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Research Article

Fixed Points of Difference Operator of Meromorphic Functions

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Let f be a transcendental meromorphic function of order less than one. The authors prove that the exact difference $\Delta f = f(z+1) - f(z)$ has infinitely many fixed points, if $a \in \mathbb{C}$ and ∞ are Borel exceptional values (or Nevanlinna deficiency values) of f. These results extend the related results obtained by Chen and Shon.

1. Introduction and Main Results

In this paper, we assume that the reader is familiar with the notations of frequency use in Nevanlinna theory (see [1–3]). Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}$. We use the notations $\sigma(f)$ to denote the order of $f(z)$, $\lambda(f,a)$, and $\lambda(1/f)$, respectively, to denote the exponent of convergence of zeros of $f(z) - a$ and poles of $f(z)$. Especially, if $a = 0$, we denote $\lambda(f,0) = \lambda(f)$. A point $z \in \mathbb{C}$ is called as a fixed point of $f(z)$ if $f(z) = z$. There is a considerable number of results on the fixed points for meromorphic functions in the plane; we refer the reader to Chuang and Yang [4]. It follows Chen and Shon [5]; we use the notation $\tau(f)$ to denote the exponent of convergence of fixed points of f that is defined as

$$\tau(f) = \limsup_{r\to\infty} \frac{\log N(r, 1/(f-z))}{\log r}. \quad (1)$$

Let f be a transcendental meromorphic function in the complex plane $\mathbb{C}$. The exact differences Δf are defined by $\Delta f = f(z+1) - f(z)$.

Recently, there are a number of papers (including [6–16]) focusing on the differences analogues of Nevanlinna's theory and its application on the complex difference equations. For the fixed points of the difference operator Δf, Chen and Shon have proved the following.

Theorem A (see [17]). *Let f be a transcendental entire function of order of growth $\sigma(f) = 1$ and have infinitely many zeros with the exponent of convergence of zeros $\lambda(f) < 1$. Then Δf has infinitely many zeros and infinitely many fixed points.*

When the order of f is less than 1, Chen and Shon have proved the following.

Theorem B (see [5]). *Let f be a transcendental meromorphic function of order of growth $\sigma(f) \le 1$. Suppose that f satisfies $\lambda(1/f) < \lambda(f) < 1$ or has infinitely many zeros (with $\lambda(f) = 0$) and finitely many poles. Then Δf has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

A natural question is, letting f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$, is there a similar result as that in Theorem B if $\lambda(1/f) \ge \lambda(f)$ or f has infinitely many zeros (with $\lambda(f) = 0$) and infinitely many poles?

In this paper, we will prove the following theorem to answer the question.

Theorem 1 (main). *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$ and $a \in \mathbb{C}$. Suppose that f satisfies $\lambda(1/f) < \sigma(f)$ and $\lambda(f,a) < \sigma(f)$. Then Δf has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

From Theorem 1, we can get the following corollary.

Corollary 2. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$. Suppose that f satisfies $\lambda(f) \le \lambda(1/f) < \sigma(f)$. Then Δf has infinitely many fixed points and satisfies the exponent of convergence of fixed points $\tau(\Delta f) = \sigma(f)$.*

In Theorem 1, we suppose that f satisfies $\lambda(1/f) < \sigma(f)$ and $\lambda(f,a) < \sigma(f)$. That is to say ∞ and a are Borel exceptional values of f. If we suppose that ∞ and a are Nevanlinna deficiency values of f, is there a similar result as that in Theorem B? In the following, we give Theorem 3 to answer this question.

Let $f(z)$ be a meromorphic function in the complex plane $\mathbb{C}$ and $a \in \mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. Nevanlinna's deficiency of f with respect to a is defined by

$$\delta(a,f) = 1 - \limsup_{r\to\infty} \frac{N(r, 1/(f-a))}{T(r,f)}. \tag{2}$$

If $a = \infty$, then one should replace $N(r, 1/(f-a))$ in the above formula by $N(r,f)$. If $\delta(a,f) > 0$, then a is called a Nevanlinna deficiency value of f.

Theorem 3 (main). *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$ and $a \in \mathbb{C}$. Suppose that f satisfies $\delta(\infty, f) = 1$ and a is a Nevanlinna deficiency value of f. Then Δf has infinitely many fixed points.*

Corollary 4. *Let f be a transcendental entire function of order of growth $\sigma(f) < 1$ and $a \in \mathbb{C}$. Suppose that $\delta(a,f) > 0$. Then Δf has infinitely many fixed points.*

2. Some Lemmas

Lemma 1 (lemma on the logarithmic derivative). *Let $f(z)$ be a meromorphic function. If the function $f(z)$ has finite order, then*

$$m\left(r, \frac{f^{(k)}}{f}\right) = O(\log r) \tag{3}$$

holds for any positive integer k.

