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Retraction

Retracted: Iterative Schemes by a New Generalized Resolvent for a Monotone Mapping and a Relatively Weak Nonexpansive Mapping

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This article in [1] has been retracted as it is essentially identical in content with a previously published paper by the same authors titled "A New Generalized Resolvent and Application in Banach Mappings." This manuscript was published in East Asian Mathematical Journal, Volume 30 (2014), No. 1, pp. 69–77.

References

[1] X. Wang, J. Chen, and H. Tong, "Iterative schemes by a new generalized resolvent for a monotone mapping and a relatively weak nonexpansive mapping," *Journal of Applied Mathematics*, vol. 2014, Article ID 603186, 6 pages, 2014.

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Research Article

Iterative Schemes by a New Generalized Resolvent for a Monotone Mapping and a Relatively Weak Nonexpansive Mapping

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We introduce a new generalized resolvent in a Banach space and discuss some of its properties. Using these properties, we obtain an iterative scheme for finding a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping. Furthermore, strong convergence of the scheme to a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping is proved.

1. Preliminaries

Let E be a real Banach space with dual E^*. We denote by J the normalized duality mapping from E into 2^{E^*}, defined by

$$Jx := \left\{ f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2 \right\}, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex, then J is single valued and if E is uniformly smooth, then J is uniformly continuous on bounded subsets of E. Moreover, if E is a reflexive and strictly convex Banach space with a strictly convex dual, then J^{-1} is single valued, one-to-one, and surjective, and it is the duality mapping from E^* into E and thus $JJ^{-1} = I_{E^*} = I^*$ and $J^{-1}J = I_E = I$ (see [1]). We note that, in a Hilbert space H, J is the identity mapping.

Let E be a smooth, reflexive, and strictly convex Banach space. We define the function $V_2 : E \times E \to R$ by

$$V_2(y, x) = \|x\|^2 - 2\langle Jy, x \rangle + \|y\|^2, \quad (2)$$

for all $x \in E, y \in E$. Let C be a nonempty closed convex subset of E. For an arbitrary point x of E, consider the set $\{z \in C : V_2(z, x) = \min_{y \in C} V_2(y, x)\}$. In 1996, Alber [2] introduced generalized projection $\Pi_C : E \to C$ from Hilbert space to uniformly convex and uniformly smooth Banach space:

$$V_2(\Pi_C x, x) = \min_{y \in C} V_2(y, x). \quad (3)$$

Such a mapping Π_C is called the generalized projection.

Applying the definitions of V_2 and J, a functional $V : E^* \times E \to R$ is defined by the following formula:

$$V(x^*, y) = V_2(J^{-1}x^*, y), \quad \forall x^* \in E^*, \ y \in E. \quad (4)$$

In the following, we will make use of the following lemmas.

Lemma 1 (see [3]). *Let E be a real smooth Banach space and let $A : E \to 2^{E^*}$ be a maximal monotone mapping; then $A^{-1}0$ is a closed and convex subset of E and the graph of A, $G(A)$, is demiclosed in the following sense, for all $x_n \in D(A)$ with $x_n \to x$ in E and for all $y_n \in Ax_n$ with $y_n \to y$ in E implying that $x \in D(A)$ and $y \in Ax$.*

Lemma 2 (see [2]). *Let C be a nonempty closed and convex subset of a real reflexive, strictly convex, and smooth Banach space E and let $x \in E$. Then, $y \in C$ and*

$$V_2(y, \Pi_C x) + \leq V_2(\Pi_C x, x) \leq V_2(y, x). \quad (5)$$

Lemma 3 (see [2]). *Let C be a convex subset of a real smooth Banach space E. Let $x \in E$ and $x_0 \in C$. Then, $V_2(x_0, x) = \inf\{V_2(z, x) : z \in C\}$ if and only if*

$$\langle z - x_0, Jx_0 - Jx \rangle \geq 0. \tag{6}$$

Lemma 4 (see [4]). *Let E be a real smooth and uniformly convex Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If either $\{x_n\}$ or $\{y_n\}$ is bounded and $V_2(x_n, y_n) \to 0$ as $n \to \infty$, then $x_n - y_n \to 0$, as $n \to \infty$.*

Let E^* be a smooth Banach space and let D^* be a nonempty closed convex subset of E^*. A mapping $R^* : D^* \to D^*$ is called generalized nonexpansive if $F(R^*) \neq \emptyset$ and

$$V\left(R^* x^*, J^{-1} y^*\right) \leq V\left(x^*, J^{-1} y^*\right), \quad \forall x^* \in D^*, \ y^* \in F\left(R^*\right), \tag{7}$$

where $F(R^*)$ is the set of fixed points of R^*.

