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TRANSLATION PLANES OF ODD ORDER AND ODD DIMENSION

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ABSTRACT. The author considers one of the main problems in finite translation planes to be the identification of the abstract groups which can act as collineation groups and how those groups can act.

The paper is concerned with the case where the plane is defined on a vector space of dimension 2d over GF(q), where q and d are odd. If the stabilizer of the zero vector is non-solvable, let G_0 be a minimal normal non-solvable subgroup. We suspect that G_0 must be isomorphic to some SL(2,u) or homomorphic to A_6 or A_7. Our main result is that this is the case when d is the product of distinct primes.

The results depend heavily on the Gorenstein-Walter determination of finite groups having dihedral Sylow 2-groups when d and q are both odd. The methods and results overlap those in a joint paper by Kallaher and the author which is to appear in Geometriae Dedicata. The only known example (besides

Desarguesian planes) is Hering's plane of order 27 (i.e., d and q are both equal to 3) which admits SL(2,13).

KEY WORDS AND PHRASES. Translation Planes, Collineation Graphs, Finite Geometries.

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1. INTRODUCTION

A translation plane of order q^d with kernel $F = GF(q)$ may be represented by a vector space of dimension 2d over F. (The plane is usually said to have dimension d over F.) Here the points are the elements of the vector space and the lines are the translates of the components of a spread. A spread is a class of d-dimensional subspaces (the components of the spread) such that each non-zero vector belongs to exactly one component.

The group of collineations fixing the zero vector is called the translation complement; the subgroup consisting of linear transformations is the linear translation complement.

The dimension and order are both assumed to be odd in this paper.

The Hering plane of order 27[8] is the only known example of a non-Desarguesian translation plane of odd order and odd dimension in which the collineation group is non-solvable. Thus, the question arises as to whether there are others and what they are like (if others do exist). The translation complement contains SL(2,13) in the case of the Hering plane. The Sylow 2-groups of the induced permutation group on ℓ_∞ are cyclic or dihedral (when dimension and order are odd); it is possible that the key non-solvable group is always SL(2,u) for some u or, perhaps, is a pre-image of A_6 or A_7. This is suggested by the Gorenstein-Walter Theorem [5]. The author [14] has previously shown that this is the case for minimal non fixed-point-free groups (see below) which are non-solvable. However, a non-solvable linear group need not have a non-solvable minimal non-f.p.f. subgroup.

Throughout this paper G is a non-solvable group of linear transformations and G_0 is a minimal normal non-solvable subgroup.

Our most important result is Theorem (3.5) which states that, if d is the product of distinct primes, a minimal non-solvable normal subgroup of the linear translation complement either has the form SL(2,u) or is a pre-image of A_6 or A_7.

We have no new examples, so the question as to whether Hering's plane of order 27 is the only one (of odd order and dimension) remains open.

We include some informal discussion to indicate the importance of the possibility that d might divide u-1 in the SL(2,u) case and then show that there are severe numerical restrictions on this case and strengthen certain results of Kallaher and the author [12].

The present paper is similar in method, spirit, and results to the joint one. Here there are more restrictions placed on d and weaker initial restrictions on the group.

The notation and language are more or less standard. Some of the terminology and even some of the facts, may not be familar to every potential reader of this paper. We finish this Introduction with a brief discussion of these matters and some remarks on notation.

A group of linear transformations is fixed point free (f.p.f.) if no non-trivial element fixes any non-zero vector. One obtains a Frobenius permutation group by adjoining the translations so that every f.p.f. linear group is a Frobenius complement [11]. For a Frobenius complement, the Sylow subgroups of odd order are cyclic; the Sylow 2-groups are cyclic or generalized quaternion. (See [15].)

A normal subgroup of a linear group G is a minimal non-f.p.f. group with respect to G if it is not fixed point free but every normal subgroup of G properly contained in it is f.p.f.

An irreducible group of linear transformations acting on a vector space V is imprimitive if V is the direct sum of subspaces which are permuted by the group. These subspaces will be called subspace of imprimitivity. An irreducible group is primitive if it is not imprimitive.

A minimal invariant subspace of a reducible group G will sometimes be called a minimal G-space. If V_1 is a subspace such that all of the minimal G-spaces in V_1 are isomorphic as G-modules, then V_1 will be called a homogeneous space.

The order of G is denoted by $|G|$. If G is the full group and G_0 is a subgroup, $\mathcal{C}(G_0)$ is the centralizer of G_0 in G.

The subgroup of G which fixes ℓ is $G(\ell)$.

If σ is a non-f.p.f. element, $V(\sigma)$ denotes the subspace consisting of all vectors fixed by σ.

Fit G denotes the Fitting subgroup of G.

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2. LINEAR GROUPS WITH DIHEDRAL 2-GROUPS.

(2.1) LEMMA. Let G be a non-solvable group of linear transformations and let G_0 be a minimal non-solvable normal subgroup. Let H be a cyclic normal subgroup of G included in G_0. Then H is in the center of G_0.

PROOF. If $G_0 \cap \mathcal{C}(H)$ is non-solvable, then G_0 centralizes H. Otherwise $G_0/G_0 \cap \mathcal{C}(H)$ is isomorphic to a non-solvable group of automorphisms of H (induced on H by conjugation). But the automorphism group of a cyclic group is solvable.

