Internat. J. Math. & Math. Sci.
VOL. 13 NO. 4 (1990) 687-708

ON GENERALIZATIONS OF THE SERIES OF TAYLOR, LAGRANGE, LAURENT AND TEIXEIRA

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* Work supported by CAUTL, Instituto de Física-Matemática, INIC.

(Received April 6, 1989 and in revised form October 30, 1989)

SUMMARY

The classical theorems of Taylor, Lagrange, Laurent and Teixeira, are extended from the representation of a complex function F(z), to its derivative $F^{(\nu)}(z)$ of complex order ν, understood as either a 'Liouville' (1832) or a 'Rieman (1847)' differintegration (Campos 1984, 1985); these results are distinct from, and alternative to, other extensions of Taylor's series using differintegrations (Osler 1972, Lavoie & Osler & Tremblay 1976). We consider a complex function F(z), which is analytic (has an isolated singularity) at ζ, and expand its derivative of complex order $F^{(\nu)}(z)$, in an ascending (ascending-descending) series of powers of an auxiliary function f(z), yielding the generalized Teixeira (Lagrange) series, which includes, for $f(z)=z-\zeta$, the generalized Taylor (Laurent) series. The generalized series involve non-integral powers and/or coefficients evaluated by fractional derivatives or integrals, except in the case $\nu=0$, when the classical theorems of Taylor (1715), Lagrange (1770), Laurent (1843) and Teixeira (1900) are regained. As an application, these generalized series can be used to generate special functions with complex parameters (Campos 1986), e.g., the Hermite and Bessel types.

KEY WORDS AND PHRASES. Fractional Derivatives, Generalized Taylor and Laurent Series, Special Functions, and Generalized Cauchy Integral.

1980 AMS SUBJECT CLASSIFICATION CODES. 26A33, 26A36.

§1 - INTRODUCTION

The ordinary concept of n-th derivative (primitive) can be extended from positive $\nu=+n$ (negative $\nu=-n$) integral order to rational, real or complex order ν, by generalizing any of the definitions in the classical theory of functions: (i) the limit of the Leibnitz-Newton incremental ratio is generalized to a limit of finite differences, which yields an algebraic definition of derivative of complex order

(Grunwald 1867; Butzer & Westphal 1974); (ii) the classical integral along the real axis, when extended to fractional order (Liouville 1832; Riemann 1847; Weyl 1971), leads to a concept of integration with complex order (Erdelyi 1940; Kober 1940); (iii) the Cauchy loop-integral can be extended to complex exponent, leading to the appearance of branch-cut(s), and requiring a suitable choice of paths of integration (Letnikov 1868; Nekrassov 1888; Lavoie & Tremblay & Osler 1974; Nishimoto 1984; Campos 1984). The theorems of expansion in series of ascending powers associated with Taylor (1715) and Lagrange (1770), occupy a central position in the theory of functions, and the series of Laurent (1843) and Teixeira (1900), which also involve descending powers, are useful to classify singularities in the complex plane. It is therefore natural, that we seek generalizations of these four classical theorems, in the context (iii) of differintegration of complex functions.

When considering the differintegration of complex functions, it is important (Lavoie & Osler & Tremblay 1976) not to confuse different systems, viz., there is a distinct system of differintegration, for each set of branch-cuts in the complex plane (Campos 1984). The 'degenerate' case when branch-cuts are absent is the ordinary derivation (Whittaker & Watson 1902). The next two simplest, and most useful cases, are (Campos 1985) the 'Liouville' (1832) ['Riemann' (1847)] systems of differintegration, which correspond to indefinite (Weyl 1917) [definite (Erdelyi 1940; Kober 1940)] real integrals (Ross 1974; Oldham & Spanier 1974), and, in the complex plane, to an Hankel (1864) path [Pochhammer (1890) double-laced loop], about a semi-infinite (Nishimoto 1984) [finite (Lavoie & Tremblay & Osler 1974)] branch-cut. Extensions of the Taylor's theorem (Osler 1971), and other series expansions involving differintegrations (Osler 1970a,b, 1972a,b, 1973), have appeared in the literature. Our approach to the generalization of Taylor's, Laurent's, Lagrange's and Teixeira's theorems, differs in two respects: (i) the former gives series expansions for the complex function F(z), whereas we represent its differintegration $F^{(\nu)}(z)$; (ii) we consider not only 'Riemann' differintegrations along closed loops, but also 'Liouville' differintegrations along open paths.

The method of generalization of the series of Taylor, Laurent, Lagrange and Teixeira, outlined in the present introduction (§1), is similar to the classical proofs, with two important differences: (i) the differintegration of of analytic functions (functions with one branch-point), requires the use of Hankel paths (teardrop loops), in the 'Liouville' ('Riemann') system (§2); (ii) in both systems of differintegration $F^{(\nu)}(z)$ of the function F(z), the generalized Cauchy integral involves non-integral powers, and binomial instead of geometric series are used. For an analytic function F(z), its differintegration $F^{(\nu)}(z)$ is expanded in ascending integral powers $\{f(z)\}^k$ of an auxiliary function, with coefficients which involve differintegrations; this yields the generalized Lagrange series (§3), which contains, as particular case $f(z)=z-\zeta$, the generalized Taylor series. For a function F(z) with an isolated singularity at $z=\zeta$, there are ascending integral and descending non-integral powers, with coefficients specified by integrals with respectively non-integral and integral powers; this yields the generalized Teixeira series (§4), which includes, as particular cases, the generalized Laurent series for $f(z)=z-\zeta$, and also, all the other series listed in the diagram, and whose regions of convergence are illustrated in Figures 1 to 4. It goes beyond the scope of the present paper to explore the extensive applications of these series expansions, so that the discussion (§5) is confined to two examples, namely, the generation of Bessel (Hermite) functions of complex order using the expansion of a 'Liouville' differintegration in a singular (regular) form of the extended Laurent (MacLaurin) series.

§2 - DIFFERINTEGRATION IN SINGLY- AND DOUBLY-CONNECTED REGIONS

We precede the derivation of series expansions for the differintegration $d^\nu F/dz^\nu$ of a complex function F(z), by recalling the definition (Campos 1984) in a suitable form; since we seek expansions in powers of an auxiliary analytic function f(z), we replace the independent variable z by f(z), and define $d^\nu\{F(z)\}/d^\nu\{f(z)\}$, i.e. the differintegration of F(z), with complex order ν, with regard to f(z). We consider first the case in which F(z) is analytic at z, and both for the 'Liouville' ('Riemann') differintegrations, in the case F(z) has no branch-points (a branch-point at b):

DEFINITION 1 (differintegration of an analytic function). The 'Liouville' ('Riemann') differintegration $d^\nu\{F(z)\}/d\{f(z)\}^\nu$, with complex order n, of the function F(z) analytic at ζ=z, and without branch-points (with a branch point at ζ=b), with regard to an auxiliary analytic function f(z), is defined by:

$$d^\nu\{F(z)\}/d\{f(z)\}^\nu \equiv \{\Gamma(1+\nu)/2\pi i\} \int_L F(\zeta)\, \{f(\zeta) - f(z)\}^{-\nu-1} f'(\zeta)\, d\zeta, \tag{1}$$

where the path of integration L is the Hankel contour (teardrop loop) in Figure 1 (Figure 2), going round ζ=z in the positive direction, and starting and ending at infinity ζ=∞ (the branch-point ζ=b), so as to surround the semi-infinite (finite) branch-cut from ζ=z to ζ=∞ (ζ=b).

Remark 1: In the case of the 'Riemann' differintegration, the teardrop loop has finite length, and no convergence problems arise. In the case of the 'Liouville' differintegration, the Hankel path extends to infinity, and an asymptotic condition is required to assure convergence, e.g., the function F(z) must decay sufficiently fast relative to f(z) in a sector about the branch-line:

$$\arg\{f(z)\} - \delta < \arg(\zeta) < \arg\{f(z)\} + \delta:\ F(\zeta) = O\left(\{f(\zeta)\}^{\mathrm{Re}(\nu)-\varepsilon}\right), \tag{2}$$

for some ε,δ>0.

Remark 2: In the 'Liouville' ('Riemann') differintegration (1) ν cannot be a negative integer (and, further, the exponent of $F(\zeta)=(\zeta-b)^\mu G(\zeta)$ at the branch-point ζ=b must satisfy Re(μ)>-1). These restrictions on ν (ν and μ) are not essential, and can be removed by modifying the Hankel path (Campos 1984) [teardrop loop (Campos 1985)] into other shapes.

