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A THEOREM ON "LOCALIZED" SELF-ADJOINTNESS OF SCHRÖDINGER OPERATORS WITH L^1_{LOC}-POTENTIALS

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ABSTRACT. We prove a result which concludes the self-adjointness of a Schrödinger operator from the self-adjointness of the associated "localized" Schrödinger operators having L^1_{Loc}-Potentials.

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1. INTRODUCTION.

In 1978, Simader [1] proved a result which concludes the self-adjointness of a Schrödinger operator from the self-adjointness of the associated "localized" Schrödinger operators. A similar result was given by Brezis [2] in 1979 which seems to be slightly more general than [1]. Both papers deal with Schrödinger operators having L^2_{loc}-potentials.

In this paper, we give an analogous result to [2] for Schrödinger operators with L^1_{loc}-potentials and show the common structure of [1] and [2]. In the proof, we use arguments due to Kato [3] and Simader [2], which are based on quadratic form methods.

We first give some notations (compare [4]). If t is a semi-bounded quadratic form with lower bound α, we denote the inner product associated with t by $(u,v)_t$: $= t[u,v] + (1 - \alpha)(u,v)$, for u,v in the form domain Q(t) of t. The associated norm will be denoted by $||\cdot||_t$. t is closed if Q(t) together with $(\cdot,\cdot)_t$ is a Hilbert space. Recall the one-to-one correspondence between semibounded quadratic forms and semibounded self-adjoint operators. If T is a self-adjoint semibounded operator, the domain of the closed form associated with T will be denoted by Q(T) and the form by $\langle u,v\rangle \longmapsto (Tu/v)$ for $u,v \in Q(T)$. The associated norm will be called the form norm of T. We will always write Q(T) for the Hilbert space of the associated form if the inner product is clear. A set which is dense in the Hilbert space Q(T) will be called a form core of T.

Let q be a real-valued function on $\mathbb{R}^n$ and assume

$$q \in L^1_{loc}(\mathbb{R}^n) \tag{C_1}$$

and

$$\begin{aligned} &Lu := -\Delta u + qu \\ \text{with} \quad &D(L) := \{u \in L^2(\mathbb{R}^n)/qu \in L^1_{loc}(\mathbb{R}^n)\} \end{aligned} \tag{1.1}$$

where the sum in (1.1) is taken in the distributional sense. Then we define a "maximal" operator in $L^2(\mathbb{R}^n)$ associated with L such that

$$\left.\begin{aligned} &T_{max}\, u := Lu \\ \text{with} \quad &D(T_{max}) := \{u \in D(L)/Lu \in L^2(\mathbb{R}^n)\}. \end{aligned}\right\} \tag{1.2}$$

Consider the quadratic form associated with L

$$t[w,v] := \int \bar{w}\, Lv\,, \qquad w,v \in C_o^\infty(\mathbb{R}^n). \tag{1.3}$$

If we assume

t is bounded from below and closable (without loss of generality $t \geq 0$), (C_2)

then there exists a semibounded self-adjoint operator T_F associated with the closure of t. Note that for $q \in L^2_{loc}(\mathbb{R}^n)$, T_F coincides with the Friedrichs extension of $T_{min} := T_{max}|\, C_o^\infty(\mathbb{R}^n)$; see [3]. $Q(T_F)$ is then the closure of $C_o^\infty(\mathbb{R}^n)$ in the sense of the norm $||\cdot||_t$ associated with the inner product $(w,v)_t := t[w,v] + (w,v)$; $w,v \in C_o(\mathbb{R}^n)$.

From (C_2), we know $\qquad T_F \geq 0. \qquad (1.4)$

Now consider $\phi \in C_o^\infty(\mathbb{R}^n)$ with $0 \leq \phi \leq 1$ such that $\phi(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\phi(x)=0$ for $|x| \geq 1$.