(2.2) LEMMA. Let G be a non-solvable group of linear transformations and let G_0 be a minimal non-solvable subgroup. If the Fitting subgroup of G_0 is fixed point free, then it is the center of G_0.

PROOF. Let Fit G_0 be the Fitting subgroup of G_0. Then Fit G_0 is the direct product of its Sylow subgroups since it is nilpotent and the Sylow subgroups of odd order are cyclic, since it is a Frobenius complement. The Sylow 2-group in Fit G_0 is either cyclic or generalized quaternion. (See Passman [15].) Suppose that the 2-group in Fit G_0 is a generalized quaternion group Q. Then $G_0/\mathcal{C}(Q) \cap G_0$ is isomorphic to a group of automorphisms of Q. The automorphism group of Q is a 2-group or S_4 (see Passman [15].) and hence is solvable. Hence $\mathcal{C}(Q) \cap G_0$ is non-solvable; by the minimal property of G_0, $G_0 = \mathcal{C}(Q) \cap G_0$. This is a contradiction since Q is a non-abelian subgroup of G_0. Thus Fit G_0 is the direct product of cyclic groups of relatively prime order and is cyclic. The rest of the argument follows from the previous Lemma. Using reasoning similar to that used above, G_0 modulo the subgroup centralizing Fit G_0 is a group of automorphisms of a cyclic group and hence is solvable, so G_0 must centralize Fit G_0. But Fit G_0 includes $Z(G_0)$, so Fit $G_0 = Z(G_0)$.

(2.3) THEOREM. If G is a non-solvable group of linear transformations on a vector space V over GF(q) with a minimal non-solvable normal subgroup G_0, if Fit G_0 is fixed point free and if the Sylow 2-groups in $\overline{G}_0$ (the factor group of G_0 modulo its center) are dihedral then G_0 is SL(2,u) or PSL(2,u) for some odd u or $\overline{G}_0 = A_6$ or A_7.

PROOF. Let H be the maximal normal subgroup of G which is included in G_0 but is not equal to G_0.

Then H is solvable and includes Fit G_0. By the previous Lemma Fit G_0 is $Z(G_0)$. We claim that H = Fit G_0. If not, H has a normal subgroup B such that $B/B \cap Z(G_0)$ is a non-trivial and abelian. The B would be nilpotent and hence in Fit G_0. Therefore, H = Fit G_0.

It follows from the definition of H that G_0/H has no proper characteristic subgroup - i.e., it is characteristically simple. Hence it is a direct product of isomorphic simple groups. (See Huppert [10], Satz 9.12.) By hypotheses, the Sylow 2-groups of $\overline{G}_0 = G_0 H = G_0/Z(G_0)$ are dihedral. It follows that $\overline{G}_0$ is a simple group with dihedral Sylow 2-groups. Furthermore, the minimal property of G_0 implies that $G_0' = G_0$ and hence H is a Schur multiplier for $\overline{G}_0$. By Gorenstein and Walter [5], $\overline{G}_0 = PSL(2,u)$ for some odd u or is equal to A_7. Furthermore, if $\overline{G}_0 = PSL(2,u)$ and $u \neq 9$ then $G_0 = SL(2,u)$. (See Huppert [10], Satz 25.7.) Note that $PSL(2,9) = A_6$.

One of the key assumptions of Theorem (2.3) was that Fit G_0 is fixed point free. The next few Lemmas develop the machinery for examining the case where Fit G_0 is not fixed point free. Dixon [2] gives a similar development for vector spaces over an algebraically closed field. We have attempted to modify Dixon's argument to apply to vector spaces over a finite field.

(2.4) LEMMA. Let G be a group of linear transformations acting irreducibly on a vector space V_F of dimension n over GF(q) = F. Suppose that the non-singular linear transformations which commute with G are all scalars. If σ is any element of GL(n,q) such that trace $\sigma\rho = 0$ for all ρ in G, then $\sigma = 0$.

PROOF. Let $\mathcal{M} = \{\sigma |$ trace $\sigma\rho = 0 \ \forall \ \rho$ in $G\}$. The ring of all linear transformations on V_F is a vector space of dimension n^2 over F and $\mathcal{M}$ is a subspace.

Furthermore G acts in a natural way as a group of linear transformations on this vector space of dimension n^2 and G leaves $\mathcal{M}$ invariant.

Let $\mathcal{L}$ be a minimal G-invariant subspace of $\mathcal{M}$. By Lemma (2.7) in Dixon [2], the dimension of $\mathcal{L}$ is n and there is a vector v in V_F such that $\langle v\mathcal{L}\rangle = V_F$.

Let $\{\mu_1, \mu_2, \ldots, \mu_n\}$ be a basis for $\mathcal{L}$. The $\{v\mu_1, v\mu_2, \ldots, v\mu_n\}$ is a basis for V_f. Label these vectors $v_1, \ldots, v_n$ respectively.

Let w be an arbitrary non-zero element of V and let $w\mu_i = \sum_j a_{ij}v_j = v \sum_j a_{ij}\mu_j$ where the $a_{ij} \varepsilon F$. Let $\lambda = \lambda(w)$ be an eigenvalue of the matrix (a_{ij}). If $\lambda \varepsilon F$, let E be an extension which contains λ. Then $\{(w - \lambda v)\mu_1, (w - \lambda v)\mu_2, \ldots, (w - \lambda v)\mu_n\}$ is a basis for the vector space $\langle (w - v)\mathcal{L}\rangle$. This is a subspace of V_E, where V_E is a vector space of dimension n over E.