Besides the differintegration of an analytic function (1), which lead to the extended Lagrange and Taylor series (§3-4), we also need, for the extended Laurent and Teixeira series (§5-6), the definition of differintegration in the neighbourhood of a singular point $\zeta=z$. For the 'Liouville' ('Riemann') differintegration, we consider two Hankel paths (Figure 5) [teardrop loops (Figure 6)], connected by lines AB and CD, taken in opposite directions. The function $F(\zeta)$ is analytic in the interior of the composite path, so that (1) holds. In the limit $A \to D$ and $B \to C$, the integrals along AB and CD cancel, and we are left with the integral (1) along the two paths, taken on opposite directions, viz., positive for the outer, and negative for the inner path:

DEFINITION 2 (differintegration of a function with an isolated singularity): The 'Liouville' (Riemann') differintegration $d^{\nu}\{F(z)\}/d\{f(z)\}^{\nu}$, with complex order ν, of the analytic function $F(\zeta)$, with an isolated singularity at $\zeta=z$, and no branch-points (a branch-point at $\zeta=b$), with regard to an auxiliary analytic function $f(z)$, is defined by:

$$\left\{2\pi i/\Gamma(1+\nu)\right\} D^{\nu}\{F(z)\}/f\{f\}^{\nu} = \int_{L} F(\zeta)\left\{f(\zeta) - f(\zeta)\right\}^{-\nu-1} f'(\zeta)\, d(\zeta) - \int_{\ell} F(\eta)\{f(\eta) - f(\eta)\}^{-\nu-1} f'(\eta)\, d(\eta) - \tag{3}$$

as the difference of the same integrand, takenin the positive directions, along an outer L and an inner ℓ Hankel path (Figure 3) [teardrop loop (Figure 4)], around the same semi-infinite (finite) branch-cut, joining $\zeta=z$ to $\zeta=\infty$ ($\zeta=b$).

Remark 3: The substance of remarks 1 and 2 applies to Definition 2 as well as to Definition 1, e.g., the asymptotic condition (2) assures convergence along both Hankel paths L and ℓ, in Figure 3.

We have thus covered all four cases of differintegration $D^{\nu}\{F(z)\}/D\{f(z)\}^{\nu}$, for (i) $F(z)$ analytic or with an isolated singularity at $\zeta=z$, and (ii) with or without branch-point at $\zeta=b$.

Besides these preliminaries on the dependent function $F(z)$, we also need some preliminaries on the independent function:

Lemma 1 (singly-connected region defined by an analytic function): The region D defined by the analytic function $f(z)$:

$$R > 0: D \equiv \{z: |f(z)| \leq R\} \tag{4}$$

has boundary D-∂D given by:

$$\partial D = \left\{\zeta: |f(\zeta)| = R\right\},\ D - \partial D = \left\{z: |f(z)| < R\right\}; \tag{5a,b}$$

the region D_{ε} defined by:

$$\partial D = \left\{\zeta: |f(z)| \leq R - \varepsilon\right\} \equiv D_{\varepsilon} \subset D \tag{6}$$

is a closed sub-region of D (Figure 7).

Proof: By the maximum modulus principle, if f(z) is analytic in a region, the maximum of $|f(z)|$ lies on the boundary. Hence, considering the region D(4), we must have $|f|=|f|_{max}=R$ on the boundary ∂D, and $|f|<R$ in the open interior $D-\partial D$. If we impose a lowerbound $|f|\leq R-\varepsilon$ with $0<\varepsilon<R$, we obtain a closed region, lying entirely within $D-\partial D$. QED.

If the function f(z) is analytic and non-zero, we can also define an inner boundary, and obtain doubly-connected region:

Lema 2 (doubly-connected region defined by an analytic function without zeros: The region D defined by the analytic function f(z) without zeros:

$$R > r > 0: D \equiv \{z: r \leq |f(z)| \leq R\}, \tag{7}$$

is doubly-connected with inner ∂E and outer ∂D boundaries:

$$\partial D = \{\zeta: |f(\zeta)| = R\}, \quad \partial E = \{\eta: |f(\eta)| = r\}, \tag{8a,b}$$

and interior:

$$D = \partial D - \partial E = \{z: r < |f(z)| < R\}; \tag{9}$$

the region:

$$0 < \varepsilon,\delta; \quad \varepsilon + \delta < R\text{-}r: \{z: r + \delta \leq |f(z)| \leq R - \varepsilon\} D_{\varepsilon,\delta} \subset D, \tag{10}$$

is a closed sub-region of D (Figure 8).

Proof: Since f(z) is analytic and non-zero, 1/f(z) is also analytic, and the maxima of $|f|$ and $1/|f|$ lie on the boundary. In the case of the region (7), the outer boundary (8a) is the maximum of $|f|$, and the inner boundary (8b) the maximum of $1/|f|$, i.e., minimum of $|f|$, leaving (9) as the interior. Choosing a lower upper bound $|f|\leq R-\varepsilon$, and a higher lower bound $|f|\geq r+\delta$, in a compatible way $R-\varepsilon>r+\delta$, leads to a closed region (10), lying entirely within D. QED.

Remark 4: The open interior (closed sub-region) will be relevant to absolute (uniform) convergence of series.

Remark 5: The singly- (doubly-) connected region of Lemma 1 (Figure 7) [Lemma 2 (Figure 8)] is relevant to analytic (singular) functions; they are drawn together in the case of compact (ring-shaped) regions, bounded by one (two) closed loop(s), in Figure 1 and 2 (3 and 4), with either Hankel paths (Figure 5) or teardrop loops (Figure 6), respectively for 'Liouville' (Figures 1 and 3) and 'Riemann' (Figure 2 and 4) differintegrations.

§3 - GENERALIZATION OF TAYLOR'S AND LAGRANGE'S THEOREMS

We consider first the expansion of the differintegration of a complex function F(z), with regard to an auxiliary function f(z), at a regular point:

THEOREM 1 (generalized Lagrange series): The differintegrations dn $F^{v}\{(z)\}/d\{f(z)\}^{v}$, with complex order v, of a complex function F(z), with regard to an auxiliary function f(z), can be expanded in a series of ascending integral powers of the latter:

$$d^{v}\{F(z)\}/d\{f(z)\}^{v} = \sum_{k=0}^{\infty} A_k \{f(z)\}^k : \tag{11}$$

with coefficients specified by a differintegration of non-integral order:

$$A_k \equiv (k!)^{-1} \lim_{\zeta\to a} (\partial/\partial\zeta)^{v+k} \left\{F(\zeta)\{f(\zeta)/(\zeta-a)\}^{-v-k-1} f'(\zeta)\right\}. \tag{12}$$

It is assumed that the dependent F(ζ) and independent f(ζ) functions are both analytic at ζ=z, and that the latter has a simple zero f(z)=0≠f '(a) at ζ=a. The series (11) converges absolutely (uniformly) in the open region D-∂D (5b) [closed sub-region D_ε (6)], where R is the largest positive real number, such that the region D (4) excludes all singularities of F(z).

Remark 6: The teardrop loop L in Figure 2, which is used in the 'Riemann' differintegration of a function F(ζ), analytic at ζ=z and with a branch-point at ζ=b, can be continuously deformed into the boundary ∂D of the region of convergence of the generalized Lagrange series. Therefore the latter is exact provided that $R \geq |z-b| + \varepsilon$, for some ε > 0.

