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Research Article

Integrodifferential Inequalities Arising in the Theory of Differential Equations

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The goal of this paper is to achieve some new results related to integrodifferential inequalities of one independent variable which can be applied as a study of qualitative and quantitative properties of solutions of some nonlinear integral equations.

1. Introduction

Integral and integrodifferential inequalities play a significant role in recent years by many authors [1–11], which provide an explicit bounds on the solutions of a class of differential and integral equations.

Lemma 1. *Pachpatte (1995) studied the following useful integral inequality: Let $u(t)$, $f(t)$, and $g(t)$ be nonnegative continuous functions defined on R_+ and c_1 and c_2 be positive constants. If*

$$u(t) \le \left(c_1 + \int_0^t f(s)\,u(s)\,ds\right)\left(c_2 + \int_0^t g(s)\,u(s)\,ds\right), \quad (1)$$

for $t \in R_+$, where

$$H(t) = 1 - c_1 c_2 \int_0^t R(s)\,Q(s)\,ds > 0, \quad (2)$$

$$R(t) = g(t)\int_0^t f(\sigma)\,d\sigma + f(t)\int_0^t g(\sigma)\,d\sigma, \quad (3)$$

$$Q(t) = \exp\left(\int_0^t \left[c_1 g(\sigma) + c_2 f(\sigma)\right] d\sigma\right), \quad (4)$$

for $t \in R_+$, then

$$u(t) \le \left[\frac{1}{H(t)}\right] c_1 c_2 Q(t), \quad (5)$$

for all $t \in R_+$.

2. Main Results

Here by using Lemma 1, we establish some new results in the form of integrodifferential inequalities instead of integral inequality.

Theorem 2. *Let $x(t)$, $x^{\bullet}(t)$, $f(t)$, and $g(t)$ be nonnegative real valued continuous functions defined for $R_+ = [0, \infty)$. Let c_1 and c_2 be positive constants. If*

$$x^{\bullet 2}(t) \le \left(c_1 + \int_0^t f(s)\,x^{\bullet 2}(s)\,ds\right)\left(c_2 + \int_0^t g(s)\,x^{\bullet 2}(s)\,ds\right), \quad (6)$$

for all $t \in R_+$, then

$$x^{\bullet}(t) \le \left[\frac{1}{\sqrt{H(t)}}\right]\sqrt{(c_1 c_2)}\sqrt{Q(t)}, \quad (7)$$

where $c_1 c_2 \ge 1$.

$H(t)$, $R(t)$, and $Q(t)$ are defined as in (2), (3), and (4), respectively, for all $t \in R_+$.

Proof. Define a function $z^2(t)$ by the right-hand side of (6), such that

$$z^2(t) = \left(c_1 + \int_0^t f(s)\,x^{\bullet 2}(s)\,ds\right)\left(c_2 + \int_0^t g(s)\,x^{\bullet 2}(s)\,ds\right), \quad (8)$$

where

$$z^2(0) = c_1 c_2. \tag{9}$$

From (6) and (8), we get

$$x^{\bullet 2}(t) \le z^2(t). \tag{10}$$

By differentiating (8) and using the fact that

$$x^{\bullet}(t) \le z(t) \tag{11}$$

we observe

$$\begin{aligned} 2z(t)z'(t) &\le (c_1 g(t) + c_2 f(t)) z^2(t) \\ &\quad + \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma \right) z^4(t) \end{aligned} \tag{12}$$

or

$$\begin{aligned} &2z^{-3}(t)z'(t) - (c_1 g(t) + c_2 f(t)) z^{-2}(t) \\ &\quad \le \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma \right). \end{aligned} \tag{13}$$

Let

$$v(t) = z^{-2}(t). \tag{14}$$

Differentiating (14) with respect to x, we get

$$v'(t) = -2z^{-3}(t)z'(t), \tag{15}$$

where

$$v(0) = (c_1 c_2)^{-1}. \tag{16}$$

By substituting (14) and (15) in (13), we have

$$v'(t) + (c_1 g(t) + c_2 f(t)) v(t) \ge -R(t). \tag{17}$$

Inequality (17) implies the estimation for $v(t)$ and by using (16), we observe that

$$\begin{aligned} v(t) &\ge (c_1 c_2)^{-1} Q^{-1}(t) \left(1 - (c_1 c_2) \int_0^t R(s) Q(s)\,ds \right) ds, \end{aligned} \tag{18}$$