Let C be a nonempty closed convex subset of E, and let T be a mapping from C into itself. We denote by $F(T)$ the set of fixed points of T. A point of p in C is said to be a strong asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges strongly to p such that the strong $\lim_{n\to\infty}(Tx_n - x_n) = 0$. The set of strong asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. A mapping T from C into itself is called weak relatively nonexpansive if $\widetilde{F}(T) = F(T)$ and $V_2(p, Tx) \leq V_2(p, x)$ for all $x \in C$ and $p \in F(T)$ (see [5]).

Let E be a smooth Banach space and let C be a nonempty closed convex subset of E. A mapping $R : C \to C$ is called generalized nonexpansive if $F(R) \neq \emptyset$ and

$$V_2(Rx, y) \leq V_2(x, y), \quad \forall x \in C, \ y \in F(R), \tag{8}$$

where $F(R)$ is the set of fixed points of R. Let E be a reflexive and smooth Banach space and let $B \subset E^* \times E$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, Ibaraki and Takahashi [6] considered the set

$$J_\lambda x := \{z \in E : x \in z + \lambda BJ(z)\}. \tag{9}$$

Such a J_λ is called the generalized resolvent and is denoted by

$$J_\lambda = (I + \lambda BJ)^{-1}. \tag{10}$$

By sunny nonexpansive retractions, they discussed the existence of a retraction R_C of E onto C such that, for any $x \in E$,

$$\langle x - R_C x, J(R_C x) - J(y) \rangle \geq 0, \quad \forall y \in C, \tag{11}$$

where E is a smooth Banach space and C is nonempty closed subset of E (see [7]).

In [7], Zegeye and Shahzad studied the following iterative scheme for finding a zero point of a maximal strongly monotone mapping A in a real uniformly smooth and uniformly convex Banach space E. Then the sequence $\{x_n\}$ generated by

$$\begin{aligned}
&x_0 \in K, chosenarbitrary,\\
&y_n = J^{-1}\left(Jx_n - \alpha_n Ax_n\right),\\
&z_n = Ty_n,\\
&H_0 = \{v \in K : \phi(v, z_0) \leq \phi(v, y_0) \leq \phi(v, x_0)\},\\
&H_n = \{v \in H_{n-1} \cap W_{n-1} : \phi(v, z_n) \leq \phi(v, y_n) \leq \phi(v, x_n)\},\\
&W_0 = E,\\
&W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle x_n - v, Jx_0 - Jx_n \rangle \geq 0\},\\
&x_{n+1} = \prod_{H_n \cap W_n} (x_0), \quad n \geq 1
\end{aligned} \tag{12}$$

converges strongly to $\Pi_{A^{-1}0\cap F(T)}(x_0)$, where $\Pi_{A^{-1}0\cap F(T)}$ is the generalized projection from E onto $A^{-1}0 \cap F(T)$.

In this paper, motivated by Alber [2], Ibaraki and Takahashi [6], and Zegeye and Shahzad [7], we first introduce the generalized resolvent and discuss its properties. Secondly, we give an iterative scheme for finding a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping. Finally, we show its convergence.

2. The Generalized Resolvent J_λ^* and Some of Its Properties

Let E^* be a reflexive and smooth Banach space and let $B \subset E \times E^*$ be a maximal monotone operator. For each $\lambda > 0$ and $x \in E$, consider the set:

$$J_\lambda^* x^* := \left\{z^* \in E^* : x^* \in z^* + \lambda BJ^{-1}(z^*)\right\}. \tag{13}$$

If $z_1^* + \lambda w_1^* = x^*$, $z_2^* + \lambda w_2^* = x^*$, $w_1^* \in BJ^{-1}(z_1^*)$, $w_2^* \in BJ^{-1}(z_2^*)$, then we have from the monotonicity of B that

$$\left\langle w_1^* - w_2^*, J^{-1}(z_1^*) - J^{-1}(z_2^*) \right\rangle \geq 0, \tag{14}$$

and hence

$$\left\langle \frac{x^* - z_1^*}{\lambda} - \frac{x^* - z_2^*}{\lambda}, J^{-1}(z_1^*) - J^{-1}(z_2^*) \right\rangle \geq 0. \tag{15}$$