Remark 7: The Hankel path L in Figure 1, which is used in the 'Liouville' differintegration of a function F(ζ) analytic at ζ=z and without branch-points extends to infinity, and as long as R is finite, cuts the boundary ∂D of the region of convergence of the Lagrange series, at two points $\zeta=\zeta_\pm$. The integrals along the paths L and ∂D, taken between the points ζ_+ and ζ_-, are equal, say to a value τ. The integral along ∂D differs from τ, by a term $O(\zeta_+ - \zeta_-)$, which vanishes as R→∞, because $\zeta_+ \to \zeta_-$. The integral (1) along L differs from τ, by a term not exceeding:

$$\begin{aligned}\Delta &\leq \left|\int_{-\infty}^{\zeta_-}\right| + \left|\int_{-\infty}^{\zeta_+}\right| F(\zeta)\,(f(\zeta) - f(z))^{-v-1} f'(\zeta)\, d\zeta . \\ &\leq 2 \int_{x\equiv\inf\{Re(\zeta_\pm)\}}^{\infty} |F(\zeta)|\,|(f(\zeta) - f(z)|^{-Re(v)-1} df ;\end{aligned} \tag{13}$$

using the asymptotic condition (2), an upper bound can be estimated for (13), viz.:

$$\begin{aligned}\Delta &\leq 2M \int_x^{\infty} f^{Re(v)-\varepsilon} |f-B|^{Re(v)-1} df \\ &\leq 4M \int_x^{\infty} f^{-1-\varepsilon} df = \{4M/(1+\varepsilon)\}\, x^{-\varepsilon},\end{aligned} \tag{14}$$

where M,B are constants. The expression (14) vanishes Δ → 0 as R → ∞, because $x \equiv \inf\{Re(\zeta_+)\} \to \infty$ and ε > 0. Thus the Lagrange series for the 'Liouville' differintegration is exact as R → ∞, i.e. for F(z) a polynomial or integral function, with infinite radius of convergence.

The remark 6 (Remark 7) indicates conditions under which we may interchange integrals along the teardrop loop (Hankel path) L and along the boundary ∂D of the region of convergence, in the proof of the generalized Lagrange series, for 'Riemann' ('Liouville') differintegrations:

Proof: The definition of differintegration (1), with complex order ν, of the function $F(z)$ with regard to the auxiliary function $f(\zeta)$, involves a non-integral power, which may be expanded in a binomial series:

$$\{f(\zeta) - f(z)\}^{-\nu-1} = \sum_{k=0}^{\infty} \binom{-\nu-1}{k} \{-f(z)\}^{k} \{f(\zeta)\}^{-\nu-1-k}, \tag{15}$$

whose radius of convergence is $|f(z)| = |f(\zeta)| \equiv R$, i.e. it converges absolutely in the open interior (5b) of (4), and uniformly in the closed sub-region (6). Therefore, the series (15) may be integrated term-by-term along the boundary ∂D of the region of uniform convergence D_ε, leading to a series (11) of integral powers of $f(z)$, with coefficients given by:

$$A_k \equiv \{(-)^k/k!\} \{\Gamma(1+\nu)\Gamma(-\nu)/2\pi i\ \Gamma(-\nu-k)\} \int_{\partial D_\varepsilon} F(\zeta) \{f(\zeta)\}^{-\nu-1-k} f'(\zeta)\, d\zeta =$$
$$\{\Gamma(1+\nu + k)/k!2\pi i\} \int_{\partial D} F(\zeta) \{f(\zeta)\}^{-\nu-1-k} f'(\zeta)\, d\zeta . \tag{16}$$

Since the auxiliary function $f(z)$ is assumed to have a simple zero at $z=a$:

$$f(a) = 0 \neq f'(a): g(\zeta,a) \equiv F(\zeta) \{f(\zeta)/(\zeta-a)\}^{-\nu-k-1} f'(\zeta), \tag{17}$$

the function (17) is analytic at $\zeta=a$, and may be used to evaluate (16):

$$A_k \equiv \{\Gamma(1+\nu + k)/k!2\pi i\} \int_{L} g(\zeta,a) (\zeta-a)^{-\nu-k-1} d\zeta = (k!)^{-1} \lim_{\zeta\to a} \partial^{\nu+k} \{g(\zeta,a)\} \partial\zeta^{\nu+k}, \tag{18}$$

where the differintegration (1) was used. Substitution of (17) into (18), leads to the coefficients (12) of the generalized Lagrange series (11), proving its uniform convergence in the closed sub-region D_ε (6). In order to prove absolute convergence in the larger open interior D-∂D (5b) of D (4), we note that the coefficients (16) have an upper bound:

$$|A_k| \le \left|\Gamma(-\nu)/\Gamma(-\nu-k)\right| (k!2\pi)^{-1} \left|f(\zeta)\right|^{-Re(\nu)-k-1} ML, \tag{19}$$

where L denotes the length of ∂D, and M an upper bound of the analytic function $G(\zeta) \equiv \Gamma(1+\nu)\, F(\zeta)\, f'(\zeta)$; thus the series of moduli of (11) is bounded by:

$$\sum_{k=0}^{\infty} |A_k| |f(z)|^k \le (ML/2\pi) \sum_{k=0}^{\infty} \left|\binom{-\nu-1}{k}\right| |f(z)|^k \left|f(\zeta)\right|^{-Re(\nu)-k-1}, \tag{20}$$

a constant times the series of moduli of (15). It follows that the absolute convergence of the binomial series (15), in the open interior D-∂D (5b) of D (4), proves the absolute convergence of the generalized Lagrange series in the same region. QED.

The simplest auxiliary function $f(\zeta)$ which is analytic, and has a simple zero at ζ=a, is:

$$\left|f(\zeta)\right| = \zeta\text{-a}: A_k \equiv (k!)^{-1} \lim_{\zeta\to a} \partial^{\nu+k} F/\partial\zeta^{\nu+k}, \tag{21}$$

for which the coefficients (12) simpllify to (21), and the generalized Lagrange series (14) reduces to:

THEOREM 2 (generalized Taylor series): The differintegration $F^{(\nu)}(z)$, with complex order ν, of a function F(z) analytic at z=a, can be expanded in an ascending series of integral powers:

$$F^{(\nu)}(z) = \sum_{k=0}^{\infty} \{(z-a)^k/k!\}\, F^{(\nu+k)}(a), \tag{22}$$

whose coefficients involve differintegrations of non-integral order at a. If $\zeta=\eta$ is the singularity of F(z) closest to ζ=a, then $R \equiv |\eta\text{-a}|$ is the radius of convergence of the series (22), i.e. it converges absolutely (uniformly) in the open circle $|z\text{-}a|<R$ (closed sub-circle $|z\text{-}a|\le R-\varepsilon$, with $0<\varepsilon<R$).

Remark 8: The Remark 6 (Remark 7) about 'Riemann' ('Liouville') differintegrations of generalized Lagrange series, also apply to the generalized Taylor series, which is the particular case of the former, in which the region of convergence becomes a circle. The former includes regions of different shapes, depending on the function f(z), e.g.: (i) for $|\sin z|<R$ we have an 'oval' region; (ii) for $|\exp(z^n)|<R$ we have a non-compact region consisting of angular sectors extending to infinity. In the generalized Lagrange series we require that the auxiliary analytic function f(z) be such that (4a) be a closed loop, e.g. f(z) = sin z is admissible but f(z) = $\exp(z^n)$ is not.

Remark 9: The original Taylor and Lagrange series are obtained setting $\nu=0$ respectively in (22) and (10,11), in which case differintegrations are not needed (they do not appear for ν any integer).

Since the Taylor series:

$$F(z) = \sum_{k=0}^{\infty} \{f^{(k)}(a)/k!\}\,(z-a)^k, \tag{23}$$

converges uniformly in $|z\text{-}a|\le R-\varepsilon$ with $0<\varepsilon<R$, the 'Riemann' differintegration of the power (Lavoie & Osler & Tremblay 1976):

$$d^\nu\{(z-a)^k\}/dz^\nu = \{k!/\Gamma(1+\nu-k)\}\,(z-a)^{k-\nu}, \tag{24}$$

may be applied term-by-term (Campos 1984) to the r.h.s. of (23). The l.h.s. of (23) is an analytic function, to which we can only apply a 'Liouville' differintegration $D^\nu F/Dz^\nu$. Since the 'Liouville' and 'Riemann' differintegrations, with non-integral order ν, are generally incompatible (Lavoie, Osler & Tremblay 1976), we may expect the two expressions to be distinct:

THEOREM 3 (extended Taylor series): The 'Liouville' differintegration $D^\nu F/Dz^\nu$, with non-integral complex order ν, of an analytic function F(z), is distinct from the extended Taylor series, of non-integral powers, about a regular point z=a:

$$\nu \in C - Z: D^\nu F/Dz^\nu \neq \sum_{m=0}^{\infty} \{F^{(m)}(a)/\Gamma(1+\nu-m)\}\,(z-a)^{m-\nu}, \tag{25}$$

which has the same region of convergence as the generalized Taylor series (22).