where $R(t)$ and $Q(t)$ are defined as in (3) and (4) and by applying (11) and (14) it is noticed that

$$x^{\bullet}(t) \le \left[\frac{1}{\sqrt{H(t)}} \right] \sqrt{(c_1 c_2)} \sqrt{Q(t)}, \tag{19}$$

where $H(t)$ is defined as in (2). This completes the proof. □

Theorem 3. *Let $x(t)$, $x^{\bullet}(t)$, $f(t)$, $g(t)$, c_1, and c_2 be defined as in Theorem 2 for $R_+ = [0, \infty)$. If*

$$\begin{aligned} &x^{\bullet 2}(t) \\ &\le \left(c_1 + \int_0^t f(s) x^{\bullet}(s)\,ds \right) \left(c_2 + \int_0^t g(s) x^{\bullet 2}(s)\,ds \right), \end{aligned} \tag{20}$$

for all $t \in R_+$, then

$$x^{\bullet}(t) \le \left[\frac{1}{H(t)} \right] \sqrt{(c_1 c_2)} Q(t), \tag{21}$$

where $c_1 c_2 \ge 1$ and

$$H(t) = 1 - \frac{\sqrt{(c_1 c_2)}}{2} \int_0^t R(s) Q(s)\,ds > 0, \tag{22}$$

$$Q(t) = \exp\left(\left(\frac{1}{2} \right) \int_0^t [c_1 g(\sigma) + c_2 f(\sigma)]\,d\sigma \right) \tag{23}$$

for all $t \in R_+$.

Proof. Define a function $z^2(t)$ by the right-hand side of (20), such that

$$\begin{aligned} &z^2(t) \\ &= \left(c_1 + \int_0^t f(s) x^{\bullet}(s)\,ds \right) \left(c_2 + \int_0^t g(s) x^{\bullet 2}(s)\,ds \right), \end{aligned} \tag{24}$$

where

$$z^2(0) = c_1 c_2. \tag{25}$$

From (20) and (24), we get

$$x^{\bullet 2}(t) \le z^2(t). \tag{26}$$

By differentiating (24) and since $z(t)$ is monotone nondecreasing function for $t \in R_+$ and using the fact that

$$x^{\bullet}(t) \le z(t) \tag{27}$$

we observe that

$$\begin{aligned} 2z(t)z'(t) &\le (c_1 g(t) + c_2 f(t)) z^2(t) \\ &\quad + \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma \right) z^3(t) \end{aligned} \tag{28}$$

or

$$\begin{aligned} &2z^{-2}(t)z'(t) - (c_1 g(t) + c_2 f(t)) z^{-1}(t) \\ &\quad \le \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma \right). \end{aligned} \tag{29}$$

Let

$$v(t) = z^{-1}(t); \quad v(0) = (c_1c_2)^{-1/2}. \tag{30}$$

By repeating the same steps from (14)–(18) in (29) with suitable modifications, the estimation for $v(t)$ implies

$$v(t) \geq (c_1c_2)^{-1/2} Q^{-1}(t) \cdot \left(1 - \frac{\sqrt{(c_1c_2)}}{2}\int_0^t R(s)Q(s)\,ds\right) ds. \tag{31}$$

From (27) and (30) in (31), we get

$$x^{\bullet}(t) \leq \left[\frac{1}{H(t)}\right]\sqrt{(c_1c_2)Q(t)} \tag{32}$$

for all $t \in R_+$, where $R(t)$, $H(t)$, and $Q(t)$ are defined as in (3), (22), and (23), respectively. This completes the proof. □

Theorem 4. *Let $x(t)$, $x^{\bullet}(t)$, $f(t)$, $g(t)$, c_1, and c_2 be defined as in Theorem 2 for $R_+ = [0,\infty)$. If*

$$x^{\bullet p}(t) \leq \left(c_1 + \int_0^t f(s)x^{\bullet p}(s)\,ds\right)\left(c_2 + \int_0^t g(s)x^{\bullet}(s)\,ds\right), \tag{33}$$

for all $t \in R_+$, then

$$x^{\bullet}(t) \leq \left[\frac{1}{H(t)}\right](c_1c_2)^{1/p}Q(t), \tag{34}$$

where $c_1c_2 \geq 1$ and $p > 0$.

$$H(t) = 1 - \frac{(c_1c_2)^{1/p}}{p}\int_0^t R(s)Q(s)\,ds > 0, \tag{35}$$

$$Q(t) = \exp\left(\left(\frac{1}{p}\right)\int_0^t [c_1 g(\sigma) + c_2 f(\sigma)]\,d\sigma\right) \tag{36}$$

for all $t \in R_+$.