So, we obtain

$$\left\langle x^* - z_1^* - (x^* - z_2^*), J^{-1}(z_1^*) - J^{-1}(z_2^*) \right\rangle \geq 0, \tag{16}$$

and hence

$$\left\langle z_2^* - z_1^*, J^{-1}(z_1^*) - J^{-1}(z_2^*) \right\rangle \geq 0. \tag{17}$$

This implies $z_1^* = z_2^*$. Then, $J_\lambda^* x^*$ consists of one point. We also denote the domain and the range of $J_\lambda^* x^*$ by $D(J_\lambda^*) = R(I^* + \lambda BJ^{-1})$ and $R(J_\lambda^*) = D(BJ^{-1})$, respectively, where I^* is the identity on E^*. Such a $J_\lambda^* : E^* \to E^*$ is called the generalized resolvent of B and is denoted by

$$J_\lambda^* = \left(I^* + \lambda BJ^{-1}\right)^{-1}. \quad (18)$$

We get some properties of J_λ^* and $(BJ^{-1})^{-1}0$.

Proposition 5. *Let E^* be a reflexive and strictly convex Banach space with a Fréchet differentiable norm and let $B \subset E \times E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then, the following hold:*

(1) *$D(J_\lambda^*) = E^*$ for each $\lambda > 0$;*

(2) *$(BJ^{-1})^{-1}0 = F(J_\lambda^*)$ for each $\lambda > 0$, where $F(J_\lambda^*)$ is the set of fixed points of J_λ^*;*

(3) *$(BJ^{-1})^{-1}0$ is closed;*

(4) *$J_\lambda^* : E^* \to E^*$ is generalized nonexpansive for each $\lambda > 0$.*

Proof. (1) From the maximality of B, we have

$$R(J + \lambda B) = E^*, \quad \forall \lambda > 0. \quad (19)$$

Hence, for each $x^* \in E^*$, there exists $x \in E$ such that $x^* \in Jx + \lambda Bx$. Since E is reflexive and strictly convex, J is bijective. Therefore, there exists $z^* \in E^*$ such that $x = J^{-1}(z^*)$. Therefore, we have

$$\begin{aligned} x^* &\in JJ^{-1}(z^*) + \lambda BJ^{-1}(z^*) \\ &= z^* + \lambda BJ^{-1}(z^*) \subset R\left(I^* + \lambda BJ^{-1}\right) = D\left(J_\lambda^*\right). \end{aligned} \quad (20)$$

This implies $E^* \subset D(J_\lambda^*)$. $D(J_\lambda^*) \subset E^*$ is clear. So, we have $D(J_\lambda^*) = E^*$.

(2) Let $\lambda > 0$. Then, we have

$$\begin{aligned} x^* \in F(J_\lambda) &\Longleftrightarrow J_\lambda^* x^* = x^* \Longleftrightarrow x^* \in x^* + \lambda BJ^{-1}(x^*) \\ &\Longleftrightarrow 0 \in \lambda BJ^{-1}(x^*) \Longleftrightarrow 0 \in BJ^{-1}(x^*) \\ &\Longleftrightarrow x^* \in \left(BJ^{-1}\right)^{-1} 0. \end{aligned} \quad (21)$$

(3) Let $\{x_n^*\} \subset (BJ^{-1})^{-1}0$ with $x_n^* \to x^*$. From $x_n^* \in (BJ^{-1})^{-1}0$, we have $J^{-1}(x_n^*) \in B^{-1}0$. Since J^{-1} is norm to norm continuous and $B^{-1}0$ is closed, we have that $J^{-1}(x_n^*) \to J^{-1}(x^*) \in B^{-1}0$. This implies $x^* \in (BJ^{-1})^{-1}0$. That is, $(BJ^{-1})^{-1}0$ is closed.

(4) Let $x^* \in E^*$, $y^* \in E^*$, $z^* \in E^*$, and $\lambda > 0$. By Definition (2) and calculating that

$$\begin{aligned} &V\left(x^*, J^{-1}z^*\right) + V\left(z^*, J^{-1}y^*\right) \\ &\quad = \|x^*\|^2 + \|z^*\|^2 - 2\left\langle x^*, J^{-1}z^*\right\rangle \\ &\qquad + \|y^*\|^2 + \|z^*\|^2 - 2\left\langle z^*, J^{-1}y^*\right\rangle \\ &\quad = V\left(x^*, J^{-1}y^*\right) + 2\left\langle z^* - x^*, J^{-1}z^* - J^{-1}y^*\right\rangle, \end{aligned} \quad (22)$$

we have that

$$\begin{aligned} V\left(x^*, J^{-1}y^*\right) &= V\left(x^*, J^{-1}z^*\right) + V\left(z^*, J^{-1}y^*\right) \\ &\quad + 2\left\langle x^* - z^*, J^{-1}z^* - J^{-1}y^*\right\rangle. \end{aligned} \quad (23)$$