Note that $w\mu_1 - \lambda v\mu_i = v[\sum_j a_{ij}\mu_j - \lambda\mu_i]$. The determinant of the matrix $(a_{ij}) - \lambda I$ is zero, so the vectors $(w - \lambda v)\mu_i$ are dependent and the dimension of $\langle (w - v)\mathcal{L}\rangle$ is less than n.

Thus $\langle (w - \lambda v)\mathcal{L}\rangle$ is a subspace of V_E which is invariant under G and is not V_E.

We had $v\mu_i = v_i$. Let $w\mu_i = w_i$. Now v_i and w_i are in V_F (which is embedded in V_E). Let the mapping τ be defined by $w_i\tau = \lambda v_i$. Then τ becomes a linear transformation on V_E by defining $(c_1w_1 + \cdots + c_nw_n)\tau$ $c_1(w_1\tau) + \cdots + c_n(w_n\tau)$ for $c_1, \ldots, c_n \varepsilon E$. (Note that τ also acts as a linear transformation on V_F.)

Let x be an arbitrary member of V_F. Then $x + x\tau$ belongs to $\langle (w - v)\mathcal{L}\rangle$. Suppose that $x_1\tau = x_2\tau$. Then $(x_1 + x_1\tau) - (x_2 + x_2\tau) = x_1 - x_2$ is a vector in V_F which belongs to $\langle (w - \lambda)\mathcal{L}\rangle$.

Now $<(w - \lambda v)\mathfrak{L}> \cap V_F$ is a G-invariant subspace of V_F and, since G is irreducible, is either the null space or all of V_F. Suppose that $V_F \subseteqq <(w - \lambda v)\mathfrak{L}>$. A basis for V_F is also a basis for V_E so in this case $<(w - \lambda v)\mathfrak{L}> = V_E$, which is a contradiction.

Hence $x_1\tau = x_2\tau$, $x_1\ x_2 \in V_F$ implies $x_1 = x_2$ so τ is a non-singular linear transformation on V_F. If $\rho \in G$ and $x \in V_F$, then $(x + x\tau)\rho = x\rho + x\rho\tau$ so τ commutes with G. Hence τ is a scalar transformation on V_F. We had $w_i\tau = \lambda v_i$. Hence $\lambda \in F$ and $E = F$.

Hence $<(w - \lambda v)\mathfrak{L}>$ is a G-invariant subspace of V_F of dimension less than n. Hence $<(w - \lambda v)\mathfrak{L}>$ is the null space for each $w \neq 0$ in V_F.

In particular this holds for $w = v_1, v_2, \ldots, v_n$. Let $\lambda(v_i) = \lambda_i$. Then $(v_1 - \lambda_1 v)\ \mathfrak{L} = 0$ for $i = 1, \ldots, n$. Hence $v_1\mu_j = \lambda_j v\mu_j = \lambda_j v_j$.

Using $v_1, \ldots, v_n$ as a basis and thinking of $\mu_1, \ldots, \mu_n$ as matrices over this basis we get

$$\mu_1 = \begin{pmatrix} \lambda_1 & 0 \ldots 0 \\ \lambda_1 & 0 \ldots 0 \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ \lambda_1 & 0 \ldots 0 \end{pmatrix} \quad \text{etc.}$$

so that, in general, trace $\mu_i = \lambda_i$. But, by definition of $\mathfrak{M}$ and $\mathfrak{L}$, trace $\mu_i\rho = 0$ for each ρ in G, including the case where ρ is the identity. Hence all of the λ_i are equal to zero and all of the μ_i are zero.

REMARK. Except for the consideration of the possibilities that the eigenvalues λ_i might be in some extension this is the proof of case I, Theorem 2.7A in Dixon [2].

(2.5) LEMMA. Under the hypotheses of (2.4) the ring generated by the linear transformations in G has dimension n^2, i.e. it is the full ring of all linear transformations on V_F.

PROOF. (See Dixon, Theorem 2.4B.) The argument goes through without change except that Dixon requires the field F to be algebraically closed in order to use Theorem 2.7A. If his Theorem 2.7A is replaced by our Lemma (2.4) his proof applies here.

(2.6) THEOREM. Let G be a group of non-singular linear transformations acting irreducibly on a vector space V of dimension n. Suppose that $G' \leq Z(G)$ and that Z(G) consists of scalars then $[G:Z(G)] = n^2$.

PROOF. If $\sigma, \rho \in G$ then $\sigma^{-1}\rho^{-1}\sigma\rho \in Z(G)$, so $\rho^{-1}\sigma\rho = \sigma\lambda$ for some λ in Z(G). Hence trace σ = trace $\rho^{-1}\sigma\rho = \lambda$ trace σ. If $\lambda = 1$ for all choices of ρ then $\sigma \in Z(G)$; otherwise trace $\sigma = 0$. Thus, the trace is zero for all elements of G not in Z(G).

By (2.5), the ring of all linear transformations on V has a basis $\sigma_1, \sigma_2, \ldots, \sigma_{n^2}$ in G. For an arbitrary σ in G, $\sigma = \sum_i a_i\sigma_i$, where $a_i \in F$. Suppose that $a_j \neq 0$. Then

$$\text{trace } \sigma\sigma_j^{-1} = \text{trace } \left[\sum_{i\neq j} a_i\sigma_i\sigma_j + a_jI\right]$$

Since σ_i and σ_j are independent for $i \neq j$, $\sigma_i\sigma_j^{-1}$ is not in Z(G) and has trace zero.