Lemma 2 (see [18]). *Let $f(z)$ be a meromorphic function with the exponent of convergence of poles $\lambda(1/f) = \lambda < +\infty$ and let c be a nonzero complex number. Then for each $\varepsilon > 0$, we have*

$$N(r, f(z+c)) = N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r). \tag{4}$$

Lemma 3. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$ and let c be a nonzero complex number. Then*

$$N(r, f(z+c)) = N(r,f) + O(\log r). \tag{5}$$

Proof. Since the order $\sigma(f) := \sigma < 1$, then $\lambda(1/f) = \lambda \le \sigma < 1$. Therefore, for any $0 < \varepsilon < 1-\sigma$, it follows from Lemma 2 that

$$\begin{aligned} N(r, f(z+c)) &= N(r,f) + O\left(r^{\lambda-1+\varepsilon}\right) + O(\log r) \\ &= N(r,f) + O(1) + O(\log r). \end{aligned} \tag{6}$$

That is,

$$N(r, f(z+c)) = N(r,f) + O(\log r). \tag{7}$$

□

Lemma 4 (see [6]). *Let f be a function transcendental and meromorphic in the plane which satisfies*

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0. \tag{8}$$

Then Δf is transcendental.

Lemma 5. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) = \sigma < 1$. Then Δf is transcendental.*

Proof. Since the order $\sigma(f) := \sigma < 1$, then, for any positive $\varepsilon (0 < \varepsilon < 1-\sigma)$, there exists $R > 0$ such that for any $r > R$ we have

$$T(r,f) \le r^{\sigma+\varepsilon}. \tag{9}$$

Therefore,

$$\liminf_{r\to\infty} \frac{T(r,f)}{r} = 0. \tag{10}$$

Lemma 5 follows Lemma 4. □

Lemma 6 (see [7]). *Let $f(z)$ be a meromorphic function of finite order, then $\sigma(\Delta f) \le \sigma(f)$.*

Lemma 7 (see [7]). *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$. Then for any $\varepsilon > 0$ and any positive integer k, there exists a set $E \subset (1,\infty)$ that depends on f and has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E \cup [0,1]$ we have*

$$\frac{\Delta^k f(z)}{f(z)} = \frac{f^{(k)}(z)}{f(z)} + O\left(r^{(k+1)(\sigma-1)+\varepsilon}\right). \tag{11}$$

It is easy to derive the following lemma from Lemma 1 and Lemma 7.

Lemma 8. *Let f be a transcendental meromorphic function of order of growth $\sigma(f) < 1$. Then for any positive integer k there exists a set $E \subset (1,\infty)$ that depends on f and has finite logarithmic measure, such that*

$$m\left(r, \frac{\Delta^k f(z)}{f(z)}\right) = O(\log r), \quad r \notin E. \tag{12}$$

3. Proof of Theorems

Proof. Since

$$\frac{1}{f}=\frac{\Delta f}{zf}-\frac{z\Delta^2 f-\Delta f}{zf}\frac{\Delta f-z}{z\Delta^2 f-\Delta f}, \tag{13}$$

then

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &\le m\left(r,\frac{\Delta f}{zf}\right)+m\left(r,\frac{z\Delta^2 f-\Delta f}{zf}\right) \\ &\quad + m\left(r,\frac{\Delta f-z}{z\Delta^2 f-\Delta f}\right)+O(1) \\ &\le 2m\left(r,\frac{\Delta f}{f}\right)+m\left(r,\frac{\Delta^2 f}{f}\right) \\ &\quad + m\left(r,\frac{\Delta f-z}{z\Delta^2 f-\Delta f}\right)+O(\log r). \end{aligned} \tag{14}$$

Applying the first fundamental theorem, we get

$$\begin{aligned} m\left(r,\frac{1}{f}\right) &= T(r,f)-N\left(r,\frac{1}{f}\right)+O(1), \\ m\left(r,\frac{\Delta f-z}{z\Delta^2 f-\Delta f}\right) &= m\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right) \\ &\quad + N\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right) \\ &\quad - N\left(r,\frac{\Delta f-z}{z\Delta^2 f-\Delta f}\right)+O(1) \\ &\le m\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right) \\ &\quad + N\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right)+O(1). \end{aligned} \tag{15}$$