Let $x^* \in E^*$, $y^* \in F(J_\lambda)$, and $\lambda > 0$. From the above formula, we have

$$\begin{aligned} V\left(x^*, J^{-1}y^*\right) &= V\left(x^*, J^{-1}J_\lambda^* x^*\right) + V\left(J_\lambda^* x^*, J^{-1}y^*\right) \\ &\quad + 2\left\langle x^* - J_\lambda^* x^*, J^{-1}J_\lambda x^* - J^{-1}y^*\right\rangle. \end{aligned} \quad (24)$$

Since $((x^* - J_\lambda^* x^*)/\lambda) \in BJ^{-1}(J_\lambda^* x^*)$ and $0 \in BJ^{-1}(y^*)$, we have

$$\left\langle x^* - J_\lambda^* x^*, J^{-1}J_\lambda^* x^* - J^{-1}y^*\right\rangle \geq 0. \quad (25)$$

Therefore, we get

$$\begin{aligned} V\left(x^*, J^{-1}y^*\right) &\geq V\left(x^*, J^{-1}J_\lambda^* x^*\right) + V\left(J_\lambda^* x^*, J^{-1}y^*\right) \\ &\geq V\left(J_\lambda^* x^*, J^{-1}y^*\right). \end{aligned} \quad (26)$$

That is, J_λ^* is generalized nonexpansive on E^*. □

Theorem 6 (see [8]). *Let E be a Banach space and let $A \subset E \times E^*$ be a maximal monotone operator with $A^{-1}0 \neq \emptyset$. If E^* is strictly convex and has a Fréchet differentiable norm, then, for each $x \in E$, $\lim_{\lambda \to \infty}(J + \lambda A)^{-1}J(x)$ exists and belongs to $A^{-1}0$.*

Using Theorem 6, we get the following result.

Theorem 7. *Let E^* be a uniformly convex Banach space with a Fréchet differentiable norm and let $B \subset E \times E^*$ be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the following hold:*

(1) *for each $x^* \in E^*$, $\lim_{\lambda \to \infty} J_\lambda^* x^*$ exists and belongs to $(BJ^{-1})^{-1}0$;*

(2) *if $R^* x^* := \lim_{\lambda \to \infty} J_\lambda^* x^*$ for each $x^* \in E^*$, then R^* is a sunny generalized nonexpansive retraction of E^* onto $(BJ^{-1})^{-1}0$.*

Proof. (1) By defining a mapping Q_λ from E to E by

$$Q_\lambda x := \left(I + \lambda J^{-1}B\right)x, \quad \forall x \in E, \ \lambda > 0, \quad (27)$$

we have, for all $x^* \in E^*, \lambda > 0$, $J_\lambda^* x^* = JQ_\lambda J^{-1}(x^*)$. In fact, define

$$x_\lambda^* := JQ_\lambda J^{-1}(x^*) = \left[J\left(I + \lambda J^{-1}B\right)J^{-1}\right]^{-1}(x^*). \quad (28)$$

Then, we have

$$x^* \in J\left(I + \lambda J^{-1}B\right)J^{-1}(x_\lambda^*) = \left(I^* + \lambda BJ^{-1}\right)x_\lambda^*, \quad (29)$$

and hence $x_\lambda^* = J_\lambda^* x^*$. From Theorem 6, we get

$$\lim_{\lambda \to \infty} Q_\lambda J^{-1}(x^*) = u \in B^{-1}0. \quad (30)$$

If E^* is uniformly convex, then E has a Fréchet differentiable norm. So, J is norm to norm continuous. Since $B^{-1}0$ is closed, we have

$$\lim_{\lambda \to \infty} J_\lambda^* x^* = \lim_{\lambda \to \infty} JQ_\lambda J^{-1}(x^*) = Ju \in JB^{-1}0 = \left(BJ^{-1}\right)^{-1}0. \quad (31)$$

(2) We define a mapping R^* from E^* to E^* by

$$R^* x^* := \lim_{\lambda \to \infty} J_\lambda^* x^*, \quad \forall x^* \in E^*. \tag{32}$$