Hence trace $\sigma\sigma_j^{-1} = na_j$. Note that we cannot have $n \equiv 0$ mod the characteristic, for otherwise the trace would be zero for all elements of G, contrary to Lemma (2.1).

Thus, for each σ in G $\exists$ j $\ni$ σ belongs to the coset $\sigma_jZ(G)$. By the independence of the σ_i, j is unique. Hence $\sigma_1, \ldots, \sigma_{n^2}$ form a set of representatives for the distinct cosets of Z(G).

(2.7) LEMMA. Let V be a vector space over $GF(q)$ where the dimension of V is the product of distinct primes. Let G be an irreducible group of linear transformations on V. Suppose that G has a normal subgroup G_0 such that: (1) the unique maximal abelian normal subgroup of G_0 consists of scalars; (2) for some prime u, a Sylow u-group S of Fit G_0 is non-abelian and $S/Z(S)$ is abelian; (3) S is faithful on its minimal invariant subspaces. Then u is one of the primes dividing the dimension and $|S/Z(S)| = u^2$.

PROOF. S is characteristic in G_0 and hence normal in G. Hence the dimension of each minimal S-space divides the dimension of V. Let n be the dimension of a minimal S-space. By the previous Theorem, $|S/Z(S)| = n^2$. But n must be a power of u and divide the dimension of V. Hence $n = u$.

(2.8) LEMMA. In the notation of (2.1) - (2.3) suppose that Fit G_0 is not fixed point free. Then G_0 includes a subgroup W which is a minimal non-f.p.f. group with respect to G, where W is a w-group from some prime w. If W_0 is the maximal normal subgroup of G included in W but not equal to W, then $W_0 \subseteq Z(G_0)$ and W/W_0 is elementary abelian.

If Fit G_0 is not fixed point free, it contains an element of prime order which is not fixed point free. Since Fit G_0 is nilpotent, it is a direct product of its Sylow subgroups so one of the Sylow subgroups of Fit G_0 is a non-f.p.f. normal subgroup of G. Indeed G has a minimal non-f.p.f. group W included in Fit G_0, where W is a w-group for some prime w.

Let W_0 be a maximal normal subgroup of G included in W but not equal to W. If w is odd, then W_0 is cyclic, since the Sylow subgroups of odd order in a Frobenius complement are cyclic. In this case, $W_0 \subseteq Z(G_0)$ by Lemma (2.1). If $w = 2$, then W_0 is either cyclic or generalized quaternion. In the latter case $\mathcal{C}(W_0) \cap G_0$ is a proper subgroup of G_0 normal in G and hence solvable. Thus $G_0/\,\mathcal{C}(W_0) \cap G_0$ is non-solvable. But this factor group is isomorphic to a group of automorphisms of a quaternion group and this automorphism group is solvable. (See Passman [15], pp. 74, 76.) We conclude that W_0 is in $Z(G_0)$.

(2.9) THEOREM. Let G be a group of linear transformations acting on a vector space V. Suppose that G is irreducible and

(1) The Sylow 2-groups of $G/Z(G)$ are diherdral.

(2) G is non-solvable with a minimal non-solvable normal subgroup G_0.

(3) The dimension of V is the produce of distinct primes.

Then either Fit G_0 is fixed point free so that G_0 satisfies the conclusion of (2.3) or Fit G_0 contains an elementary abelian group W which is a minimal non-f.p.f. group with respect to G.

PROOF. The theorem holds if W of (2.8) is non-trivial and W_0 is trivial, so suppose W_0 is non-trivial. We wish to apply (2.7). For this purpose, we can restrict our attention to a minimal G_0-space V_1. Note that the dimension of a minimal G_0-space is also the product of distinct primes.

Now all of the minimal $Z(G_0)$ spaces in a G_0-space are isomorphic as $Z(G_0)$-modules. As in Hering [7] Hilfssatz 5, there is a field K so that the additive group of V_1 is a vector space over K and the elements of $Z(G_0)$ become scalars.

Now consider the action of W on a minimal W-space in V_1. Again, the dimension is a product of distinct primes, so (2.7) implies that $|W/W_0| = w^2$ and w is one of the primes dividing the dimension of V over F.

Let $\hat{G} = G/C(w)$. Then $\hat{G}$ induces, by conjugation, a group of automorphisms of $\hat{W}$. That is, there is a homomorphism from $\hat{G}$ into $GL(2,w)$, since $\hat{W}$ is elementary abelian of order w^2. The kernel of the homomorphism is the subgroup of G which centralizes $\hat{W}$. If the subgroup of $\hat{G}_0$ which centralizes $\hat{W}$ were non-solvable, then $\hat{G}_0$ would centralize W. We wish to show that this cannot be the case.

The reader may verify that $\hat{\lambda}$ centralizes $\hat{W}$ if $\lambda^{-1}\sigma^{-1}\lambda\sigma \in \mathcal{C}(W)$ for all σ in W. (Here λ is a pre-image of $\hat{\lambda}$.) In particular, if $\lambda \in G_0$, $\sigma \in W$ we must have $\lambda^{-1}\sigma^{-1}\lambda\sigma = \nu$ for some ν in $\mathcal{C}(W) \cap G_0$. Then $\lambda^{-1}\sigma^{-1}\lambda = \nu\sigma^{-1}$. But $\lambda^{-1}\sigma^{-1}\lambda \in W$, so $\nu\sigma^{-1} \in W$ and $\nu \in W$. Thus $\nu \in W_0 \subseteq Z(G_0)$.