Combining (14)-(15) we have

$$\begin{aligned} T(r,f) &\le N\left(r,\frac{1}{f}\right)+2m\left(r,\frac{\Delta f}{f}\right)+m\left(r,\frac{\Delta^2 f}{f}\right) \\ &\quad + m\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right) \\ &\quad + N\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right)+O(\log r) \\ &\le N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Delta f-z}\right)+N\left(r,z\Delta^2 f-\Delta f\right) \\ &\quad + 2m\left(r,\frac{\Delta f}{f}\right) \\ &\quad + m\left(r,\frac{\Delta^2 f}{f}\right)+m\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right)+O(\log r). \end{aligned} \tag{16}$$

Since

$$\begin{aligned} \Delta^2 f &= \Delta\left(f(z+1)-f(z)\right) \\ &= f(z+2)-2f(z+1)+f(z), \\ \Delta(\Delta f-z) &= \Delta\left(f(z+1)-f(z)-z\right) \\ &= f(z+2)-2f(z+1)+f(z)-1, \end{aligned} \tag{17}$$

then, we can get

$$\begin{aligned} z\Delta^2 f-\Delta f &= zf(z+2)-2zf(z+1)+zf(z) \\ &\quad - f(z+1)+f(z). \\ z\Delta(\Delta f-z)-(\Delta f-z) &= zf(z+2)-2zf(z+1) \\ &\quad + zf(z)-f(z+1)+f(z). \end{aligned} \tag{18}$$

Therefore,

$$\begin{aligned} \frac{z\Delta^2 f-\Delta f}{\Delta f-z} &= \frac{z\Delta(\Delta f-z)-(\Delta f-z)}{\Delta f-z} \\ &= \frac{z\Delta(\Delta f-z)}{\Delta f-z}-1, \end{aligned} \tag{19}$$

$$\begin{aligned} N\left(r,z\Delta^2 f-\Delta f\right) &\le N(r,f(z+2))+N(r,f(z+1)) \\ &\quad + N(r,f(z)). \end{aligned} \tag{20}$$

Thus from Lemma 3 and (20), we deduce

$$N\left(r,z\Delta^2 f-\Delta f\right)\le 3N(r,f(z))+O(\log r). \tag{21}$$

By Lemmas 5 and 6, we know that $\Delta f-z$ is a transcendental meromorphic function of order of growth $\sigma(\Delta f-z)\le\sigma(f)<1$. It follows from Lemma 8 and (19) that there exists a set $E\subset(1,\infty)$ that has finite logarithmic measure, such that for any $r\notin E$ we have

$$\begin{aligned} m\left(r,\frac{\Delta f}{f}\right) &= O(\log r), \\ m\left(r,\frac{\Delta^2 f}{f}\right) &= O(\log r), \\ m\left(r,\frac{z\Delta^2 f-\Delta f}{\Delta f-z}\right) &= O(\log r). \end{aligned} \tag{22}$$

From (16) and (21)-(22), we have

$$\begin{aligned} T(r,f) &\le 3N(r,f)+N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\Delta f-z}\right) \\ &\quad + O(\log r), \quad r\notin E. \end{aligned} \tag{23}$$

Denoting $g \equiv f - a$ by (23) we derive,

$$\begin{aligned} T(r,f) &\le T(r,g) + O(1) \\ &\le 3N(r,g) + N\left(r,\frac{1}{g}\right) + N\left(r,\frac{1}{\Delta g - z}\right) \\ &\quad + O(\log r) \\ &\le 3N(r,f) + N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{\Delta f - z}\right) \\ &\quad + O(\log r), \quad r \notin E. \end{aligned} \tag{24}$$

□

3.1. The Rest of the Proof of Theorem 1. By Lemma 6, we know that $\tau(\Delta f) \le \sigma(f)$. If $\tau(\Delta f) < \sigma(f)$, by $\lambda(1/f) < \sigma(f)$ and $\lambda(f,a) < \sigma(f)$, there exists a number $\eta < \sigma(f)$, such that for any sufficient r we have

$$\begin{gathered} N(r,f) < r^{\eta}, \qquad N\left(r,\frac{1}{f-a}\right) < r^{\eta}, \\ N\left(r,\frac{1}{\Delta f - z}\right) < r^{\eta}. \end{gathered} \tag{25}$$

Combining (24) and (25), we can get a contradiction. Therefore, we have $\tau(\Delta f) = \sigma(f)$.

