 3.2 and obtain the following results.

Theorem 3.5 *Assume that*

(H_7) $\sum_{i=1}^m |a_i| - 1 < 0$ *and the assumption* (H_1) *is true*.

Then (1.3) ($g(t,x) = g(x)$) *has at least one even anti-periodic solution*.

Theorem 3.6 *Suppose that the assumptions* (H_4), (H_7) *are true*. *Then* (1.3) ($g(t,x) = g(x)$) *has at least one odd anti-periodic solution*.

Basing on the proof of Theorem 2 in [10], for the possible anti-periodic solution $x(t)$ of (3.1) ($g(t,x) = g(x)$), the hypothesis (H_7) yields that there exists *a prior bounds* in $C^{2m+1,\pi}$, *i.e.*, $x(t)$ satisfies

$$\|x\|_{C^{2m+1}} \leq T_3,$$

where T_3 is a positive constant independent of λ. The remainder of the proof work of Theorem 3.5 and Theorem 3.6 is quite similar to the proof of Theorem 3.1 and Theorem 3.2, so we omit the details.

4 Anti-periodic solutions with symmetry of (1.5)

In this section, we will give some existence results on anti-periodic solutions with symmetry of (1.5).

Theorem 4.1 *Assume that*

(H_8) *there exist non-negative functions* $\alpha_2, \beta_2 \in C(\mathbb{R}, \mathbb{R}^+)$ *such that*

$$\left|g(t,x)\right| \le \alpha_2(t)|x|^{p-1} + \beta_2(t), \quad \forall t, x \in \mathbb{R};$$

(H_9) $\|\alpha_2\|_0 \lambda_1^{-(m+1)} - 1 < 0$ *and the assumption* (H_5) *is true*.

Then (1.5) *has at least one even anti-periodic solution*.

Proof We consider the following homotopic equation of (1.5):

$$\left(\phi_p\left(x^{(m+1)}\right)\right)^{(m+1)} = -\lambda g(t,x) + \lambda e(t), \quad \lambda \in [0,1]. \tag{4.1}$$

Define the operator $D_{03} : D(D_{03}) \subset C_0^{0,\pi} \longrightarrow L^1([0,2\pi], \mathbb{R})$ by

$$(D_{03}x)(t) = \left(\phi_p\left(x^{(m+1)}(t)\right)\right)^{(m+1)}, \quad \forall t \in \mathbb{R},$$

where

$$D(D_{03}) = \Big\{x \in C_0^{2m+1,\pi} : \left(\phi_p\left(x^{(m+1)}(t)\right)\right)^{(m)} \text{ is absolutely continuous on } \mathbb{R}\Big\}.$$

Let $N_{03} : C_0^{0,\pi} \longrightarrow L^1([0,2\pi], \mathbb{R})$ be the Nemytskii operator

$$(N_{03}x)(t) = -g\left(t, x(t)\right) + e(t), \quad \forall t \in \mathbb{R}.$$

Obviously, the operator D_{03} is invertible and the problem of even anti-periodic solutions for (4.1) is equivalent to the operator equation

$$D_{03}x = \lambda N_{03}x, \quad x \in D(D_{03}).$$

From hypotheses (H_8), (H_9) and (3.8) in [15], for the possible even anti-periodic solution $x(t)$ of (4.1), there exists *a prior bounds* in $C_0^{0,\pi}$, *i.e.*, $x(t)$ satisfies

$$\|x\|_{C^0} \le T_4,$$

where T_4 is a positive constant independent of λ. So, our problem is reduced to construct one completely continuous operator Q_λ, which sends $C_0^{0,\pi}$ into $C_0^{0,\pi}$, such that the fixed points of operator Q_1 in some open bounded set are the even anti-periodic solutions of (1.5).

With this in mind, let us define the set as follows:

$$\Omega_{03} = \left\{x \in C_0^{0,\pi} : \|x\|_{C^0} < T_4 + 1\right\}.$$

By hypothesis (H_5), it is easy to see that

$$(N_{03}x)(-t) \equiv (N_{03}x)(t), \quad \forall x \in C_0^{0,\pi}.$$

Hence, the operator N_{03} sends $C_0^{0,\pi}$ into $C_0^{0,\pi}$. Define the completely continuous operator $Q_\lambda : \overline{\Omega_{03}} \longrightarrow C_0^{0,\pi}$ by

$$\begin{aligned} Q_\lambda x &= \underbrace{J_1 J_0 \cdots J_0 J_1}_{m+1} \phi_q \underbrace{J_0 J_1 \cdots J_1 J_0}_{m+1} \lambda N_{03} x \\ &= \phi_q(\lambda) D_{03}^{-1} N_{03} x, \quad \lambda \in [0,1] \text{ (if } m = 2n, n = 1,2,\ldots) \end{aligned}$$

or

$$\begin{aligned} Q_\lambda x &= \underbrace{J_1 J_0 \cdots J_1 J_0}_{m+1} \phi_q \underbrace{J_1 J_0 \cdots J_1 J_0}_{m+1} \lambda N_{03} x \\ &= \phi_q(\lambda) D_{03}^{-1} N_{03} x, \quad \lambda \in [0,1] \text{ (if } m = 2n-1, n = 1,2,\ldots). \end{aligned}$$