More particular, let λ be an element of G_0 such that $|\lambda|$ is a prime distinct from w. Then $\sigma^{-1}\lambda\sigma = \lambda\nu$; $|\lambda\nu| = |\lambda|$. Since $\nu \in Z(G_0)$ and $|\nu|$ is a power of w this implies that $\nu = 1$. That is, if $\lambda^{-1}\sigma^{-1}\lambda\sigma \in \mathcal{C}(W)$, then λ commutes with λ if $|\lambda|$ is a prime distinct from W.

The subgroup of G_0 generated by all elements λ of prime order $\neq$ w is a characteristic subgroup of G_0. Call this subgroup G_2. If $G_2 = G_0$, then G_0 centralizes W, but W is a non-abelian normal subgroup of G_0. This is a contradiction, so G_2 must be a proper subgroup of G_0 and hence solvable. But, except for the w-groups, the Sylow sugbroups of G_0 are included in G_2, so G_0/G_2 is a w-group. But if G_2 is solvable, G_0/G_2 must be non-solvable, so we again get a contradiction. Thus W_0 must be trivial if W exists and W must be abelian.

(2.10) LEMMA. Suppose that the Sylow 2-groups in $G/Z(G)$ are dihedral. Let H be a maximal normal subgroup of G included in G_0 but not equal to G_0. Then G_0/H is simple and either $H = W[\mathcal{C}(W) \cap G_0]$ or $W \subseteq Z(G_0)$.

PROOF. By much the same argument we just used, G_0/H is a direct product of isomorphic simple groups which, however, must be non-solvable this time. A group with cyclic Sylow 2-groups is solvable. Since H includes $Z(G_0)$ the Sylow 2-groups of G_0/H must be dihedral. The direct product of dihedral groups is not dihedral, so G_0/H is simple.

Also $H = H[\mathcal{C}(H) \cap G_0]$ so G_0/H is isomorphic to a group of outer automorphisms induced on H by conjugation. This group of automorphisms leaves W invariant; the subgroup acting as inner automorphisms of W is induced by $W[\mathcal{C}(W) \cap G_0]$. The outer automorphisms of H which act as inner automorphisms of W will be a normal subgroup. But G_0/H is simple so this normal subgroup is either trivial or is the whole thing.

Suppose that every element of G_0/H corresponds to an inner automorphism of W. Then every element of G_0 belongs to $W[\mathcal{C}(W) \cap G_0]$. But $W[\mathcal{C}(W) \cap G_0]$ is solvable unless $\mathcal{C}(W) \cap G_0$ is non-solvable. In the latter case $W \subseteq Z(G_0)$.

Otherwise, the group of outer automorphisms induced on W is isomorphic to the group of outer automorphisms induced on H so $G_0/H \cong G_0/W[\mathcal{C}(W) \cap G_0]$. Hence $H = W[\mathcal{C}(W) \cap G_0]$ in this case.

3. TRANSLATION PLANES OF ODD ORDER AND DIMENSION

The results of Theorems (2.3) and (2.8) have implications for translation planes because of the following result of Hering [9], Theorem 1.

(3.1) LEMMA. Let Π be a translation plane of order q^d with kernel $GF(q)$, where q and d are odd. Let G be a subgroup of the linear translation complement and let $\overline{G}$ be the induced permutation group on ℓ_∞ - i.e. $\overline{G}$ is the factor group of G modulo the scalars. Then the Sylow 2-groups in $\overline{G}$ are cyclic or dihedral.

REMARK. Groups with cyclic Sylow 2-groups are solvable. (See Burnside [1], p. 326, Theorem II.)

Another result of Hering is pertinent. (See [9] Theorem 2.)

(3.2) LEMMA. Under the assumptions of (3.1), let $G(\ell)$ be the subgroup of G stabilizing a component ℓ. Then $G(\ell)$ is solvable.

(3.3) LEMMA. Suppose that W of (2.8) is abelian and non-trivial, and G is irreducible. Then Π (as a vector space) is a direct sum $V_1 \oplus \cdots \oplus V_k$ of homogeneous W-spaces. G induces a transitive permutation group on $V_1, \cdots, V_K$; W is not faithful on V_i, $i = 1, \cdots, k$, and k is odd.

PROOF. If W_0 is trivial, W is elementary abelian by (2.9). Thus if $\lambda \in W$ and the subspace $V(\lambda)$ pointwise fixed by λ is non-trivial, then W leaves $V(\lambda)$ invariant. Hence W is not faithful on its minimal spaces. Since each homogeneous space is a direct sum of minimal spaces that are isomorphic as W-models, W is not faithful on its minimal spaces.

The rest of the Lemma follows from Clifford's Theorem. (See [4].) Furthermore k must divide the dimension $2d$ of Π as a vector space.

Let $W(V_i)$ be the subgroup of W which fixes V_i pointwise. Note that the fact that W is abelian and V_i is a homogeneous W-space implies that each element of W which is not f.p.f. on V_i is in $W(V_i)$. Furthermore a fixed point free w-group must be cyclic and W is elementary abelian so $W/W(V_i)$ is cyclic of order w. Thus if $W(V_i) = W(V_j)$ then V_i and V_j must be isomorphic as W-modules. But this is not the case if V_i, V_j are homogeneous W-spaces.