Proof. Define a function $z^p(t)$ by the right-hand side of (33), such that

$$z^p(t) = \left(c_1 + \int_0^t f(s)x^{\bullet p}(s)\,ds\right)\left(c_2 + \int_0^t g(s)x^{\bullet}(s)\,ds\right), \tag{37}$$

where

$$z^p(0) = c_1c_2. \tag{38}$$

From (33) and (37), we get

$$x^{\bullet p}(t) \leq z^p(t) \tag{39}$$

or

$$x^{\bullet}(t) \leq z(t). \tag{40}$$

By differentiating (37) and since $z(t)$ is monotone nondecreasing function for $t \in R_+$, we observe that

$$\begin{aligned} pz^{p-1}(t)z'(t) &\leq (c_1 g(t) + c_2 f(t))z^p(t) \\ &\quad + \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma\right)z^{p+1}(t) \end{aligned} \tag{41}$$

or

$$\begin{aligned} &pz^{-2}(t)z'(t) - (c_1 g(t) + c_2 f(t))z^{-1}(t) \\ &\quad \leq \left(f(t)\int_0^t g(\sigma)\,d\sigma + g(t)\int_0^t f(\sigma)\,d\sigma\right). \end{aligned} \tag{42}$$

Let

$$v(t) = z^{-1}(t); \quad v(0) = (c_1c_2)^{-1/p}. \tag{43}$$

By repeating the same steps from (14)–(18) in (42) with suitable modifications, the estimation for $v(t)$ implies

$$v(t) \geq (c_1c_2)^{-1/p} Q^{-1}(t) \cdot \left(1 - \frac{(c_1c_2)^{1/p}}{p}\int_0^t R(s)Q(s)\,ds\right) ds. \tag{44}$$

From (40) and (43) in (44), we get

$$x^{\bullet}(t) \leq \left[\frac{1}{H(t)}\right](c_1c_2)^{1/p}Q(t), \tag{45}$$

for all $t \in R_+$, where $R(t)$, $H(t)$, and $Q(t)$ are defined as in (3), (35), and (36), respectively. This completes the proof. □

Theorem 5. *Let $x(t)$, $x^{\bullet}(t)$, $f(t)$, $g(t)$, c_1, and c_2 be defined as in Theorem 2 for $R_+ = [0,\infty)$. If*

$$x^{\bullet p}(t) \leq \left(c_1 + \int_0^t f(s)[x^{\bullet}(t) + x^{\bullet p}(s)]\,ds\right) \cdot \left(c_2 + \int_0^t g(s)x^{\bullet}(s)\,ds\right), \tag{46}$$

for all $t \in R_+$, then

$$x^{\bullet}(t) \leq \left[\frac{1}{H(t)}\right](c_1c_2)^{1/p}Q(t), \tag{47}$$

where $c_1c_2 \geq 1$ and $p > 0$.

$$H(t) = 1 - \frac{(c_1c_2)^{1/p}}{p} \int_0^t R(s) Q(s) ds > 0, \tag{48}$$

$$Q(t) = \exp\left(\left(\frac{1}{p}\right) \int_0^t \left[c_1 g(\sigma) + 2c_2 f(\sigma) + f(\sigma) \int_0^s g(\eta) d\eta + g(\sigma) \int_0^s f(\eta) d\eta\right] d\sigma\right) \tag{49}$$

for all $t \in R_+$.