The remainder of the proof work is quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

Theorem 4.2 *Suppose that the assumptions* (H_6), (H_8), (H_9) *are true. Then* (1.5) *has at least one odd anti-periodic solution.*

Proof We consider the homotopic equation (4.1) of (1.5). Define the operator $D_{13} : D(D_{13}) \subset C_1^{0,\pi} \longrightarrow L^1([0,2\pi],\mathbb{R})$ by

$$(D_{13}x)(t) = \big(\phi_p\big(x^{(m+1)}(t)\big)\big)^{(m+1)}, \quad \forall t \in \mathbb{R},$$

where

$$D(D_{13}) = \big\{x \in C_1^{2m+1,\pi} : \big(\phi_p\big(x^{(m+1)}(t)\big)\big)^{(m)} \text{ is absolutely continuous on } \mathbb{R}\big\}.$$

Let $N_{13} : C_1^{0,\pi} \longrightarrow L^1([0,2\pi],\mathbb{R})$ be the Nemytskii operator

$$(N_{13}x)(t) = -g\big(t, x(t)\big) + e(t), \quad \forall t \in \mathbb{R}.$$

Thus, the problem of odd anti-periodic solutions for (4.1) is equivalent to the operator equation

$$D_{13}x = \lambda N_{13}x, \quad x \in D(D_{13}).$$

Our problem is reduced to construct one completely continuous operator W_λ, which sends $C_1^{0,\pi}$ into $C_1^{0,\pi}$, such that the fixed points of operator W_1 in some open bounded set are the odd anti-periodic solutions of (1.5). With this in mind, let us define the following set:

$$\Omega_{13} = \big\{x \in C_1^{0,\pi} : \|x\|_{C^0} < T_4 + 1\big\}.$$

By hypothesis (H_6), it is easy to see that

$$(N_{13}x)(-t) \equiv -(N_{13}x)(t), \quad \forall x \in C_1^{0,\pi}.$$

Hence, the operator N_{13} sends $C_1^{0,\pi}$ into $C_1^{0,\pi}$. Define the completely continuous operator $W_\lambda : \overline{\Omega_{13}} \longrightarrow C_1^{0,\pi}$ by

$$\begin{aligned} W_\lambda x &= \underbrace{J_0 J_1 \cdots J_1 J_0}_{m+1} \phi_q \underbrace{J_1 J_0 \cdots J_0 J_1}_{m+1} \lambda N_{13} x \\ &= \phi_q(\lambda) D_{13}^{-1} N_{13} x, \quad \lambda \in [0,1] \text{ (if } m = 2n, n = 1, 2, \ldots) \end{aligned}$$

or

$$\begin{aligned} W_\lambda x &= \underbrace{J_0 J_1 \cdots J_0 J_1}_{m+1} \phi_q \underbrace{J_0 J_1 \cdots J_0 J_1}_{m+1} \lambda N_{13} x \\ &= \phi_q(\lambda) D_{13}^{-1} N_{13} x, \quad \lambda \in [0,1] \text{ (if } m = 2n-1, n = 1, 2, \ldots). \end{aligned}$$

The remainder of the proof work is quite similar to the proof of Theorem 3.1, so we omit the details. The proof is complete. □

Theorem 4.3 *Assume that $g(t,x)$ has the decomposition*

$$g(t,x) = u(t,x) + v(t,x)$$

such that

(H_{10}) *there exist non-negative constants γ, r with $r > p$, such that*

$$(-1)^{m+1} x u(t,x) \ge \gamma |x|^r, \quad \forall t, x \in \mathbb{R};$$

(H_{11}) *there are non-negative functions $\alpha_3, \beta_3 \in C(\mathbb{R}, \mathbb{R}^+)$ such that*

$$\left| v(t,x) \right| \le \alpha_3(t) |x|^{r-1} + \beta_3(t), \quad \forall t, x \in \mathbb{R};$$

(H_{12}) $\|\alpha_3\|_0 - \gamma \le 0$ *and the assumption* (H_5) *is true.*

Then (1.5) *has at least one even anti-periodic solution.*

Theorem 4.4 *Suppose that the assumptions* (H_6), (H_{10}), (H_{11}), (H_{12}) *are true. Then* (1.5) *has at least one odd anti-periodic solution.*

Basing on the proof of Theorem 3.2 in [15], for the possible anti-periodic solution $x(t)$ of (4.1), the hypotheses (H_{10}), (H_{11}), (H_{12}) yield that there exists *a prior bounds* in $C^{2m+1,\pi}$, *i.e.*, $x(t)$ satisfies

$$\|x\|_{C^0} \le T_5,$$

where T_5 is a positive constant independent of λ. The remainder of the proof work of Theorem 4.3 and Theorem 4.4 is quite similar to the proof of Theorem 4.1 and Theorem 4.2, so we omit the details.

Remark Assumptions (H_{10}), (H_{11}), (H_{12}) guarantee that the degree with respect to x of $g(t,x)$ is allowed to be greater than $p-1$, which is different from the hypothesis (H_8) of Theorem 4.1 and Theorem 4.2.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
HP carried out the theoretical analysis. JY drafted the manuscript. All authors read and approved the final manuscript.

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