Let $u^* \in (BJ^{-1})^{-1}0 = F(J_\lambda^* x^*)$. Then, $R^* u^* = \lim_{\lambda\to\infty} J_\lambda^* u^* = \lim_{\lambda\to\infty} u^* = u^*$. Therefore, R^* is a retraction of E^* onto $(BJ^{-1})^{-1}0$. Since $x^* \in J_\lambda^* x^* + \lambda BJ^{-1}(J_\lambda^* x^*)$, we have

$$\left\langle \frac{x^* - J_\lambda^* x^*}{\lambda}, J^{-1}\left(J_\lambda^* x^*\right) - J^{-1}\left(z^*\right) \right\rangle \geq 0, \quad \forall z^* \in \left(BJ^{-1}\right)^{-1} 0, \tag{33}$$

and hence

$$\left\langle x^* - J_\lambda^* x^*, J^{-1}\left(J_\lambda^* x^*\right) - J^{-1}\left(z^*\right) \right\rangle \geq 0. \tag{34}$$

Letting $\lambda \to 0$, we get

$$\left\langle x^* - R^* x^*, J^{-1}\left(R^* x^*\right) - J^{-1}\left(z^*\right) \right\rangle \geq 0, \quad \forall z^* \in \left(BJ^{-1}\right)^{-1} 0. \tag{35}$$

From Proposition 5, R^* is sunny and generalized nonexpansive. This implies that R^* is a sunny generalized nonexpansive retraction of E^* onto $(BJ^{-1})^{-1}0$. □

3. An Iterative Scheme for Finding a Zero Point of a Monotone Mapping by J_λ^*

Now we construct an iterative scheme which converges strongly to a point which is a fixed point of relatively weak nonexpansive mapping and a zero of monotone mapping.

Theorem 8. *Let E^* be a uniformly convex Banach space and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator. Let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a relatively weak nonexpansive mapping with $A^{-1}0 \cap F(T) \neq \emptyset$. Assume that $0 \leq \alpha_n < a < 1$ is a sequence of real numbers. Then, the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \qquad \lambda_n \longrightarrow +\infty, \\ y_n = J^{-1}\left(\alpha_n J x_n + (1-\alpha_n) J_{\lambda_n}^* J x_n\right), \\ J_{\lambda_n}^* = \left(I^* + \lambda_n A J^{-1}\right)^{-1}, \\ z_n = T y_n, \\ H_0 = \left\{v \in C : V_2\left(v, z_0\right) \leq V_2\left(v, y_0\right) \leq V_2\left(v, x_0\right)\right\}, \\ H_n = \left\{v \in H_{n-1} \cap W_{n-1} : V_2\left(v, z_n\right) \leq V_2\left(v, y_n\right) \leq V_2\left(v, x_n\right)\right\}, \\ W_0 = C, \\ W_n = \left\{v \in H_{n-1} \cap W_{n-1} : \left\langle v - x_n, J x_0 - J x_n \right\rangle \leq 0\right\}, \\ x_{n+1} = \prod_{H_n \cap W_n} \left(x_0\right), \quad n \geq 1 \end{gathered} \tag{36}$$

converges strongly to $\Pi_{A^{-1}0\cap F(T)}(x_0)$, where $\Pi_{A^{-1}0\cap F(T)}$ is the generalized projection from E onto $A^{-1}0 \cap F(T)$.

Proof. We first show that H_n and W_n are closed and convex for each $n \geq 0$. From the definition of H_n and W_n, it is obvious that H_n is closed and W_n is closed and convex for each $n \geq 0$. We show that H_n is convex. Since

$$\begin{aligned} H_n = &\left\{v \in H_{n-1} \cap W_{n-1} : V_2\left(v, z_n\right) \leq V_2\left(v, y_n\right)\right\} \\ &\cap \left\{v \in H_{n-1} \cap W_{n-1} : V_2\left(v, y_n\right) \leq V_2\left(v, x_n\right)\right\}, \end{aligned} \tag{37}$$

$V_2(v, y_n) \leq V_2(v, x_n)$ is equivalent to

$$2\left\langle v, Jx_n - Jy_n \right\rangle + \left\|y_n\right\|^2 + \left\|x_n\right\|^2 \leq 0, \tag{38}$$

and $V_2(v, z_n) \leq V_2(v, y_n)$ is equivalent to

$$2\left\langle v, Jy_n - Jz_n \right\rangle + \left\|z_n\right\|^2 + \left\|x_n\right\|^2 \leq 0, \tag{39}$$

it follows that H_n is convex.