Let $V_i{}^*$ be the subspace pointwise fixed by $W(V_i)$. Then $V_i{}^*$ is a direct sum of homogeneous W-spaces. If $W(V_1)$, $W(V_2)$, $\cdots$ are distinct subgroups (not necessarily disjoint) then we must have $V_i{}^* = V_i$. But $V_i{}^*$ must be a subplane or a subspace of a component of the spread.

If $V_i{}^* \subseteq \ell$ for some component ℓ, then ℓ is invariant under W. If W leaves just one or two components invariant then G must fix or interchange these two and cannot be non-solvable. If W has 3 invariant components every non-f.p.f. element of W must fix a subplane pointwise.

Hence V_i is a subplane and has even dimension. This implies that k is odd, since $2d = k \dim V_1$.

(3.4) THEOREM. Let G be a non-solvable and irreducible subgroup of the linear translation complement of a translation plane Π of order q^d with kernel $GF(q)$, where q and d are odd. If W of (2.8) is non-trivial, then

W_0 must be non-trivial.

PROOF. Suppose that W_0 is trivial, so that (3.3) holds. Let $G(V_1)$ be the stabilizer of V_1 in G_1. The index of $G(V_1)$ in G is equal to k, so a Sylow 2-group of $G(V_1)$ is a Sylow 2-group of G.

Let $\tilde{G}(V_1)$ be the induced group on V_1 - i.e. $\tilde{G}(V_1)$ may be identified with the factor group obtained by taking $G(V_1)$ modulo the subgroup fixing V_1 pointwise. Then $\tilde{W}$ is a normal subgroup of order w in $\tilde{G}(V_1)$, and all of the minimal $\tilde{W}$ spaces in V_1 are isomorphic as $\tilde{W}$-modules. As in Hering [7], $\tilde{G}(V_1) \subseteq \Gamma L(s,q^t)$ and the subgroup centralizing $\tilde{W}$ is isomorphic to $\tilde{G}(V_1) \cap GL(s,q^t)$ for some s, t such that st = dim V_1. Thus the index of $\tilde{\mathcal{C}}(W) \cap \tilde{G}(V_1)$ divides t and is not divisible by 4.

Hence the index of $G(V_1) \cap \mathcal{C}(W)$ in $G(V_1)$ is not divisible by 4. Let S be a Sylow 2-group of $G(V_1)$. As pointed out at the beginning of the proof, S is then a Sylow 2-group of G. Hence $S/S \cap \mathcal{C}(W)$ is a Sylow 2-group of $G/\mathcal{C}(W)$ and its order is 1 or 2. This implies that $G/\mathcal{C}(W)$ is solvable. Hence $G_0/G_0 \cap \mathcal{C}(W)$ is solvable. This is a contradiction since $G_0 \cap \mathcal{C}(W)$ is solvable and G_0 is non-solvable. We conclude that W_0 must be non-trivial.

(3.5) THEOREM. Let Π be a translation plane of order q^d with kernel GF(q), where q and d are odd. Let G be a subgroup of the linear translation complement. Suppose that G is non-solvable and irreducible with a minimal normal non-solvable G_0 and that d is the product of distinct primes. Then either $G_0 \cong SL(2,u)$ for some odd u or $\overline{G}_0 = A_6$ or A_7. Here $\overline{G}_0 = G_0/Z(G_0)$.

PROOF. This is a consequence of (3.4), (2.9), and (2.3), except for the possibility that we might have $G_0 = PSL(2,u)$. But PSL(2,u) contains an elementary abelian group of order 4 in which all three involutions are conjugate. In a translation plane of odd order and dimension all three involutions would be affine homologies. This cannot happen.

It may be worth while to take a look at some aspects of the ways that $G_0 = SL(2,u)$ can act on a translation plane. One possibility is that u is a power of the characteristic p and that the p-elements are affine elations. If G_0 contains affine homologies of prime order greater than 5 then a result of the author [13] shows that G_0 contains affine elations. The group generated by these elations will be normal in G_0 and, in fact, equal to G_0.

(3.6) LEMMA. Suppose that $G_0 = SL(2,u)$, $u > 3$, is a normal subgroup of the linear translation complement. Suppose that u is prime and that r is a prime factor of $u(u + 1)$. Then, if G_0 contains a non-f.p.f. element λ of order r, at least one of the following holds:

(a) For some component ℓ, $G(\ell) \rhd SL(2,3)$ and $r = 3$.

(b) For some component ℓ fixed by λ, $G(\ell)$ is reducible on ℓ.

(c) For some component ℓ fixed by λ, $G(\ell)$ is not faithful on ℓ.

PROOF. If λ is not fixed point free, then λ fixes some component ℓ and is not fixed point free on ℓ. If (c) does not hold, we may assume that $G(\ell)$ is faithful on ℓ.

Then $G(\ell) \rhd G(\ell) \cap G_0$ and $G(\ell) \cap G_0$ is a solvable subgroup of G_0, since $G(\ell)$ is solvable. If $r > 3$, $< \lambda >$ will be characteristic in the maximal solvable subgroup of G_0 which contains λ so that $< \lambda >$ will be normal in $G(\ell)$. The subspace of ℓ which is pointwise fixed by λ will then be invariant under $G(\ell)$ so that $G(\ell)$ is reducible on ℓ.