Proof: Since the extended Taylor series (25) has the same region of convergence as the generalized Taylor series (22), they may be compared at all points. For ν a complex number other than an integer, the series cannot coincide, because one has integral (22) and the other has non-integral (25) powers. QED.

Remark 10: If ν=n is a positive integer, then the series (25) starts with m=n, and on substitution m=k+n coincides with (22), because in this case the 'Liouville' D^n/Dz^n and 'Riemann' d^n/dz^n differintegrations coincide with the ordinary n-th derivative. The generalized (22) [extended (25)] Taylor series, with ν a non-integer complex number differ in (i) having integral (non-integral) powers, (ii) coefficients involving differintegrations (ordinary derivatives), and in (iii) having no branch-cuts (a branch-cut from z=a to z=∞).

§4 - EXTENSIONS OF LAURENT'S AND TEIXEIRA'S SERIES

Having considered (in §3) power series expansions, for the differintegration $d^\nu F/df^\nu$ of a complex function F(z) with regard to an auxiliary analytic function f(z), at a regular point, we proceed (§4) to consider an isolated singularity:

THEOREM 4 (generalized Teixeira series): The differintegration $d^\nu\{F(z)\}/\{d\{f(z)\}^\nu$, with complex order ν, of a complex function F(z), with regard to an auxiliary analytic function, can be expanded, in the neighbourhood of an isolated singularity z=a, in a combined series of ascending integral (descending non-integral) powers of the auxiliary function:

$$d^\nu\{F(z)\}/d\{f(z)\}^\nu = \sum_{k=0}^{\infty} A_k\,\{f(z)\}^k + e^{i\pi\nu}\sum_{k=1}^{\infty} A_k\,\{f(z)\}^{-\nu-k}, \tag{26}$$

with coefficients A_k (A_{-k}) with k=0,1,...(k=1,2,...) involving an integration of non-integral (integral) powers of the auxiliary function, along the outer ∂D (8a) [inner ∂E (8b)] boundary of the region D (7) of convergence:

$$k=0,1,\ldots,\infty:\ A_k \equiv \{\Gamma(1+\nu+k)/k!2\pi i\}\int_{\partial D} F(\zeta)\,\{f(\zeta)\}^{-\nu-k-1} f'(\zeta)\,d\zeta,$$

$$= \{\Gamma(\nu+k)/k!2\pi i\}\int_{\partial D} F'(\zeta)\,\{f(\zeta)\}^{-\nu-k}\,df(\zeta), \tag{27}$$

$$k=1,\ldots,\infty:\ A_{-k} \equiv \{\Gamma(\nu+k)/(k-1)!2\pi i\} \int_{\alpha} F(\eta)\, \{f(\eta)\}^{k-1} f'(\eta)\, d\eta\,,$$
$$= \{\Gamma(\nu+k)/k!2\pi i\} \int_{\alpha} F'(\eta)\, \{f(\eta)\}^{k} df(\eta). \tag{28}$$

The auxiliary function f(z) is assumed to be analytic, and to have a simple zero $f(a)=0\neq f'(a)$ at the isolated singularity z=a of F(z). The series (26) converges absolutely (uniformly) in the open interior $D-\partial D-\partial E$ (9) [closed sub-region $D_{\varepsilon,\delta}$ (10)] of D (6), where R is the largest real positive number such that D (6) does not contain any singularity of F(z), and r satisfies $0<r<R$.

Remark 10: The Remark 6 (Remark 7) concerning the 'Riemann' ('Liouville') differintegration for the generalized Lagrange series, apply equally well to the outer teardrop loop (Hankel path) L and boundary ∂D of the region D of convergence of the generalized Teixeira series; for the latter, similar conditions apply to the inner teardrop loop (Hankel path) ℓ and inner boundary ∂E of the region of convergence D of the generalized Teixeira series, as illustrated in Figure 4 (Figure 3). In the case of the 'Liouville' differintegration we could let $r,R \to \infty$ with a constant ratio $R/r \equiv \text{const} > 1$.

The Remark 10 indicates the conditions in which the inner ℓ (outer L) path of differintegration (3) may be replaced by the inner ∂E (outer ∂D) boundary of the region of convergence, in the proof of the generalized Teixeira series, which is similar to that of the generalized Lagrange series (in §3), with two integrals in (3) instead of one in (1):

Proof: The first (second) integral on the r.h.s. of the differintegration (3) near a singular point z=a, involves a non-integral power, which can be expanded in the ascending (15) [descending (29)] binomial series:

$$\{f(\eta) - f(z)\}^{-\nu-1} = -e^{i\pi\nu} \sum_{k=0}^{\infty} \binom{-\nu-1}{k} \{f(\eta)\}^{k} \{f(z)\}^{-\nu-k-1}. \tag{29}$$

The ascending (15) [descending (29)] binomial series converge absolutely in the interior (exterior) of the outer ∂D (8a) [inner ∂E (8b)] boundary, and converge uniformly in a closed sub-region defined by $|f(z)\leq R-\varepsilon(|f(z)|\geq r+\delta)$, with $\varepsilon,\delta>0$ and $\varepsilon+\delta<R-r$. Thus, in the closed sub-region $D_{\varepsilon,\delta}$ (10), both series (15) and (29) converge uniformly, and they can be integrated term-by-term, after substitution respectively on the first and second integrals on the r.h.s. of the singular differintegration (3); this leads to the generalized Teixeira series (26), with coefficients A_k (A_{-k}) of the ascending (descending) integral (non-integral) powers, specified by:

$$k=0,\ldots,\infty:\ A_k = B \int_{\partial D} f(\zeta)\, \{f(\zeta)\}^{-\nu-k-1} f'(\zeta)\, d\zeta, \tag{30}$$

$$k=1,\ldots,\infty:\ A_{-k} = B \int_{\partial E} f(\eta)\, \{f(\eta)\}^{k} f'(\eta)\, d\eta, \tag{31}$$

where the constant B is given by:

$$B \equiv \{(-)^k/k!\}\,\{\Gamma(1+\nu)\Gamma(-\nu)/2\pi i\,\Gamma(-\nu-k)\} = \Gamma(1+\nu+k)/k!2\pi i. \tag{32}$$

The expressions (30) and (31) with (32), coincide with the first expressions in respectively (27) and (28); the second expressions in (27) and (28) are obtained from the first via an integration by parts. Having proved the uniform convergence of the generalized Teixeira series (26) in the closed sub-region $D_{\varepsilon,\delta}$ (10), we proceed to prove the absolute convergence in the open interior (9); the latter follows by noting that (15) and the ascending series in (26) converge absolutely inside the outer boundary ∂D, and (29) and the descending series in (26) converge absolutely outside the inner boundary ∂E. The proof of the second statement uses (29) and (31), in the same way as the proof (19) and (20) of the first statement, viz.: (i) the coefficient (31) has an upper bound:

$$|A_{-k}| \leq |\Gamma(-\nu)/\Gamma(-\nu-k)|\,(k!2\pi)^{-1}\,|f(\eta)|^k m\,\ell, \tag{33}$$

where l is the length of the loop ∂E, and m a bound on the analytic function $G(\eta) \equiv f'(\eta)\,\Gamma(1+\nu)$; (ii) the series of moduli of the second term on the r.h.s. of (26):

$$\sum_{k=0}^{\infty} |A_{-k-1}|\,|f(z)|^{-Re(\nu)-k-1} \leq (ml/2\pi) \sum_{k=0}^{\infty} |\binom{-\nu-1}{k}|\,|f(\eta)|^k\,|f(z)|^{-Re(\nu)-k-1}, \tag{34}$$

is bounded by the series of moduli of (29) multiplied by a constant; (iii) thus the absolute convergence of (29) outside the inner boundary ∂E, implies that of the second term on the r.h.s. of (26). QED.