Proof. Define a function $z^p(t)$ by the right-hand side of (46), such that

$$z^p(t) = \left(c_1 + \int_0^t f(s) [x^{\bullet}(t) + x^{\bullet p}(s)] ds\right) \cdot \left(c_2 + \int_0^t g(s) x^{\bullet}(s) ds\right), \tag{50}$$

where

$$z^p(0) = c_1c_2. \tag{51}$$

From (46) and (50), we get

$$x^{\bullet p}(t) \le z^p(t) \tag{52}$$

or

$$x^{\bullet}(t) \le z(t). \tag{53}$$

By differentiating (50) and since $z(t)$ is monotone nondecreasing function for $t \in R_+$, we observe that

$$pz^{p-1}(t) z'(t) \le \left(c_1 g(\sigma) + 2c_2 f(\sigma) + f(\sigma) \int_0^s g(\eta) d\eta + g(\sigma) \int_0^s f(\eta) d\eta\right) z^p(t) + R(t) z^{p+1}(t) \tag{54}$$

or

$$pz^{-2}(t) z'(t) - \left(c_1 g(\sigma) + 2c_2 f(\sigma) + f(\sigma) \int_0^s g(\eta) d\eta + g(\sigma) \int_0^s f(\eta) d\eta\right) z^{-1}(t) \le R(t). \tag{55}$$

Let

$$v(t) = z^{-1}(t); \quad v(0) = (c_1c_2)^{-1/p}. \tag{56}$$

By repeating the same steps from (14)–(18) in (55) with suitable modifications, the estimation for $v(t)$ implies

$$v(t) \ge (c_1c_2)^{-1/p} Q^{-1}(t) \cdot \left(1 - \frac{(c_1c_2)^{1/p}}{p} \int_0^t R(s) Q(s) ds\right) ds. \tag{57}$$

From (53) and (56) in (57), we get

$$x^{\bullet}(t) \le \left[\frac{1}{H(t)}\right] (c_1c_2)^{1/p} Q(t), \tag{58}$$

for all $t \in R_+$, where $R(t)$, $H(t)$, and $Q(t)$ are defined as in (3), (48), and (49), respectively. This completes the proof. □

Theorem 6. *Let $x(t)$, $x^{\bullet}(t)$, $f(t)$, $g(t)$, c_1, and c_2 be defined as in Theorem 2 for $R_+ = [0, \infty)$. If*

$$x^{\bullet p}(t) \le \left(c_1 + \int_0^t f(s) x^{\bullet q}(s) ds\right)\left(c_2 + \int_0^t g(s) x^{\bullet}(s) ds\right), \tag{59}$$

for all $t \in R_+$, *then*

$$x^{\bullet}(t) \le \left[\frac{1}{Q(t)}\right]^{1/(p-q)} \left[(c_1c_2)^{p-q/p} + \left(\frac{p-q}{p}\right) \int_0^t (c_1 f(s) + c_2 g(s)) Q(s) ds\right]^{1/(p-q)}, \tag{60}$$

where $c_1c_2 \ge 1$, $p > q \ge 1$, *and* $p - q \ge 1$.

$$Q(t) = \exp\left(\frac{-(p-q)}{p} \int_0^t R(s) ds\right) \tag{61}$$

for all $t \in R_+$.

Proof. Define a function $z^p(t)$ by the right-hand side of (59), such that

$$z^p(t) = \left(c_1 + \int_0^t f(s) x^{\bullet q}(s) ds\right)\left(c_2 + \int_0^t g(s) x^{\bullet}(s) ds\right), \tag{62}$$

where

$$z^p(0) = c_1c_2. \tag{63}$$

From (59) and (62), we get

$$x^{\bullet p}(t) \le z^p(t) \tag{64}$$

or

$$x^{\bullet}(t) \le z(t). \tag{65}$$

By differentiating (62) and since $z(t)$ is monotone nondecreasing function for $t \in R_+$, we observe that

$$\begin{aligned} & pz^{p-1}(t)\, z'(t) \\ & \quad \le (c_1 g(t) + c_2 f(t))\, z^q(t) \\ & \qquad + \left(f(t) \int_0^t g(\sigma)\, d\sigma + g(t) \int_0^t f(\sigma)\, d\sigma \right) z^{1+q}(t) \end{aligned} \tag{66}$$

or

$$pz^{p-q-1}(t)\, z'(t) - R(t)\, z^{p-q}(t) \le (c_1 g(t) + c_2 f(t)). \tag{67}$$

Let

$$\begin{aligned} v(t) &= z^{p-q}(t); \\ v(0) &= (c_1 c_2)^{(p-q)/p}. \end{aligned} \tag{68}$$