If $r = 2$, λ is the unique involution in G_0. For a plane of odd dimension i.e., d is odd, a non-f.p.f. involution in the translation complement is a homology. This would come under conclusion (c); actually it cannot happen since the axis of the homology would be invariant under the non-solvable group G_0.

If $r = 3$, we have the possibility (a) with SL(2,3) characteristic in

$G_0 \cap G(\ell)$.

(3.7) COROLLARY. If r divides u + 1, the conclusions of (3.6) hold even if u is not prime.

REMARK. The cases where u is not prime or where G contains affine homologies of order 3 or 5 were handled in the Kallaher-Ostrom paper [12] under the assumption that a certain p-primitive divisor of $q^d - 1$ (which turned out to be u) divided the order of the group induced on ℓ by $G(\ell)$. Note that when (3.6) holds and there are no affine perspectivities, the orders of the non-f.p.f. elements in G_0 will divide u - 1 if $G(\ell)$ is irreducible.

Let $G(\ell)^*$ denote the group induced on ℓ by $G(\ell)$ - i.e. $G(\ell)^*$ is the factor group modulo the subgroup which fixes ℓ pointwise. If $G(\ell)^*$ has a normal subgroup whose order is a prime q-primitive divisor of $q^d - 1$, then $G(\ell)^*$ has an abelian irreducible normal subgroup. In this case $G(\ell)^* \cong \Gamma L(1,q^d)$. This is what happened in the Kallaher-Ostrom paper but this situation may arise without reference to primitive divisors. Hence we prove the following Lemma.

(3.8) LEMMA. Suppose that $G(\ell)^*$ is isomorphic to a subgroup of $\Gamma L(1,q^d)$ and contains an element σ^* such that (a) $|\sigma^*|$ is prime. (b) σ^* fixes at least one point $\neq 0$ on ℓ. Then $|\sigma^*|$ divides d.

PROOF. $\Gamma L(1,q^d)$, in its action on a vector space of dimension d over GF(q) has a cyclic normal fixed point free subgroup of order $q^d - 1$ and index d.

REMARK. In the Kallaher-Ostrom paper [12], Theorem 6.1, it turned out that d divides u - 1. A subgroup of a Frobenius complement whose order is the produce of two distinct primes must be cyclic. SL(2,u) has a subgroup of over u(u - 1) which is not fixed point free for u > 5. Putting this together with (3.6), it appears that an important subcase for the possible action of $SL(2,u)$ is the one where the orders of the non-fixed point free elements divide $u - 1$. In the context of (3.8), especially if d is prime, we again arrive at a situation where d divides $u - 1$.

(3.9) LEMMA. If u is not a power of p, if p and d are both odd, and if d divides $u - 1$ then $d = \frac{1}{2}(u - 1)$ or $d = \frac{1}{4}(u - 1)$.

PROOF. By Harris and Hering [6] $u \leqq 2(2d) + 1$. (The vector space has dimension $2d$.) Thus $d \geqq \frac{1}{4}(u - 1)$ and $d \neq u - 1$ if u and d are both odd.

(3.10) LEMMA. If $d = \frac{1}{4}(u - 1)$ then $G_0 = SL(2,u)$ is absolutely irreducible; if $d = \frac{1}{2}(u - 1)$ then either G_0 is absolutely irreducible or has an absolutely irreducible representation of dimension d over some extension of $GF(q)$.

PROOF. By Harris and Hering, the dimension of any representation (irreducible or not) is at least $\frac{1}{2}(u - 1)$.

(3.11) THEOREM. Let Π be a translation plane of order q^d and kernel $GF(q)$. Suppose that the linear translation complement of G of Π has a normal non-solvable sugbroup G_0 isomorphic to $SL(2,u)$ and that

(a) q and d are odd.

(b) d divides $u - 1$.

(c) For each component ℓ, $G(\ell)$ is irreducible.

(d) G_0 contains no affine homologies or elations.

(e) $G_0 \cap G(\ell)$ has no normal subgroup isomorphic to $SL(2,3)$.

(f) $(|G_0|, p) = 1$.

Then either $u + 1$ is a power of 2 and $d = \frac{1}{2}(u - 1)$ or $\frac{1}{2}(u + 1)$ is an odd prime and $d = \frac{1}{4}(u - 1)$.

PROOF. If $(|G_0|, p) = 1$ and if G_0 is absolutely irreducible, it has a complex representation of dimension $2d = (u - 1)$ and the representation we are using can be obtained from the complex representation. (See Dixon [2].)

In the other case, G_0 is reducible over a field extension but G_0 can be obtained from a complex representation of dimension u - 1.

Suppose that u + 1 has an odd prime factor r. Let λ be an element of order r in G_0. Under the hypotheses, it follows from (3.7) that λ is fixed point free.