Using the simplest auxiliary analytic function $f(z)=z-a$ with a simple zero at $z=a$, the generalized Teixeira series (26;27,28) simplifies to:

THEOREM 5 (generalized Laurent series): The differintegration $F^{(\nu)}(z)$, with complex order ν, of a complex function $F(z)$, in the neighbourhood of an isolated singularity $z=a$, can be expanded in a combined ascending (descending) series of integral (non-integral) powers of $z-a$:

$$F^{(\nu)}(z) \sum_{k=0}^{\infty} A_k\,(z-a)^k + e^{i\pi\nu} \sum_{k=1}^{\infty} A_{-k}\,(z-a)^{-\nu-k}, \tag{35}$$

with coefficients A_k (A_{-k}) given by an integral along an outer (inner) circle of radius $R(r<R)$, of a non-integral (integral) power:

$$k=0,\dots,\infty:\ A_k \equiv \{\Gamma(1+\nu+k)/k!2\pi i \int_{|\zeta-a|=R} (\zeta-a)^{-\nu-k-1}\,F(\zeta)\,d\zeta, \tag{36}$$

$$k=1,\dots,\infty:\ A_{-k} \equiv \{\Gamma(\nu+k)/(k-1)!2\pi i \int_{|\eta-a|=r} (\eta-a)^{k-1}\,F(\eta)\,d\eta, \tag{37}$$

If $z=c$ is the singularity of $F(z)$ closest to $z=a$, we set $0<r<R \equiv |c-a|$, the series (35) converges absolutely (uniformly) in the open ring $r<|z-a|<R$ (closed sub-ring $r+\varepsilon \leq |z-a| \leq R-\delta$, with $\varepsilon,\delta>0$ and $\varepsilon+\delta<R-r$).

Remark 11: The Gamma functions disappear from the coefficients (36,37) in the case $\nu=0$:

$$k=0,\dots,\infty:\ B_k \equiv (2\pi i)^{-1} \int_{|\zeta-a|=R} (\zeta-a)^{-k-1} F(\zeta)\, d\zeta\,, \tag{38}$$

$$k=1,\dots,\infty:\ B_{-k} \equiv (2\pi i)^{-1} \int_{|\eta-a|=r} (\eta-a)^{k-1} F(\eta)\, d\eta\,, \tag{39}$$

of the original Laurent series:

$$F(z) = \sum_{k=-\infty}^{+\infty} B_k (z-a)^k, \tag{40}$$

which satisfies the same convergence conditions as the generalized Laurent series in Theorem 5.

Remark 12: The region of validity of the generalized Laurent series is a circular annulus, whereas the generalized Teixeira series holds. As in Remark 8, the auxiliary analytic function f(z) must be such that the conditions $|f(z)|=r$ and $|f(z)|=R$ specify closed loops, with the former inside the latter for $r<R$, and the series converges in the annulus in between them.

Since the series (40) converges uniformly for $r+\varepsilon\le|z-a|\le R-\delta$, it may be differintegrated term-by-term, in the 'Riemann' system (23), leading to an extended Laurent series, which is distinct from the 'Liouville' differintegration of F(z), with non-integral order ν:

THEOREM 6 (extended Laurent series): The 'Liouville' differintegration $D^\nu F/Dz^\nu$, with non-integral (integral) order ν, of a complex function F(z), with an isolated singularity at z=a, is distinct from (equal to) the extended Laurent series, combining ascending and descending non-integral powers of z-a:

$$\sum_{m=-\infty}^{\infty} \{k!/\Gamma(1+k-\nu)\}\, B_m (z-a)^{m-\nu} \begin{Bmatrix} \neq F^{(\nu)}(z) & \text{if } \nu \in C - Z, \\ \neq F^{(\nu)}(z) & \text{if } \nu \in Z, \end{Bmatrix} \tag{41}$$

with coefficients given by (38,39), and having the same region of convergence as the generalized Laurent series (35,36,37).

Proof: The extended (41;38,39) and generalized (35;36,37) Laurent series have the same region of convergence D, i.e. can be compared at all points of D. For ν not an integer the series cannot coincide, for one (35) has some integral powers and the other (41) none. In the case $\nu=n$ an integer, the series (41) starts at m=n and proceeds in ascending powers, as does (35) starting at k=-n; setting m=-k the two series are found to coincide. QED.

Remark 13: The relationship between the generalized (35;36,37) [extended (41;38,39)] Laurent series, is similar to the Taylor case in Remark 9, i.e. they differ in (i) having some (all) powers integral (non-integral), in (ii) some (all) coefficients being integrals of non-integral (integral) powers, and in (iii) having no branch-cut (a branch-cut from $z=a$ to $z=\infty$).

§5 - DISCUSSION

The consistency of the singular (§4) [regular (§3)] series, both of the generalized Teixeira (Theorem 4) [Lagrange (Theorem 1)] type, and in the case of generalized Laurent (Theorem 2) [Taylor (Theorem 5)] series, and extended Laurent (Theorem 3) [Taylor (Theorem 6)] series, can be verified by noting that: (i) if the function $F(z)$ is analytic at $z=a$, the inner loop ∂E (circle $|z-a|=r$) can be shrunk to zero, and so vanish the coefficients A_{-k} (28) [A_{-k} (37), B_{-k} (39)] of the descending powers of the generalized Teixeira (26) [generalized (35), extended (41) Laurent] series: (ii) these reduce to the generalized Lagrange (11) [generalized (22), extended (25) Taylor] series, since the coefficients A_k (27) [A_k (36), B_k (38)] of the ascending integral powers can be evaluated, using the differintegration (1), as A_k (12) [A_k (21), $B_k=F^{(k)}(a)/k!$ in (23)]. This result implies the following hierarchy of series, represented in the diagram: (i) the generalized Teixeira series is the most general, since it represents a singular differintegration in powers of an auxiliary function; (ii) the second level consists of the generalized Laurent (Lagrange) series, which is obtained by imposing one restriction, namely, the auxiliary function $f(z)=z-a$ (a regular differintegration); (iii) imposing both restrictions leads to the third level, namely, the generalized Taylor series; (iv) the generalized Laurent (Taylor) series are not equivalent to extended Laurent (Taylor) series, which are at the same level, viz., the second (third). Note that in the diagram, the series on the left (right) are regular (singular). Some of the consequences of these four valid series expansion theorems, e.g. the rule of implicit differintegration, the identification of the principal parts of a differintegration near a pole or essential singularity, and the generalized Mittag-Leffler series of fractions for the differintegration of meromorphic functions, are presented elsewhere (Campos 1989a,b). We conclude with two examples of application to special functions, namely, the use of the generalized Laurent [MacLaurin (1742)] series, with 'Liouville' differintegration, to obtain a generating function for Bessel (Hermite) functions of complex order.

We start with the Hermite function:

Property 1 (Hermite function as 'Liouville' differintegration): The Hermite function $H_\nu(z)$, of complex order ν and variable z, is specified by a 'Liouville' differintegration of the Gaussian function:

$$H_\nu(z) = e^{i\pi\nu} e^{z^2} D^\nu\left\{e^{-z^2}\right\}/Dz^\nu . \tag{42}$$

Proof: By (1) with $f(z)=z$, the 'Liouville' differintegration is equivalent to the integral representation for the Hermite function:

$$H_\nu(z) = \{\Gamma(1+\nu)/2\pi i\}\, e^{z^2} \int_{\infty \exp\{i \arg(z)\}}^{(z+)} (z-\zeta)^{-\nu-1} e^{-\zeta^2} d\zeta, \tag{43}$$

which is known (Courant & Hilbert 1953). QED.

An alternative representation is:

Property 2 (Hermite function as limit of a differintegration): The Hermite function is fied by the following limit of a 'Liouville' differintegration:

$$H_\nu(z) = \lim_{\zeta \to 0} \partial^\nu \{\exp(-\zeta^2 + 2\zeta z)\}/d\zeta^\nu. \tag{44}$$

Proof: Perform in (43) the change of variable $\eta = z - \zeta$, to obtain the integral representation:

$$H_\nu(z) = \{T(1+\nu)/2\pi i \int_{\infty \exp\{i0\}}^{(0+)} \eta^{-\nu-1} e^{-\nu-1} e^{-\eta^2 + 2\eta z} d\eta, \tag{45}$$

which may be interpreted as the limit as $\eta \to 0$ of a 'Liouville' differintegration with order ν of the exponential. QED.