By repeating the same steps from (14)–(18) in (67) with suitable modifications, the estimation for $v(t)$ implies

$$\begin{aligned} v(t) \le \left[\frac{1}{Q(t)} \right]^{1/(p-q)} & \left[(c_1 c_2)^{p-q/p} \right. \\ & \left. + \left(\frac{p-q}{p} \right) \int_0^t (c_1 f(s) + c_2 g(s))\, Q(s)\, ds \right]^{1/(p-q)}. \end{aligned} \tag{69}$$

From (65) and (68) in (69), we get

$$\begin{aligned} x^{\bullet}(t) \le \left[\frac{1}{Q(t)} \right]^{1/(p-q)} & \left[(c_1 c_2)^{p-q/p} \right. \\ & \left. + \left(\frac{p-q}{p} \right) \int_0^t (c_1 f(s) + c_2 g(s))\, Q(s)\, ds \right]^{1/(p-q)}, \end{aligned} \tag{70}$$

for all $t \in R_+$, where $R(t)$ and $Q(t)$ are defined as in (3) and (61), respectively. This completes the proof. □

3. Application

As an application, the explicit bounds of some of the integral inequalities can be found by some examples.

Example 1. Let us consider the explicit bound on the solution of the nonlinear integrodifferential equation

$$\begin{aligned} & x^{\bullet 2}(t) \\ & \quad \le \left(1 + \int_0^t f(s)\, x^{\bullet 2}(s)\, ds \right) \left(1 + \int_0^t g(s)\, x^{\bullet}(s)\, ds \right), \end{aligned} \tag{71}$$

where $x^{\bullet}(s)$ is a nonnegative real valued continuous function and every solution of $x^{\bullet}(s)$ of (71) exists for R_+.

By using the application of Theorem 4 to (71), we observe that

$$x^{\bullet}(t) \le \left[\frac{1}{H(t)} \right] Q(t), \tag{72}$$

where

$$R(t) = \int_0^t ds + \int_0^t ds = 2t, \tag{73}$$

$$\begin{aligned} H(t) &= 1 - \frac{1}{2} \int_0^t R(s)\, Q(s)\, ds > 0 = 1 - \frac{1}{2} \int_0^t 2se^s ds \\ &= 1 - \int_0^t se^s ds = -te^t + e^t, \end{aligned} \tag{74}$$

$$Q(t) = \exp\left(\left(\frac{1}{2} \right) \int_0^t 2\, ds \right) = \exp\left(\int_0^t ds \right) = e^t. \tag{75}$$

Therefore the right-hand side of (74) provides the bound of the solution of (75) of known quantities

$$x^{\bullet} t \le \frac{e^t}{e^t + te^t} \le \frac{1}{1-t} \tag{76}$$

for $0 \le t < 1$.

Example 2. Let us consider the nonlinear integrodifferential equation of the form

$$\begin{aligned} x^{\bullet p}(t) \le & \left(1 + \int_0^t f(s) \left[x^{\bullet}(t) + x^{\bullet 2}(s) \right] ds \right) \\ & \cdot \left(1 + \int_0^t g(s)\, x^{\bullet}(s)\, ds \right), \end{aligned} \tag{77}$$

where $x^{\bullet}(s)$ is a nonnegative real valued continuous function and every solution of $x^{\bullet}(s)$ of (77) exists for R_+.

By using the application of Theorem 5 to (77), we observe that

$$x^{\bullet}(t) \le \left[\frac{1}{H(t)} \right] Q(t), \tag{78}$$

where

$$R(t) = \int_0^t ds + \int_0^t ds = 2t, \tag{79}$$

$$\begin{aligned} H(t) &= 1 - \frac{1}{2} \int_0^t R(s)\, Q(s)\, ds > 0 \\ &= 1 - \frac{1}{2} \int_0^t 2se^{(3/2)s + (1/2)s^2} ds \\ &= 1 - \int_0^t se^{((3/2)s + (1/2)s^2)} ds, \end{aligned} \tag{80}$$

$$\begin{aligned} Q(t) &= \exp\left(\left(\frac{1}{2} \right) \int_0^t \left(1 + 2 + \int_0^t d\sigma + \int_0^t d\sigma \right) ds \right) \\ &= \exp \frac{1}{2} (3t + t^2) = e^{((3/2)t + (1/2)t^2)}. \end{aligned} \tag{81}$$

Therefore the right-hand side of (80) provides the bound of the solution of (77) of known quantities

$$x^{\bullet}(t) \le \frac{e^{((3/2)t + (1/2)t^2)}}{1 - \int_0^t se^{((3/2)s + (1/2)s^2)} ds} \tag{82}$$

for $0 \le t < 1$.