Next, we show that $F =: A^{-1}0 \cap F(T) \subset H_n \cap W_n$ for each $n \geq 0$. Let $p \in F$; then relatively weak nonexpansiveness of T and generalized nonexpansiveness of J_λ^* give that

$$\begin{aligned} V_2\left(p, z_0\right) &= V_2\left(p, Ty_0\right) \leq V_2\left(p, y_0\right) \\ &= V_2\left(p, J^{-1}\left(\alpha_0 J x_0 + (1-\alpha_0) J_{\lambda_0}^* J x_0\right)\right) \\ &= \|p\|^2 + \left\|\alpha_0 J x_0 + (1-\alpha_0) J_{\lambda_0}^* J x_0\right\|^2 \\ &\quad - 2\left\langle p, \alpha_0 J x_0 + (1-\alpha_0) J_{\lambda_0}^* J x_0 \right\rangle \\ &\leq \|p\|^2 - 2\alpha_0 \left\langle p, Jx_0 \right\rangle - 2(1-\alpha_0)\left\langle p, J_{\lambda_0}^* J x_0 \right\rangle \\ &\quad + \alpha_0 \left\|Jx_0\right\|^2 + (1-\alpha_0)\left\|J_{\lambda_0}^* J x_0\right\|^2 \\ &= \alpha_0\left(\|p\|^2 - 2\alpha_0 \left\langle p, Jx_0 \right\rangle + \left\|x_0\right\|^2\right) \\ &\quad + (1-\alpha_0)\left(\|p\|^2 - 2\left\langle p, J_{\lambda_0}^* J x_0 \right\rangle + \left\|J_{\lambda_0}^* J x_0\right\|^2\right) \\ &= \alpha_0 V_2\left(p, x_0\right) + (1-\alpha_0) V_2\left(p, J^{-1} J_{\lambda_0}^* J x_0\right) \\ &= \alpha_0 V_2\left(p, x_0\right) + (1-\alpha_0) V\left(p, J_{\lambda_0}^* J x_0\right) \\ &\leq \alpha_0 V_2\left(p, x_0\right) + (1-\alpha_0) V\left(p, J x_0\right) \\ &\leq \alpha_0 V_2\left(p, x_0\right) + (1-\alpha_0) V_2\left(p, x_0\right) = V_2\left(p, x_0\right). \end{aligned} \tag{40}$$

Thus, we give that $p \in H_0$. On the other hand, it is clear that $p \in C$. Thus, $F \subset H_0 \cap W_0$ and, therefore, $x_1 = \Pi_{H_0 \cap W_0}$ is well defined. Suppose that $F \subset H_{n-1} \cap W_{n-1}$ and $\{x_n\}$ is well defined. Then, the methods in (40) imply that $V_2(p, z_n) \leq V_2(p, y_n) \leq V_2(p, x_n)$ and $p \in H_n$. Moreover, it follows from Lemma 3 that

$$\left\langle p - x_n, Jx_n - Jx_0 \right\rangle \geq 0, \tag{41}$$

which implies that $p \in W_n$. Hence $F \subset H_n \cap W_n$ and $x_{n+1} = \Pi_{H_n \cap W_n}$ is well defined. Then, by induction, $F \subset H_n \cap W_n$ and the sequence generated by (36) is well defined for each $n \geq 0$.

Now, we show that $\{x_n\}$ is a bounded sequence and converges to a point of F. Let $p \in F$. Since $x_{n+1} = \Pi_{H_n \cap W_n}(x_0)$ and $H_n \cap W_n \subset H_{n-1} \cap W_{n-1}$ for all $n \geq 1$, we have

$$V_2(x_n, x_0) \leq V_2(x_{n+1}, x_0) \tag{42}$$

for all $n \geq 0$. Therefore, $\{V_2(x_n, x_0)\}$ is nondecreasing. In addition, it follows from definition of W_n and Lemma 3 that $x_n = \Pi_{W_n}(x_0)$. Therefore, by Lemma 2 we have

$$\begin{aligned} V_2(x_n, x_0) &= V_2\left(\prod_{W_n}(x_0), x_0\right) \\ &\leq V_2(p, x_0) - V_2(p, x_n) \leq V_2(p, x_0), \end{aligned} \tag{43}$$

for each $p \in F(T) \subset W_n$ for all $n \geq 0$. Therefore, $\{V_2(x_n, x_0)\}$ is bounded. This together with (40) implies that the limit of $\{V_2(x_n, x_0)\}$ exists. Put $\lim_{n\to\infty} V_2(x_n, x_0) = d$. From Lemma 2, we have, for any positive integer m, that

