

## Research Article

# A Note on the Generalized $q$ -Bernoulli Measures with Weight $\alpha$

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We discuss a new concept of the  $q$ -extension of Bernoulli measure. From those measures, we derive some interesting properties on the generalized  $q$ -Bernoulli numbers with weight  $\alpha$  attached to  $\chi$ .

## 1. Introduction

Let  $p$  be a fixed prime number. Throughout this paper  $\mathbb{Z}_p$ ,  $\mathbb{Q}_p$ , and  $\mathbb{C}_p$  will, respectively, denote the ring of  $p$ -adic rational integers, the field of  $p$ -adic rational numbers, and the completion of algebraic closure of  $\mathbb{Q}_p$ . Let  $\mathbb{N}$  be the set of natural numbers and  $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ . Let  $\nu_p$  be the normalized exponential valuation of  $\mathbb{C}_p$  with  $|p|_p = p^{-\nu_p(p)} = 1/p$  (see [1–14]).

When we talk of  $q$ -extension,  $q$  is variously considered as an indeterminate, a complex number  $q \in \mathbb{C}$ , or a  $p$ -adic number  $q \in \mathbb{C}_p$ . Throughout this paper we assume that  $q \in \mathbb{C}_p$  with  $|1 - q|_p < 1$ , and we use the notation of  $q$ -number as

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (1.1)$$

(see [1–14]). Thus, we note that  $\lim_{q \rightarrow 1} [x]_q = x$ .

In [2], Carlitz defined a set of numbers  $\xi_k = \xi_k(q)$  inductively by

$$\xi_0 = 1, \quad (q\xi + 1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.2)$$

with the usual convention of replacing  $\xi^k$  by  $\xi_k$ .

These numbers are  $q$ -extension of ordinary Bernoulli numbers  $B_k$ . But they do not remain finite when  $q = 1$ . So he modified (1.2) as follows:

$$\beta_{0,q} = 1, \quad q(q\beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.3)$$

with the usual convention of replacing  $\beta^k$  by  $\beta_{k,q}$ .

The numbers  $\beta_{k,q}$  are called the  $k$ -th Carlitz  $q$ -Bernoulli numbers.

In [1], Carlitz also considered the extended Carlitz's  $q$ -Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \quad q^h(q\beta^h + 1)^k - \beta_{k,q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases} \quad (1.4)$$

with the usual convention of replacing  $(\beta^h)^k$  by  $\beta_{k,q}^h$ .

Recently, Kim considered  $q$ -Bernoulli numbers, which are different extended Carlitz's  $q$ -Bernoulli numbers, as follows: for  $\alpha \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ ,

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q(q^\alpha \tilde{\beta}^{(\alpha)} + 1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases} \quad (1.5)$$

with the usual convention of replacing  $(\tilde{\beta}^{(\alpha)})^k$  by  $\tilde{\beta}_{k,q}^{(\alpha)}$  (see [3]).

The numbers  $\tilde{\beta}_{k,q}^{(\alpha)}$  are called the  $k$ -th  $q$ -Bernoulli numbers with weight  $\alpha$ .

For fixed  $d \in \mathbb{Z}_+$  with  $(p, d) = 1$ , we set

$$X = X_d = \lim_{\leftarrow \mathbb{N}} \left( \frac{\mathbb{Z}}{dp^N \mathbb{Z}} \right), \quad X_1 = \mathbb{Z}_p, \\ X^* = \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \quad (1.6)$$

$$a + dp^N \mathbb{Z}_p = \{x \in X \mid x \equiv a \pmod{dp^N}\},$$

where  $a \in \mathbb{Z}$  satisfies the condition  $0 \leq a < dp^N$ .