For $k \in \mathbb{N}$, let

$$\phi_k(x) := \phi\left(\frac{x}{k}\right). \tag{1.5}$$

We now assume, for any k, there exists a "localized" operator associated with L; i.e., for $k \in \mathbb{N}$ there exist a $q_k \in L^1_{loc}(\mathbb{R}^n)$ and a L_k such that

(i) $L_k u := -\Delta u + q_k u$ (C_3)

with $D(L_k) := \{u \in L^2(\mathbb{R}^n)/q_k u \in L^1_{loc}(\mathbb{R}^n)\}$

and

(ii) $q_k \phi_k u = q\phi_k u$ for $u \in D(L)$.

We define also a "maximal" operator in $L^2(\mathbb{R}^n)$ associated with L_k; i.e., for $k \in \mathbb{N}$,

$$\left.\begin{aligned} &T_k u := L_k u \\ &\text{with} \\ &D(T_k) := \{u \in D(L_k)/L_k u \in L^2(\mathbb{R}^n)\}. \end{aligned}\right\} \tag{1.6}$$

Note, that (C_3) is not really a restriction; see Corollary 1 and Corollary 2. Denote $q_k^+ := \max\{q_k,0\}$, $q_k^- := \max\{-q_k,0\}$, $q^+ := \max\{q,0\}$, $q^- := \max\{-q,0\}$.

2. MAIN RESULTS.

THEOREM. Let $k \in \mathbb{N}$. Assume (C_1), (C_2), and (C_3) and define T_{max} and T_k as in (1.2) and (1.6). If we assume additionally,

$$T_k \text{ is self-adjoint;} \tag{C_4}$$

and

$$C_o^\infty(\mathbb{R}^n) \text{ is a form core of } T_k \text{ and there exists a } c_k > 0 \tag{C_5}$$

such that

$$(-\Delta w,w) + (q_k^+ w/w) \le c_k[(T_k w/w) + ||w||^2], \quad w \in C_o^\infty(\mathbb{R}^n), \tag{2.1}$$

then T_{max} is self-adjoint.

PROOF. First we note that, by (C_5), T_k is bounded from below by -1. Thus $Q(T_k)$ is well defined.

Now we proceed in 5 steps.

Step 1. We show that for $k \in \mathbb{N}$, $u \in D(T_{max})$ implies $\phi_k u \in Q(T_k)$, and thus, by (C_5), $\phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+)$ and $q_k u \in L^1_{loc}(\mathbb{R}^n)$ (making use of the semiboundedness of T_k).

By $H^1(\mathbb{R}^n)$, we denote the closure of $C_o^\infty(\mathbb{R}^n)$ in the usual Sobolev norm $||u||_{H_1} := (||\nabla u||^2 + ||u||^2)^{1/2}$. We have the continuous inclusions (compare Kato [3]),

$$D(T_k) \subset Q(T_k) \subset H^1(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset H^{-1}(\mathbb{R}^n) \subset Q(T)^*.$$

By $H^{-1}(\mathbb{R}^n)$ and $Q(T_k)^*$, we denote the antidual spaces of $H^1(\mathbb{R}^n)$ and $Q(T_k)$. $T_k + 2$ maps $D(T_k)$ onto $L^2(\mathbb{R}^n)$ and it is well known (see [4]) that this can be extended to a bicontinuous map $T_k' + 2$ from $Q(T_k)$ onto $Q(T_k)^*$. Actually, $T_k' + 2$ is a restriction of $L_k + 2$ to $Q(T_k)$ since, by (2.1) and the semiboundedness of T_k, $v \in Q(T_k)$ implies $q_k v \in L^1_{loc}(\mathbb{R}^n)$. Now let $u \in D(T_{max})$. Using (C_3), we get in the distributional sense

$$L_k\phi_k u = \phi_k T_{max} u - 2\,\nabla\phi_k\,\nabla u - (\Delta\phi_k)u. \tag{2.2}$$

Since $\nabla\phi_k\, u \in H^{-1}(\mathbb{R}^n)$ and all other terms on the right hand side of (2.2) are in $L^2(\mathbb{R}^n)$, we have

$$L_k\phi_k u \in H^{-1}(\mathbb{R}^n) \subset Q(T_k)^*.$$

Since $T_k' + 2$ is bijective, we conclude in the same way as Kato [3, Lemma 2] that $\phi_k u \in Q(T_k)$.