The expression (44) suggests that we consider the generalized MacLaurin series, i.e. generalized Taylor series (22) about the origin a=0, for the 'Liouville' differintegration of the function in curly brackets in (44), viz.:

$$\partial^\nu \{\exp(-\zeta^2 + 2\zeta z)\}/\partial\zeta^\nu = \sum_{k=0}^{\infty} (\zeta^k/k!) \lim_{\zeta \to 0} (\partial/\partial\zeta)^{\nu+k} e^{-\zeta^2 + 2\zeta z}; \tag{46}$$

since the exponential is an integral function, the radius of convergence is $R=\infty$, and thus the series (46) for the 'Liouville' differintegration is exact, according to Remark 7. Substituting (44) into (46), we obtain:

Property 3 (generating differintegration for Hermite functions): The 'Liouville' differintegration:

$$\partial^\nu \{\exp(-\zeta^2 + 2\zeta z)\}/\partial\zeta^\nu = \sum_{k=0}^{\infty} (\zeta^k/k!)\, H_{\nu+k}(z), \tag{47}$$

when expanded in a generalized MacLaurin series (47), with infinite radius of convergence, specifies as coefficients of the powers ζ^k of the parameter ζ, the Hermite functions of complex orders $\nu, \nu+1, \ldots,$

Remark 14: The differintegrations are not needed $\nu=0$ for the generation of Hermite (1884) polynomials:

$$\exp(-\zeta^2 + 2\zeta z) = \sum_{k=0}^{\infty} (\zeta^k/k!)\, H_k(z). \tag{48}$$

In the case of Bessel functions, we start from the integral representations:

Property 4 (integral representations for Bessel functions): The Bessel function $J_\nu(z)$ of complex order ν and variable z, has the integral representations:

$$J_\nu(z) = \{z^\nu/2\pi i\} \int_{-\infty}^{(0+)} \zeta^{-\nu-1} \exp\{(\zeta - z^2/\zeta)/2\}\, d\zeta =$$

$$(2\pi i)^{-1} \int_{-\infty}^{(0+)} \eta^{-\nu-1} \exp\{z(\eta - 1/\eta)/2\}\, d\eta \ . \tag{49}$$

Proof: The first expression appears in the literature (Watson 1944), and the second can be deduced from the first by means of a change of variable $\zeta=\eta z$. QED.

The integral representation of the Bessel (49) [Hermite (41,42)] function involes an integrand which has no branch-point at $\zeta=0$, implying that we should use the 'Liouville' differintegration in both cases; also, the integrand is a polimorphic (analytic) function, because $\eta=0$ is an esential singularity (regular point) of the integrand in (49) [in (45)], implying that the generalized Laurent (MacLaurin) series should be used, for the differintegration which generates Bessel (Hermite) functions of complex order:

Property 5 (generating differintegration for Bessel functions): The 'Liouville' differintegration:

$$\partial^\nu\{\exp(z(-\zeta - 1/\zeta)/2\}/\partial\zeta^\nu = \sum_{k=0}^{\infty} \{\Gamma(1+\nu+k)/k!\}\, \zeta^k J_{\nu+k}(z) +$$

$$+ e^{i\pi\nu} = \sum_{k=1}^{\infty} \{\Gamma(\nu+k)/(k-1)!\}\, \zeta^{-\nu-k} J_{-k}(z)\, , \tag{50}$$

when expanded in generalized Laurent series about the origin, with infinite radius of convergence, specifies, as coefficients of the ascending integral (descending non-integral) powers of the parameter ζ, the Bessel functions (coefficients) of complex (integral) orders $\nu,\nu+1,\ldots$ $(-1,-2,\ldots)$.

Proof: Since the exponential in the integrand of (49), has no singularities other than two essential singularities at the origin $\zeta=0$ and infinity $\zeta=\infty$, its generalized Laurent series expansions about the origin has infinite radius of convergence, and, by Remark 7, the corresponding 'Liouville' differintegration, leads to an exact series:

$$\partial^\nu\{\exp(z(\zeta - 1/\zeta)/2\}/\partial\zeta^\nu = \sum_{k=0}^{\infty} a_k \zeta^k + \sum_{k=1}^{\infty} a_{-k}\, \zeta^{-\nu-k} e^{i\pi\nu}. \tag{51}$$

The coefficients a_k (a_{-k}):

$$k=0,\ldots,\infty:\ a_k \equiv \{\Gamma(1+\nu+k)/k!2\pi i \int_{-\infty}^{(0+)} \zeta^{-\nu-k-1} e^{z(\zeta-1/\zeta)/2} d\zeta\, ,$$

$$= \{\Gamma(1+\nu+k)/k!\}\, J_{\nu+k}(z), \tag{52}$$

$$k=1,\ldots,\infty:\ a_{-k} \equiv \{\Gamma(\nu+k)/(k-1)!2\pi i \int_{-\infty}^{(0+)} \eta^{k-1} e^{z(\eta-1/\eta)/2} d\eta\, ,$$

$$= \{\Gamma(\nu+k)/(k-1)!\}\, J_{-k}(z), \tag{53}$$

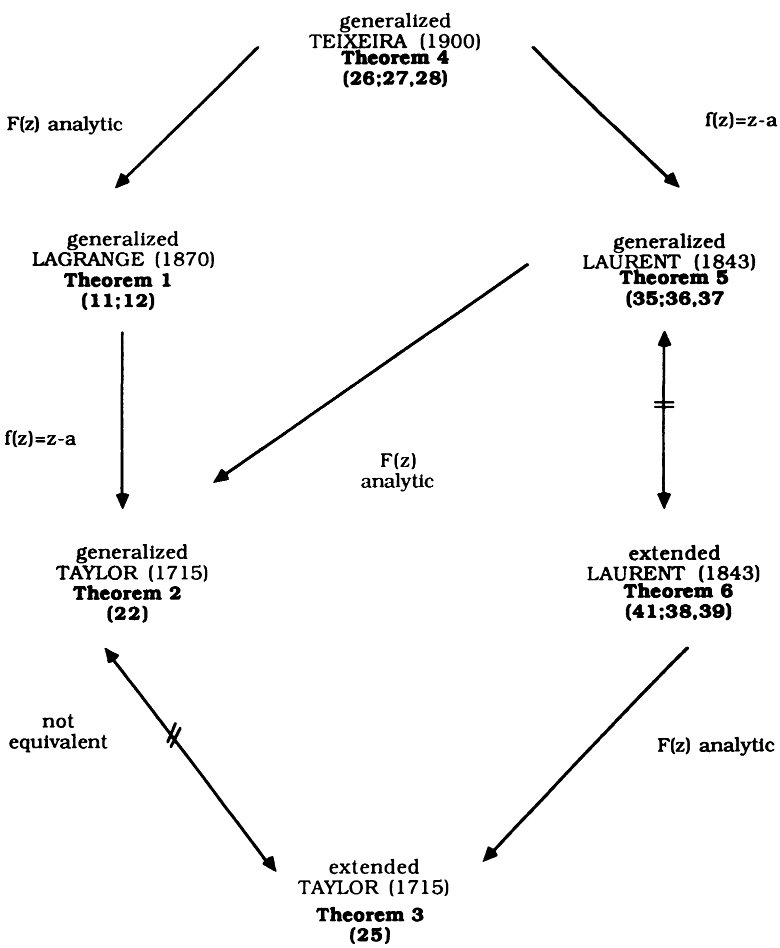
are evaluated in terms of Bessel functions (coefficients) using (49). Substituting (52) and (53) in (51) leads to (50). QED.

Remark 15: The differintegrations are not needed ν=0, for:

$$\exp\{z(\zeta-1/\zeta)/2\} = \sum_{k=-\infty}^{+\infty} \zeta^{k} J_k(z) \tag{54}$$

the generation (Schlomilch 1857) of Bessel coefficients.