$$\begin{aligned} V_2(x_{n+m}, x_n) &= V_2\left(x_{n+m}, \prod_{W_n}(x_0)\right) \leq V_2(x_{n+m}, x_0) \\ &\quad - V_2\left(\prod_{W_n}(x_0), x_0\right) \\ &= V_2(x_{n+m}, x_0) - V_2(x_n, x_0), \end{aligned} \tag{44}$$

for all $n \geq 0$. The existence of $\lim_{n\to\infty} V_2(x_n, x_0)$ implies that $\lim_{n\to\infty} V_2(x_{m+n}, x_n) = 0$. Thus, Lemma 4 implies that

$$x_{m+n} - x_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{45}$$

and hence $\{x_n\}$ is a Cauchy sequence. Therefore, there exists a point $q \in E$ such that $x_n \to q$ as $n \to \infty$. Since $x_{n+1} \in H_n$, we have $V_2(x_{n+1}, z_n) \leq V_2(x_{n+1}, y_n) \leq V_2(x_{n+1}, x_n)$. Thus by Lemma 4 and (45) we get that

$$x_{n+1} - z_n \longrightarrow 0, \qquad x_{n+1} - y_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{46}$$

and hence $\|x_n - y_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - y_n\| \to 0$ as $n \to \infty$. Furthermore, since J is uniformly continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = \lim_{n\to\infty} \|Jx_n - Jy_n\| = 0, \tag{47}$$

which implies that

$$\|Jx_{n+1} - JTy_n\| \longrightarrow \quad \text{as } n \longrightarrow \infty. \tag{48}$$

Since J^{-1} is also uniformly norm-continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - Ty_n\| = \lim_{n\to\infty} \left\|J^{-1}Jx_{n+1} - J^{-1}JTy_n\right\| = 0. \tag{49}$$

Therefore, from (46), (49), and $\|y_n - Ty_n\| \leq \|x_{n+1} - Ty_n\| + \|x_n - y_n\|$, we obtain that $\lim_{n\to\infty}\|y_n - Ty_n\| = 0$. This together with the fact that $\{x_n\}$ (and hence $\{y_n\}$) converges strongly to $q \in E$ and the definition of relatively weak nonexpansive mapping implies that $q \in F(T)$. Furthermore, from (36) and (47), we have that $(1-\alpha_n)\|J^*_{\lambda_n} Jx_n - Jx_n\| = \|Jx_n - Jy_n\| \to 0$ as $n \to \infty$. Thus, from $\lim_{n\to\infty} J^*_{\lambda_n} Jx_n = \lim_{n\to\infty} Jx_n = Jq \in JA^{-1}0 = (AJ^{-1})^{-1}0$, we obtain that $q \in A^{-1}0$.

Finally, we show that $q = \Pi_{A^{-1}0\cap F(T)}(x_0)$ as $n \to \infty$. From Lemma 2, we have

$$V_2\left(q, \prod_{A^{-1}0\cap F(T)}(x_0)\right) + V_2\left(\prod_{A^{-1}0\cap F(T)}(x_0), x_0\right) \leq V_2(q, x_0). \tag{50}$$

On the other hand, since $x_{n+1} = \Pi_{H_n\cap W_n}(x_0)$ and $F \subset H_n \cap W_n$ for all $n \geq 0$, we have by Lemma 2 that

$$\begin{aligned} V_2&\left(\prod_{A^{-1}0\cap F(T)}(x_0), x_{n+1}\right) + V_2(x_{n+1}, x_0) \\ &\leq V_2\left(\prod_{A^{-1}0\cap F(T)}(x_0), x_0\right). \end{aligned} \tag{51}$$

Moreover, by the definition of $V_2(x, y)$, we get that

$$\lim_{n\to\infty} V_2(x_{n+1}, x_0) = V_2(q, x_0). \tag{52}$$

By combining (50) and (52), we obtain that $V_2(q, x_0) = V_2(\Pi_{A^{-1}0\cap F(T)}(x_0), x_0)$. Therefore, it follows from the uniqueness of $\Pi_{A^{-1}0\cap F(T)}(x_0)$ that $q = \Pi_{A^{-1}0\cap F(T)}(x_0)$. This completes the proof. □

Remark 9. If in Theorem 8 we have that $T = I$, the identity map on E, then we get the following.