Step 2. We show that, for $k \in \mathbb{N}$, $u \in D(T_{max})$ implies $\phi_k u \in Q(T_F)$. Let $u \in D(T_{max})$. From Step 1, we know $\phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+)$. Then, because of (C_3), we also have

$$\phi_k u \in Q(q^+).$$

From a theorem due to Simon [5, Theorem 2.1] (see also [6] for generalizations), we know that $C_o^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n) \cap Q(q^+)$ in the sense of the norm

$$||w||_{t_+} := \{||\nabla w||^2 + (q^+w/w) + ||w||^2\}^{1/2}, \quad w \in H^1(\mathbb{R}^n) \cap Q(q^+).$$

Therefore, we can find a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $C_o^\infty(\mathbb{R}^n)$ such that

$$||v_n - \phi_k u||_{t_+} \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.3}$$

Then, because of (1.4), we have

$$\phi_k u \in Q(q^-) \text{ and}$$

$$(q^-(v_n - \phi_k u)/(v_n - \phi_k u)) \longrightarrow 0 \quad (n \longrightarrow \infty). \tag{2.4}$$

(2.3) and (2.4) imply $\phi_k u \in Q(T_F)$.

Step 3. We show that, for $k \in \mathbb{N}$, $v \in Q(T_k)$ implies $\phi_k v \in Q(T_k) \cap Q(T_F)$ and

$$u \in Q(T_F) \text{ implies } \phi_k u \in Q(T_k). \tag{2.5}$$

Let $v \in Q(T_k)$. Then, because of (C_5), there exists a sequence $\{v_n\}_{n\in\mathbb{N}}$ in $C_o^\infty(\mathbb{R}^n)$ such that

$$||v_n - v||_{t_k} \longrightarrow 0 \quad (n \longrightarrow \infty), \tag{2.6}$$

where $||\cdot||_{t_k}$ denotes the form of T_k.

For $\alpha_k := 1 + \sup |\nabla\phi_k|$, we have

$$||\nabla\phi_k(v_n - v)|| \le \alpha_k\{||\nabla(v_n - v)|| + ||v_n - v||\} \quad (2.7)$$

and

$$\int q_k^+ |\phi_k(v_n - v)|^2 \le \int q_k^+ |(v_n - v)|^2; \quad (2.8)$$

because of the semiboundedness of T_k, we have

$$(q_k^-\phi_k(v_n-v)/\phi_k(v_n-v)) \le ||\nabla\phi_k(v_n-v)||^2 + \int q_k^+|\phi_k(v_n-v)|^2 + ||\phi_k(v_n - v)||^2. \quad (2.9)$$

(2.9), together with (2.6), (2.7) and (2.8), yields

$$\phi_k v \in Q(T_k) \quad (2.10)$$

and

$$||\phi_k v_n - \phi_k v||_{t_k} \longrightarrow 0 \quad (n \longrightarrow \infty).$$

Since, by (C_3), we have

$$||\phi_k v_n||_t^2 = ||\phi_k v_n||^2_{t_k} - ||\phi_k v_n||^2 \quad (n \in \mathbb{N}).$$

($||\cdot||_t$ denotes the form norm of T_F).

We can conclude

$$||\phi_k(v_n - v_m)||_t \longrightarrow 0 \quad (n,m \longrightarrow \infty)$$

and thus

$$\phi_k v \in Q(T_F). \quad (2.11)$$

(2.10) and (2.11) prove the first part of Step 3.