HIERARCHY OF SERIES



DIAGRAM

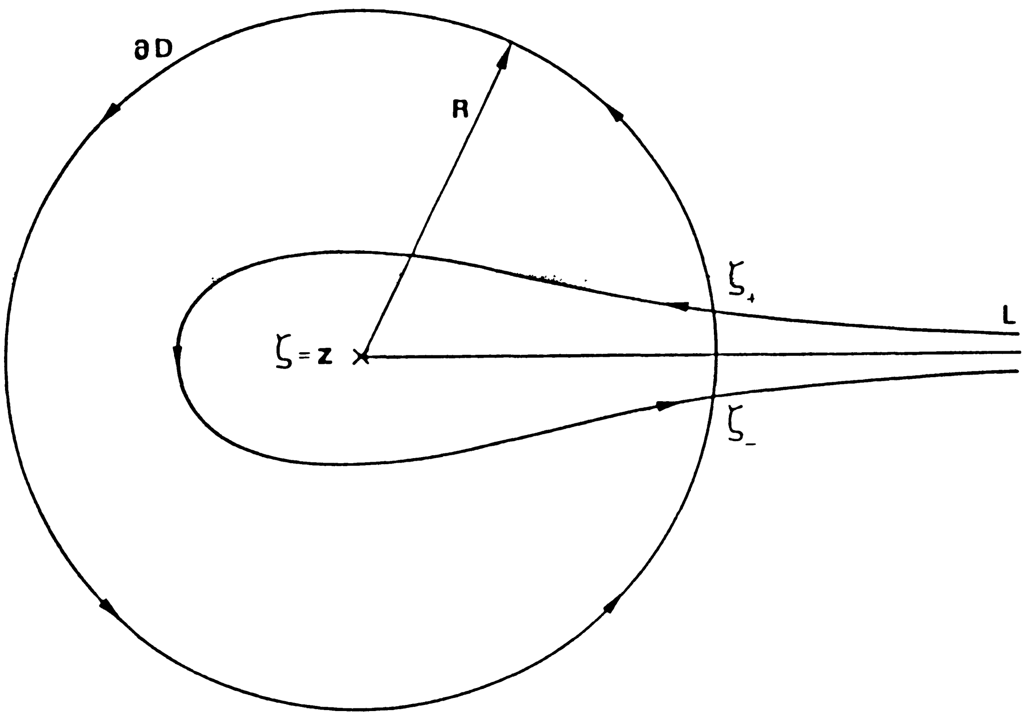


Figure 1 - The Hankel loop L used in the 'Liouville' differintegration of a function analytic at ζ=z, touches at the points $\zeta_{\pm}$, the boundary ∂D of the region of convergence of the generalized Lagrange series, which is assumed to be a closed loop, which becomes a circle of radius R in the particular case of the generalized Taylor series.

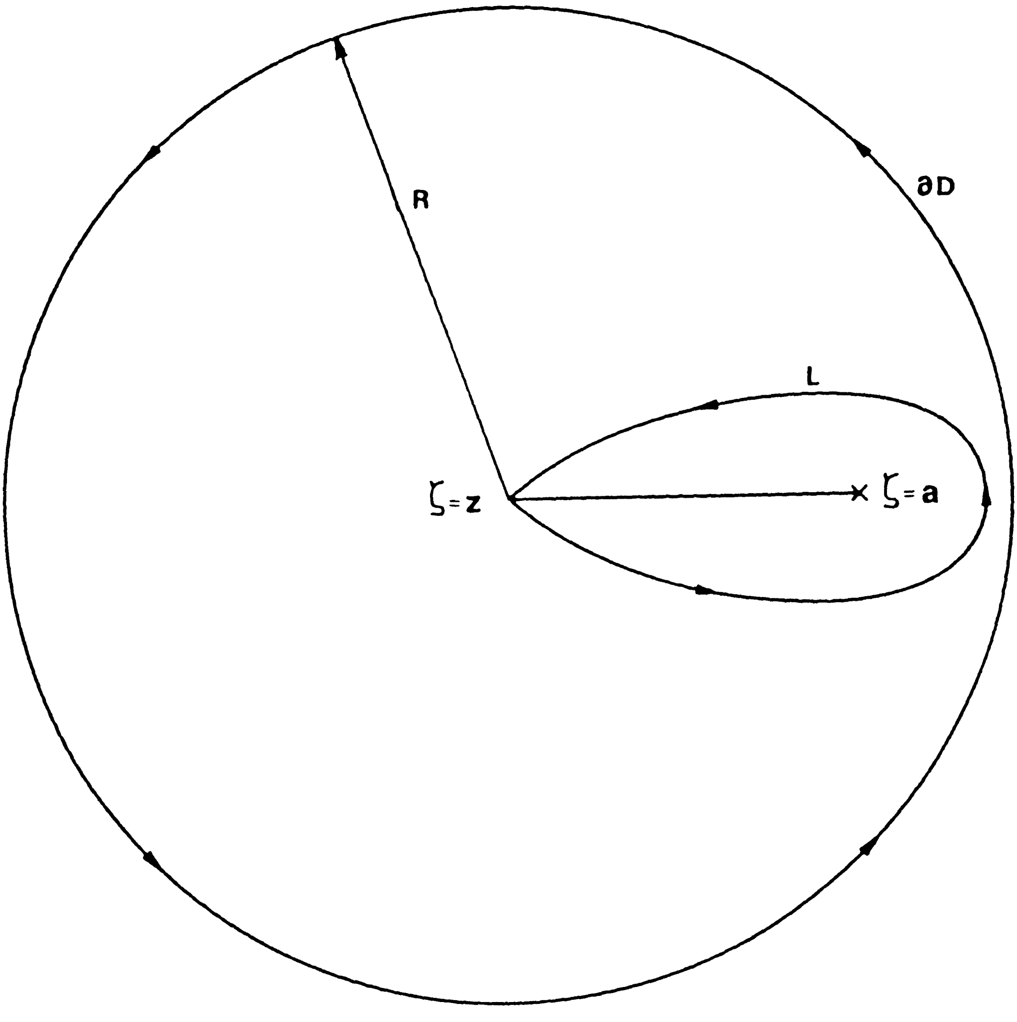


Figure 2 - The same closed loop (circle) of convergence of the generalized Lagrange (Taylor) series D, does not touch, and can be continuously deformed onto, the teardrop loop L, used in the 'Riemann' differintegration of a function analytic at $\zeta=z$, and with a branch-point at $\zeta=b$.

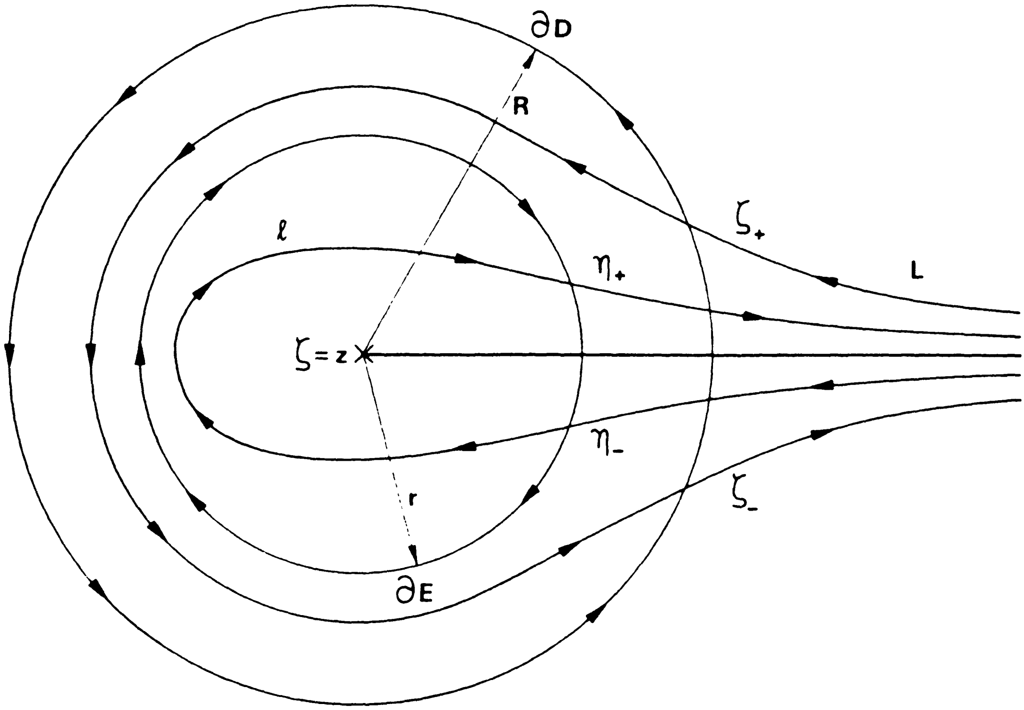


Figure 3 - The 'Liouville' differintegration of a function with an isolated singularity at ζ=z, uses an inner ℓ and an outer L Hankel path, both about the same semi-infinite branch-cut joining ζ=z to infinity, which intersect the inner ∂E and outer ∂D closed-loop boundaries, of the doubly-connected regions of convergence of the generalized Teixeira (Laurent) series, respectively at the points $\zeta_\pm$ and $\eta_\pm$.