3.2. The Rest of the Proof of Theorem 3. Since $\delta(\infty, f) = 1$, then $N(r,f) = o(T(r,f))$. By (24), we can get

$$\begin{aligned} (1 - o(1))\, T(r,f) \le N\left(r,\frac{1}{f-a}\right) + N\left(r,\frac{1}{\Delta f - z}\right) \\ + O(\log r), \quad r \notin E. \end{aligned} \tag{26}$$

Since $\delta(a,f) > 0$, then there is a positive number $\theta < 1$ such that

$$N\left(r,\frac{1}{f-a}\right) < \theta T(r,f). \tag{27}$$

If Δf has only a finite number of fixed points, then from (26) and (27) we would have

$$(1 - o(1) - \theta)\, T(r,f) \le O(\log r), \quad r \notin E. \tag{28}$$

This contradicts f being transcendental. Therefore, Δf has infinitely many fixed points.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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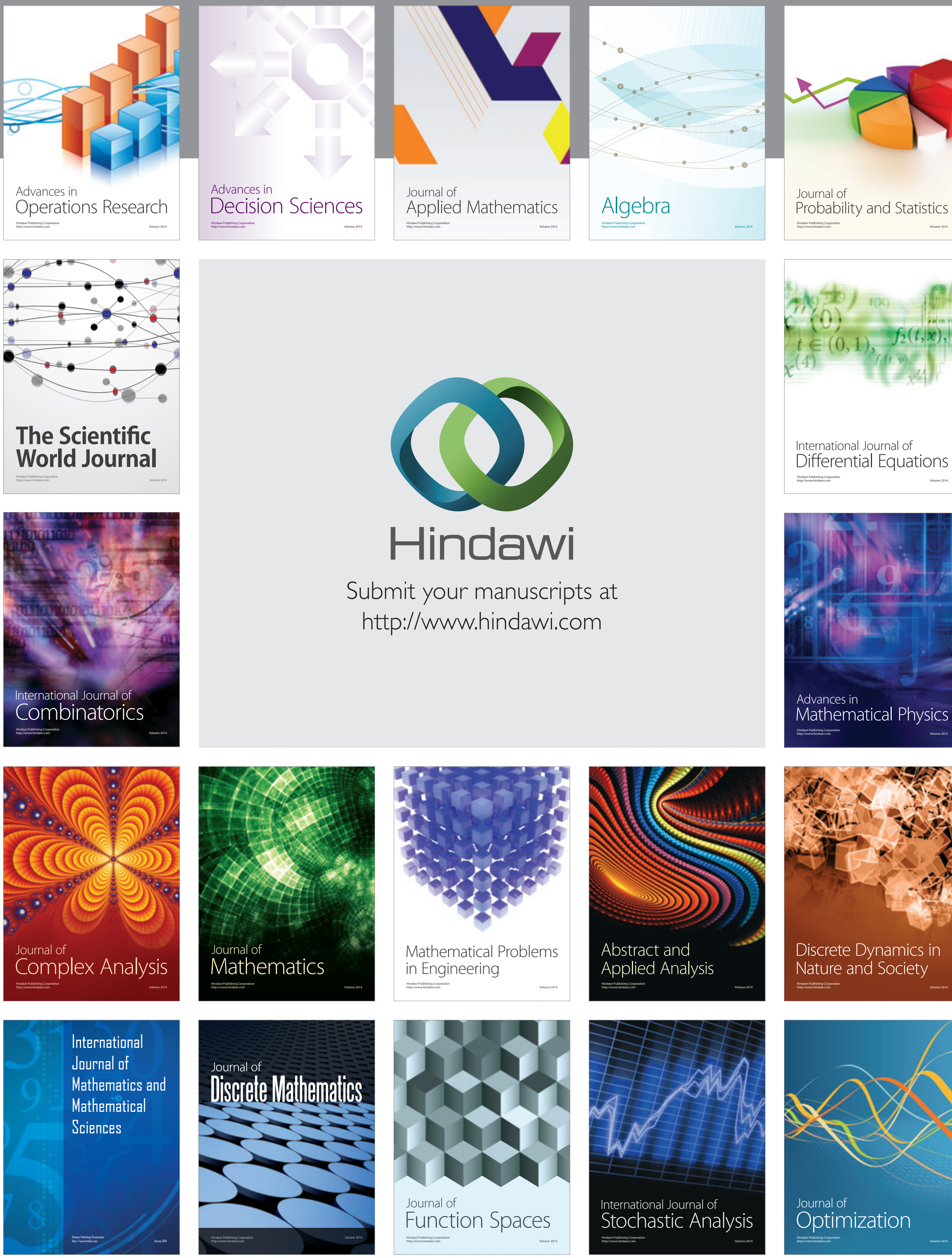
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