Now, let $u \in D(T_F)$ and $v \in Q(T_K)$. Then $\phi_k v \in Q(T_k) \cap Q(T_F)$ as proved above and there exist sequences $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_m\}_{m\in\mathbb{N}}$ in $C_o^\infty(\mathbb{R}^n)$ such that

$$||u_j - u||_t \longrightarrow 0 \text{ and } ||v_m - v||_{t_k} \longrightarrow 0 \quad (j,m \longrightarrow \infty).$$

Thus,

$$(T_F u,\phi_k v) = \lim_{j,m\to\infty} (T_F u_j,\phi_k v_m) = \lim_{j,m\to\infty} (Lu_j,\phi_k v_m). \quad (2.12)$$

Using (C_3), we have

$$(Lu_j,\phi_k v_m) = (L_k\phi_k u_j,v_m) - 2(u_j,\nabla\phi_k\nabla v_m) - (u_j,v_m\Delta\phi_k). \quad (2.13)$$

(2.12) and (2.13) yields, for a suitable constant $\gamma \in \mathbb{R}$,

$$\lim_{j\to\infty} (\phi_k u_j,v)_{t_k} = \lim_{j\to\infty} (T_k\phi_k u_j/v) = (T_F u,\phi_k v) + 2(u,\nabla\phi_k\nabla v) + \gamma(u,v).$$

Thus the limit of $\{\phi_k u_j\}_{j\in\mathbb{N}}$ exists weakly in the Hilbert space $Q(T_k)$ and since

$$||\phi_k u_j - \phi_k u|| \longrightarrow 0 \quad (j \longrightarrow \infty),$$

we conclude

$$\phi_k u \in Q(T_k),$$

which proves the second part of Step 3.

Step 4. We show $T_F \subseteq T_{max}$.

Let $u \in D(T_F)$. Then, for $k \in \mathbb{N}$ from Step 3, we know $\phi_k u \in Q(T_k)$ and therefore, by (C_5),

$$\phi_k u \in H^1(\mathbb{R}^n) \cap Q(q_k^+).$$

As in Step 1, we conclude that

$$qu \in L^1_{loc}(\mathbb{R}^n).$$

Thus $u \in D(L)$ and, from

$$T_F u = Lu \in L^2(\mathbb{R}^n),$$

we have

$$u \in D(T_{max}) \text{ and } T_F u = T_{max} u.$$

Step 5. We show $T_F = T_{max}$.

In view of Step 4, we have to show

$$D(T_{max}) \subseteq D(T_F).$$

Let $v \in D(T_{max})$ and

$$v' := (T_F + 1)^{-1} (T_{max} + 1)v.$$

Thus, $v' \in D(T_{max})$ by Step 4 and

$$(T_{max} + 1)v = (T_F + 1)v' = (T_{max} + 1)v'.$$

With

$$u := v - v' \in D(T_{max}) ,$$

we conclude $(T_{max} + 1)u = 0$ and therefore

$$((T_{max} + 1)u,w) = 0 \quad \text{for} \quad w \in C_o^\infty(\mathbb{R}^n). \tag{2.14}$$

We will show that (2.14) implies $u = 0$; then, Step 5 will be proven.

We argue in the following as Simander does in [1]. Since T_{max} is a real operator, we may assume u to be real-valued. From Step 1, we know that $\phi_k u \in Q(T_k)$ and thus, by (C_3) and the semiboundedness of T_k,

$$\phi_k u \in H^1(\mathbb{R}^n) \cap Q(q^+) \cap Q(q^-).$$

If we replace w in (2.14) by $\phi_k^2 w$, we get, after some partial integrations,

$$(\nabla\phi_k u,\nabla\phi_k w) + (q^+\phi_k u/\phi_k w) - (q^-\phi_k u/\phi_k w) + (\phi_k u,\phi_k w) = ((\nabla\phi_k)^2 u,w) - ((u\nabla w - w\nabla u,\phi_k\nabla\phi_k). \tag{2.15}$$

Since

$$u \in H^1_{loc}(\mathbb{R}^n) \quad \text{and} \quad q^\pm|\phi_k u| \in L^1(\mathbb{R}^n),$$

we can, by using an approximation, replace w in (2.15) by $u^{(m)} \in H^1_{loc}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, defined by

$$u^{(m)} := \begin{cases} u(x) & \text{for } |u(x)| \le m \\ m\,\text{sign}(u(x)) & \text{for } |u(x)| > m \end{cases}$$

for $m \in \mathbb{N}$.