Let Θ be a complex rth root of 1. Then the character of λ has the form $a_0 + a_1\Theta + \cdots + a_{r-1}\Theta^{r-1}$ where a_i is the multiplicity of the eigenvalue Θ^i, so that a_0, a_1, ... are non-negative integers. If e is the dimension of the complex representation, $a_0 + a_1 + \cdots + a_{r-1} = e$. First, suppose $e = \frac{u-1}{2}$. From character table (see Dornhoff [3]) the character of λ is equal to -1. But $1 + \Theta + \cdots + \Theta^{r-1} = 0$. Thus $a_0 + a_1\Theta + \cdots + a_{r-1}\Theta^{r-1} = \Theta + \Theta^2 + \cdots + \Theta^{r-1}$ so $a_0 + (a_1 - 1)\Theta + \cdots + (a_{r-1})\Theta^{r-1} = 0$. But λ is fixed point free and thus has no eigenvalues = 1. Hence $a_0 = 0$ in this case. But $\Theta, \cdots, \Theta^{r-1}$ are linearly independent over the integers since the polynomial $1 + x + \cdots + x^{r-1}$ is irreducible over the integers. Hence $a_1 = a_2 = \cdots = a_{r-1} = 1$ and

$$a_0 + \cdots + a_{r-1} = r - 1 = e = \frac{u-1}{2} \text{ and } r = \frac{u+1}{2}.$$

Now suppose e = u - 1. Then the character of λ has the form $-(\Theta^i + \Theta^{-i})$. Hence $a_0 + a_1\Theta + \cdots + a_{r-1}\Theta^{r-1} = -\Theta^i - \Theta^{-i}$. Again $a_0 = 0$ since λ is f.p.f. If we take the subscripts a_1, a_2, etc., mod r, we may write

$$a_1\Theta + \cdots + (a_i + 1)\Theta^i + \cdots + (a_{-i} + 1)\Theta^i + a_{r-1}\Theta^{r-1} = 0$$

if i = 0. This has no solutions for non-negative a_i.

Suppose i = 0, so

$$a_1\Theta + \cdots + a_{r-1}\Theta^{r-1} = -2 = 2(\Theta + \Theta^2 + \cdots + \Theta^{r-1})$$
$$(a_1 - 2)\Theta + \cdots + (a_{r-1}-2)\Theta^{r-1} = 0 \quad \text{implies}$$
$$a_1 = a_2 = \cdots = a_{r-1} = 2$$

so

$$a_0 + \cdots + a_{r-1} = 2(r - 1) = u - 1$$

then

$$r - 1 = \frac{u - 1}{2}, \ r = \frac{u + 1}{2} \text{ as before.}$$

Recall that, by (3.9) $d = \frac{1}{2}(u - 1)$ or $\frac{1}{4}(u - 1)$. If r exists, so that $\frac{1}{2}(u + 1)$ is odd, then $u - 1 \equiv 0 \bmod 4$ so that if d is odd, d cannot be equal to $\frac{1}{2}(u - 1)$. Thus if $\frac{1}{2}(u + 1)$ is prime, $d = \frac{1}{4}(u - 1)$.

If r does not exist, so that $u + 1$ is a power of 2, then $\frac{1}{4}(u - 1)$ is not integer so that $d = \frac{1}{2}(u - 1)$ in this case.

We can use (3.11) to make a slight improvement in Theorem 6.1 of [12]. In the following corollary, u is a prime p-primitive divisor of $q^d - 1$ where p is prime and $q = p^k$. G has the usual meaning of this paper and G_0 is a minimal non-solvable normal subgroup of G. Here and earlier in this paper when reference is made to the group induced on ℓ by $G(\ell)$, it should be understood that the subgroup fixing ℓ pointwise has been factored out. We continue to assum q and d are both odd.

(3.12) COROLLARY. Suppose that, for each component ℓ, the order of the group induced on ℓ by $G(\ell)$ is divisible by u and that G_0 exists and is non-trivial. Suppose that Π is non-Desarguesian. Then at least one of the following holds.

(a) $u = 13$, $d = 3$, $q = 3$, $G_0 = SL(2,13)$

(b) $u = 2d + 1$, $q = p$, $u + 1$ is a power of 2 and $G_0 = SL(2,u)$.

(c) $u = 2d + 1$, $q = p$, p divides d and $G_0 = SL(2,u)$.

(d) $u = 7$, $d = 3$, $q = p$ and $\overline{G} = A_7$.

PROOF. In Theorem (6.1) of [12], it is shown that, under the present hypotheses we have case (a) or (d) or $G_0 = SL(2,u)$ $q = p$, $u = 2d + 1$.

Condition (d) of (3.11) is also satisfied if $G_0 = SL(2,u)$. (It is not clear from the statement of Theorem (6.1) but the possibility that G_0 might contain affine elations is disposed of by Lemma (6.3) of [12].)

In Theorem (6.1) of [12], G is assumed to be generated by its Sylow u-groups and the conclusion is that $G = G_0$. Thus $G_0 \cap G(\ell)$ is faithful for each ℓ and its order is divisible by the p-primitive divisor u of $q^d - 1$. This implies that $G(\ell) \subset \Gamma L(1,q^d)$ and that condition (e) of (3.11) is met.

If $(|G_0|, p) = 1$ and $u = 2d + 1$, we get case (b) of the conclusion of our corollary from (3.11).

If p divides $|G_0|$ Lemma (3.6) tells us that, under present circumstance p must divide $u - 1$. But $u - 1 = 2d$ and p is odd so p must divide d.

REMARK. The factor group $G/G_0\mathcal{C}(G_0)$ is isomorphic to a group of outer automorphisms of G_0. If $G \triangleright G_0 = SL(2,u)$ where u is a prime, the outer automorphism group is trivial, so that G is isomorphic to $G_0\mathcal{C}(G_0)$. If G_0 is irreducible, then $\mathcal{C}(G_0)$ must be fixed point free.

REMARK. Later work of Kallaher and the author has shown that case (d) of (3.12) does not happen.

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