Corollary 10. *Let E^* be a uniformly convex Banach space and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator. Let C be a nonempty closed convex subset of E with $A^{-1}0 \neq \emptyset$. Assume that $0 \leq \alpha_n < a < 1$ is a sequence of real numbers. Then, the sequence $\{x_n\}$ generated by*

$$\begin{gathered} x_0 \in C, \qquad \lambda_n \longrightarrow +\infty, \\ y_n = J^{-1}\left(\alpha_n Jx_n + (1-\alpha_n) J^*_{\lambda_n} Jx_n\right), \quad J^*_{\lambda_n} = \left(I^* + \lambda_n AJ^{-1}\right)^{-1}, \\ H_0 = \{v \in C : V_2(v, z_0) \leq V_2(v, y_0) \leq V_2(v, x_0)\}, \\ H_n = \{v \in H_{n-1} \cap W_{n-1} : V_2(v, z_n) \leq V_2(v, y_n) \leq V_2(v, x_n)\}, \\ W_0 = C, \\ W_n = \{v \in H_{n-1} \cap W_{n-1} : \langle v - x_n, Jx_0 - Jx_n\rangle \leq 0\}, \\ x_{n+1} = \prod_{H_n \cap W_n}(x_0), \quad n \geq 1 \end{gathered} \tag{53}$$

converges strongly to $\Pi_{A^{-1}0}$, where $\Pi_{A^{-1}0}$ is the generalized projection from E onto $A^{-1}0$.

Remark 11. We have compared the results of [2, 6, 7] with the result in this paper.

(1) In [6], Ibaraki and Takahashi introduced the generalized resolvent $J_\lambda : E \to E$, which was denoted by

$$J_\lambda = (I + \lambda BJ)^{-1}. \quad (54)$$

In this paper, we introduce the generalized resolvent $J_\lambda^* : E^* \to E^*$, which is denoted by

$$J_\lambda^* = \left(I^* + \lambda BJ^{-1}\right)^{-1}. \quad (55)$$

(2) In [6], Ibaraki and Takahashi defined a sunny generalized nonexpansive retraction R_C of E onto $BJ^{-1}0$:

$$Rx := \lim_{\lambda \to \infty} J_\lambda x, \quad \forall x \in E. \quad (56)$$

In this paper, we define a sunny generalized nonexpansive retraction R^* of E^* onto $(BJ^{-1})^{-1}0$:

$$R^* x^* := \lim_{\lambda \to \infty} J_\lambda^* x^*, \quad \forall x \in E^*. \quad (57)$$

(3) In [7], Zegeye and Shahzad proved the strong convergence theorem of the sequence $\{x_n\}$ generated by (12). Using J_λ^*, in this paper, we construct an iterative scheme in E^*, which converges strongly to a point which is a fixed point of a relatively weak nonexpansive mapping and a zero of a monotone mapping.

The results we have obtained in this paper are studied in E^*, which is different from others.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] I. Cioranescu, *Geometry of Banach Spaces, Duality Mapping and Nonlinear Problems*, Klumer Academic Publishers, Amsterdam, The Netherlands, 1990.

[2] Ya. Alber, "Metric and generalized projection operators in Banach spaces: prop- erties and applications," in *Theory and Applications of Nonlinear Operators of Monotonic and Accretive Type*, A. Kartsatos, Ed., pp. 15–50, Marcel Dekker, New York, NY, USA, 1996.

[3] D. Pascali, *Sburlan, Nonlinear Mappings of Monotone Type*, Editura Academiae, Bucaresti, Romania, 1978.

[4] J. Li, "On the existence of solutions of variational inequalities in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 295, no. 1, pp. 115–126, 2004.

[5] Y. Su and X. Qin, "Monotone CQ iteration processes for nonexpansive semigroups and maximal monotone operators," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 68, no. 12, pp. 3657–3664, 2008.

[6] T. Ibaraki and W. Takahashi, "A new projection and convergence theorems for the projections in Banach spaces," *Journal of Approximation Theory*, vol. 149, no. 1, pp. 1–14, 2007.

[7] H. Zegeye and N. Shahzad, "Strong convergence theorems for monotone mappings and relatively weak nonexpansive mappings," *Nonlinear Analysis: Theory, Methods and Applications*, vol. 70, no. 7, pp. 2707–2716, 2009.

[8] S. Reich, "Constructive techniques for accretive and monotone operators," in *Applied Nonlinear Analysis*, V. Lakshmikan, Ed., pp. 335–345, Academic Press, NewYork, NY, USA, 1979.