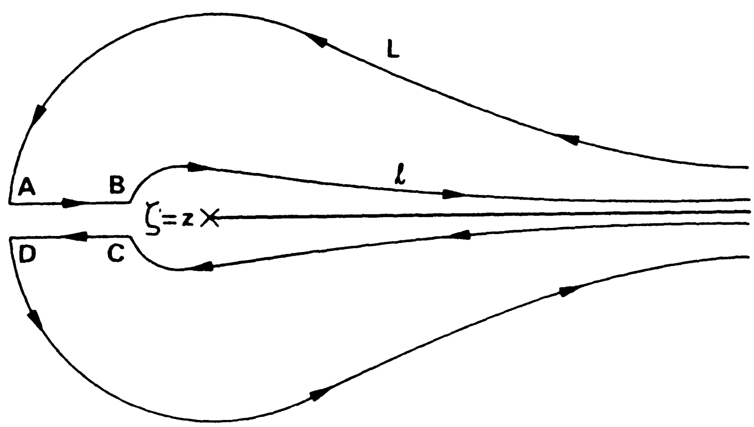


Figure 5 - One of the elements of Figure 3, is the pair of inner ℓ (outer L) Hankel paths, taken in the negative (positive) directions, around the same semi-infinite branch-cut joining infinity to ζ=z, which is a singularity surrounded by a composite path, which also includes the segments AB and CD, which cancel in the integration of an univalent function.

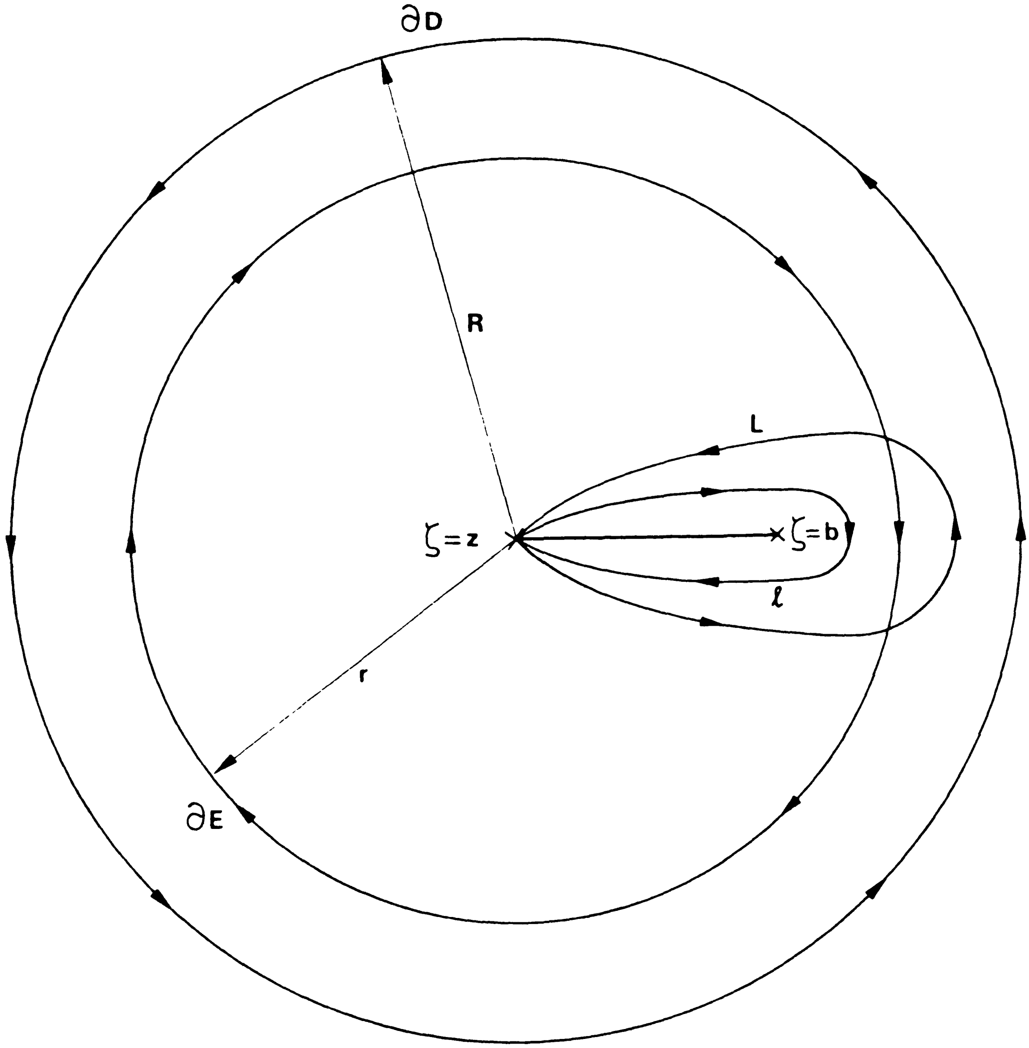


Figure 4 - The 'Riemann' differintegration of a function with an isolated singularity at ζ=z, and a branch-point at ζ=a, involves an inner ℓ and an outer L teardrop loop, both about the same finite branch-cut joining ζ=z to ζ=b, which can be deformed continuously, respectively onto the inner ∂E and outer ∂D boundary, of the region of convergence of the generalized Teixeira (Laurent) series, whose boundary is a closed loop (circle).

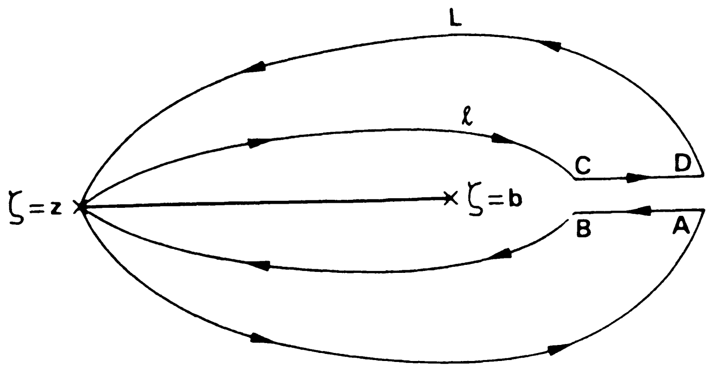


Figure 6 - One of the elements of Figure 4, is the pair of inner ℓ (outer L) teardrop loops, going clock- (counterclock-) wise around the same finite branch-cut joining $\zeta=z$ to $\zeta=b$, which is surrounded by a composite loop, also including the segments AB and CD, which cancel in the integration of an univalent function.

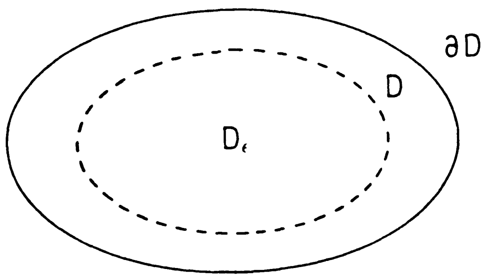


Figure 7 - The other element in Figures 1 and 2, is region D of convergence of the generalized Lagrange (Taylor) series, has a closed loop (circle) as boundary ∂D and hence is simply-connected, leading to absolute convergence in the open interior D-∂D, and uniform convergence in a closed sub-region D_ε.

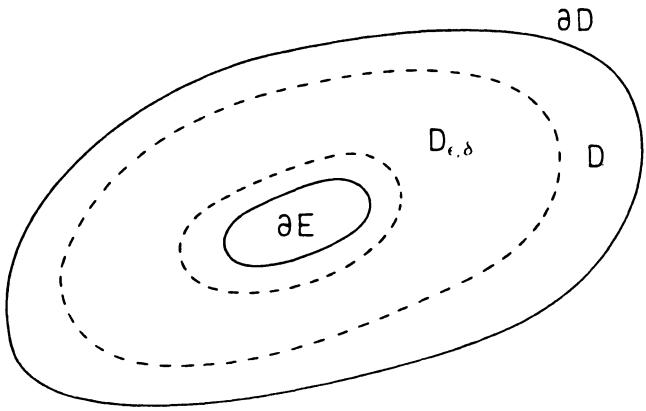


Figure 8 - The second element in Figures 3 and 4, is the ring-shaped (annulus) D of convergence of the generalized Teixeira (Laurent) series, which has closed non-intersecting loops (concentric circles) as inner ∂E and outer ∂D boundaries, and hence is doubly-connected, leading to absolute convergence in the open interior D-∂D-∂E, and uniform convergence in a closed sub-region $D_{\varepsilon,\delta}$.

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