Let  $\text{UD}(\mathbb{Z}_p)$  be the space of uniformly differentiable functions on  $\mathbb{Z}_p$ . For  $f \in \text{UD}(\mathbb{Z}_p)$ , the  $p$ -adic  $q$ -integral on  $\mathbb{Z}_p$  is defined by Kim as follows:

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \quad (1.7)$$

(see [3, 4, 15, 16]). By (1.5) and (1.7), the Witt's formula for the  $q$ -Bernoulli numbers with weight  $\alpha$  is given by

$$\int_{\mathbb{Z}_p} [x]_{q^\alpha}^n d\mu_q(x) = \tilde{\beta}_{n,q}^{(\alpha)}, \quad \text{where } n \in \mathbb{Z}_+. \tag{1.8}$$

The  $q$ -Bernoulli polynomials with weight  $\alpha$  are also defined by

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \sum_{l=0}^n \binom{n}{l} [x]_{q^\alpha}^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(\alpha)}. \tag{1.9}$$

From (1.7), (1.8), and (1.9), we can derive the Witt's formula for  $\tilde{\beta}_{n,q}^{(\alpha)}(x)$  as follows:

$$\int_{\mathbb{Z}_p} [x+y]_{q^\alpha}^n d\mu_q(y) = \tilde{\beta}_{n,q}^{(\alpha)}(x), \quad \text{where } n \in \mathbb{Z}_+. \tag{1.10}$$

For  $n \in \mathbb{Z}_+$  and  $d \in \mathbb{N}$ , the distribution relation for the  $q$ -Bernoulli polynomials with weight  $\alpha$  are known that

$$\tilde{\beta}_{n,q}^{(\alpha)}(x) = \frac{[d]_{q^\alpha}^n}{[d]_q} \sum_{a=0}^{d-1} q^a \tilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{x+a}{d}\right), \tag{1.11}$$

(see [3]). Recently, several authors have studied the  $p$ -adic  $q$ -Euler and Bernoulli measures on  $\mathbb{Z}_p$  (see [8, 9, 11, 13, 14]). The purpose of this paper is to construct  $p$ -adic  $q$ -Bernoulli distribution with weight  $\alpha$  (=  $p$ -adic  $q$ -Bernoulli unbounded measure with weight  $\alpha$ ) on  $\mathbb{Z}_p$  and to study their integral representations. Finally, we construct the generalized  $q$ -Bernoulli numbers with weight  $\alpha$  and investigate their properties related to  $p$ -adic  $q$ - $L$ -functions.

## 2. $p$ -Adic $q$ -Bernoulli Distribution with Weight $\alpha$

Let  $X$  be any compact-open subset of  $\mathbb{Q}_p$ , such as  $\mathbb{Z}_p$  or  $\mathbb{Z}_p^*$ . A  $p$ -adic distribution  $\mu$  on  $X$  is defined to be an additive map from the collection of compact open set in  $X$  to  $\mathbb{Q}_p$ :

$$\mu\left(\bigcup_{k=1}^n U_k\right) = \sum_{k=1}^n \mu(U_k) \text{ (additivity)}, \tag{2.1}$$

where  $\{U_1, U_2, \dots, U_n\}$  is any collection of disjoint compact opensets in  $X$ .

The set  $\mathbb{Z}_p$  has a topological basis of compact open sets of the form  $a + p^n \mathbb{Z}_p$ .

Consequently, if  $U$  is any compact open subset of  $\mathbb{Z}_p$ , it can be written as a finite disjoint union of sets

$$U = \bigcup_{j=1}^k (a_j + p^n \mathbb{Z}_p), \tag{2.2}$$

where  $n \in \mathbb{N}$  and  $a_1, a_2, \dots, a_k \in \mathbb{Z}$  with  $0 \leq a_i < p^n$  for  $i = 1, 2, \dots, k$ .

Indeed, the  $p$ -adic ball  $a + p^n \mathbb{Z}_p$  can be represented as the union of smaller balls

$$a + p^n \mathbb{Z}_p = \bigcup_{b=0}^{p-1} (a + bp^n + p^{n+1} \mathbb{Z}_p). \quad (2.3)$$

**Lemma 2.1.** *Every map  $\mu$  from the collection of compact-open sets in  $X$  to  $\mathbb{Q}_p$  for which*

$$\mu(a + p^N \mathbb{Z}_p) = \bigcup_{b=0}^{p-1} (a + bp^N + dp^{N+1} \mathbb{Z}_p) \quad (2.4)$$

holds whenever  $a + p^N \mathbb{Z}_p \subset X$ , extends to a  $p$ -adic distribution (=  $p$ -adic unbounded measure) on  $X$ .