Example 3. Now let us consider the boundedness and asymptotic behaviour of the solutions of nonlinear Volterra integrodifferential inequality of the form

$$x^{\bullet p}(t) = \left(a_1(t) + \int_0^t A(t-s)\, x^{\bullet q}(s)\, ds\right) \cdot \left(a_2(t) + \int_0^t B(t-s)\, x^{\bullet}(s)\, ds\right); \tag{83}$$

$x^{\bullet}(t)$ is nonnegative real valued continuous function defined on R_+ and a_1, a_2, A, B are real valued continuous function defined on R_+.

We assume that every solution of $x^{\bullet}(t)$ in (81) exists on R_+, and p and q are defined as in Theorem 6. Define the following hypotheses on the function of (81) as

$$\left|a_1(t)\right| \le c_1, \qquad \left|a_2(t)\right| \le c_2. \tag{84}$$

Also

$$|A(t-s)| \le M_1 f_1(s), \tag{85}$$

$$|B(t-s)| \le N_1 g_1(s), \tag{86}$$

$$D(t) = \left[\frac{1}{Q(t)}\right]^{1/(p-q)} \left[(c_1 c_2)^{p-q/p} + \left(\frac{p-q}{p}\right) \int_0^t (c_1 g(s) + c_2 f(s))\, Q(s)\, ds\right]^{1/(p-q)} < \infty \tag{87}$$

for all $0 \le s \le t$, $s, t \in R_+$. M, N, c_1, c_2 are nonnegative real constants and f_1, g_1 are nonnegative real valued continuous function defined on R_+.

Proof. For the boundedness of the solution of nonlinear integrodifferential equation (83), let us suppose that the hypotheses (84), (85), and (86) are satisfied and let $x^{\bullet}(t)$ be a solution of (83); then we observe that

$$\left|x^{\bullet p}(t)\right| \le \left(\left|a_1(t)\right| + \int_0^t |A(t-s)| \left|x^{\bullet q}(s)\right| ds\right) \cdot \left(\left|a_2(t)\right| + \int_0^t |B(t-s)| \left|x^{\bullet}(s)\right| ds\right). \tag{88}$$

Replacing f by $M_1 f_1$ and g by $N_1 g_1$ and applying the same proof with some modifications of Theorem 6 in (88) and with $R(t)$ and $Q(t)$ being the same as defined in Theorem 6, we noticed that every solution of $x^{\bullet}(t)$ of (88) that exists on R_+ is bounded; that is,

$$\left|x^{\bullet p}(t)\right| \le D(t). \tag{89}$$

For the asymptotic behaviour of the solution of nonlinear integrodifferential equation (83), assume the following hypotheses

$$\left|a_1(t)\right| \le c_1 e^{-\mu t}, \qquad \left|a_2(t)\right| \le c_2 e^{-\mu t}. \tag{90}$$

Also

$$|A(t-s)| \le M_1 f_1(s)\, e^{-\mu(t-2s)}, \qquad |B(t-s)| \le N_1 g_1(s)\, e^{-\mu(t-2s)} \tag{91}$$

are satisfied. Let $x^{\bullet}(t)$ be a solution of (83); then

$$\left|x^{\bullet p}(t)\right| \le e^{-2\mu t} \left(c_1 + \int_0^t M_1 f_1(s) \left|x^{\bullet q}(s)\right| e^{2\mu s} ds\right) \cdot \left(c_2 + \int_0^t N_1 g_1(s) \left|x^{\bullet}(s)\right| e^{2\mu s} ds\right). \tag{92}$$

Let $z^m(t)$ be $|x^{\bullet p}(t)| e^{2\mu t}$ for $m > 0$ in (92) and by applying the same proof with some changes of Theorem 6 in (92), we get

$$z(t) \le D(t) \Longrightarrow \left|x^{\bullet}(t)\right| \le D(t)\, e^{-2\mu t}. \tag{93}$$

Therefore the solution $x^{\bullet}(t)$ of (83) is asymptotically stable. □

Competing Interests

There is no conflict of interests regarding the publication of this paper.

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