Then, the limits of both sides of (2.15) exist and we get

$$(\nabla\phi_k u,\nabla\phi_k u) + (q^+\phi_k u/\phi_k u) - (q^-\phi_k u/\phi_k u) + (\phi_k u,\phi_k u) = ((\nabla\phi_k)^2 u,\phi_k u) + ((u\nabla\phi_k-\phi_k\nabla u),\phi_k\nabla\phi_k). \tag{2.16}$$

Since, from Step 2, we know $\phi_k u \in Q(T_F)$, we conclude from (2.16) and from $T_F+1\ge 1$ that

$$||\phi_k u||^2 \le ((T_F + 1)\phi_k u/\phi_k u) = \text{RHS of (2.16)} \longrightarrow 0 \quad (k \longrightarrow \infty).$$

Thus u = 0, which proves Step 5.

Since T_F is self-adjoint by Step 5, the theorem is proven.

COROLLARY 1. Let $k \in \mathbb{N}$. Assume (C_1) and (C_2). Set $q_k^+ := q^+$;

$$q_k^-(x) := \begin{cases} q^-(x) & \text{if } |x| \le k \\ 0 & \text{if } |x| > k \end{cases}$$

$q_k := q_k^+ - q_k^-$;

and define T_k and T_{max} as in (1.6) and (1.2). Assume additionally

$$T_k \text{ is self-adjoint} \tag{C_4}$$

and

$$\text{there exist } 0 \le a_k < 1 \text{ and } b_k \ge 0 \text{ such that} \tag{C_5}$$

$$|(q_k^- w/w)| \le a_k(-\Delta w,w) + b_k||w||^2, \quad w \in C_o^\infty(\mathbb{R}^n). \tag{2.17}$$

Then T_{max} is self-adjoint.

PROOF. (C_3) holds trivially. From (2.17), we deduce

$$(-\Delta w,w) + (q_k^+ w/w) \le \frac{1}{1-a_k}\{(T_k w/w) + (b_k + 1)||w||^2\}$$

which implies (2.1). Since $C_o^\infty(\mathbb{R}^n)$ is dense in $H^1(\mathbb{R}^n) \cap Q(q^+)$ in the sense of the norm $||\cdot||_{t_+}$ (as we know from [5], see Step 2 above), (2.17) implies that $C_o^\infty(\mathbb{R}^n)$ is a form core of T_k. Therefore, (C_5) holds and, by the theorem, self-adjointness of T_{max} follows.

Note that, for $q \in L^2_{loc}(\mathbb{R}^n)$, Corollary 1 implies the result of Simader [1] since then $T_{min}^* = T_{max}$ where

$$T_{min} := T_{max}|_{C_o^\infty(\mathbb{R}^n)}.$$

COROLLARY 2. Let $k \in \mathbb{N}$. Assume (C_1) and (C_2). Set

$$q_k(x) := \begin{cases} q(x) & \text{if } |x| \le k \\ 0 & \text{if } |x| > k \end{cases}$$

and define T_k and T_{max} as in (1.6) and (1.2). Assume additionally (C_4) and (C_5). Then T_{max} is self-adjoint. The proof follows immediately from the theorem.

In the case $q \in L^2_{loc}(\mathbb{R}^n)$, Corollary 2 implies the result of Brézis [2] by the same arguments as above. We also should note that, if $q_k^+ = q^+$ and $q_k^- = q^-$ $(k \in \mathbb{N})$ and if q^- is form-bounded relative to the form of $(-\Delta + q^+)$ with bound < 1, our theorem is Kato's [3] result for the semibounded case. In fact, our proof is a variant of Kato's proof of his main theorem in [3].

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