Now we define a map  $\mu_{k,q}^{(\alpha)}$  on the balls in  $\mathbb{Z}_p$  as follows:

$$\mu_{k,q}^{(\alpha)}(a + p^n \mathbb{Z}_p) = \frac{[p^n]_q^k}{[p^n]_q} q^a f_{k,q^{p^n}}^{(\alpha)} \left( \frac{\{a\}_n}{p^n} \right), \quad (2.5)$$

where  $\{a\}_n$  is the unique number in the set  $\{0, 1, 2, \dots, p^n - 1\}$  such that  $\{a\}_n \equiv a \pmod{p^n}$ .

If  $a \in \{0, 1, 2, \dots, p^n - 1\}$ , then

$$\begin{aligned} \sum_{b=0}^{p-1} \mu_{k,q}^{(\alpha)}(a + bp^n + p^{n+1} \mathbb{Z}_p) &= \sum_{b=0}^{p-1} \frac{[p^{n+1}]_q^k}{[p^{n+1}]_q} q^{a+bp^n} f_{k,q^{p^{n+1}}}^{(\alpha)} \left( \frac{a + bp^n}{p^{n+1}} \right) \\ &= q^a \frac{[p^n]_q^k}{[p^n]_q} \frac{[p]_q^k}{[p]_q^{p^n}} \sum_{b=0}^{p-1} q^{bp^n} f_{k,(q^{p^n})^p}^{(\alpha)} \left( \frac{(a/p^n) + b}{p} \right). \end{aligned} \quad (2.6)$$

From (2.6), we note that  $\mu_{k,q}^{(\alpha)}$  is  $p$ -adic distribution on  $\mathbb{Z}_p$  if and only if

$$\frac{[p]_q^k}{[p]_q^{p^n}} \sum_{b=0}^{p-1} q^{bp^n} f_{k,(q^{p^n})^p}^{(\alpha)} \left( \frac{(a/p^n) + b}{p} \right) = f_{k,q^{p^n}}^{(\alpha)} \left( \frac{a}{p^n} \right). \quad (2.7)$$

**Theorem 2.2.** *Let  $\alpha \in \mathbb{N}$  and  $k \in \mathbb{Z}_+$ . Then we see that  $\mu_{k,q}^{(\alpha)}(a + p^n \mathbb{Z}_p)$  is  $p$ -adic distribution on  $\mathbb{Z}_p$  if and only if*

$$\frac{[p]_q^k}{[p]_q^{p^n}} \sum_{b=0}^{p-1} q^{bp^n} f_{k,(q^{p^n})^p}^{(\alpha)} \left( \frac{(a/p^n) + b}{p} \right) = f_{k,q^{p^n}}^{(\alpha)} \left( \frac{a}{p^n} \right). \quad (2.8)$$

One sets

$$f_{k,q^{p^n}}^{(\alpha)}(x) = \tilde{\beta}_{k,q^{p^n}}^{(\alpha)}(x). \quad (2.9)$$

From (2.5) and (2.9), one gets

$$\mu_{k,q}^{(\alpha)}(a + p^n \mathbb{Z}_p) = \frac{[p^n]_q^k}{[p^n]_q} q^a \tilde{\beta}_{k,q^{p^n}}^{(\alpha)}\left(\frac{a}{p^n}\right). \quad (2.10)$$

By (1.11), (2.10), and Theorem 2.2, we obtain the following theorem.

**Theorem 2.3.** Let  $\mu_{k,q}^{(\alpha)}$  be given by

$$\mu_{k,q}^{(\alpha)}(a + dp^N \mathbb{Z}_p) = \frac{[dp^N]_q^k}{[dp^N]_q} q^a \tilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right). \quad (2.11)$$

Then  $\mu_{k,q}^{(\alpha)}$  extends to a  $\mathbb{Q}(q)$ -valued distribution on the compact open sets  $U \subset X$ .

From (2.11), one notes that

$$\begin{aligned} \int_X d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{dp^{N-1}} \mu_{k,q}^{(\alpha)}(x + dp^N \mathbb{Z}_p) \\ &= \lim_{N \rightarrow \infty} \frac{[dp^N]_q^k}{[dp^N]_q} \sum_{a=0}^{dp^{N-1}} q^a \tilde{\beta}_{k,q^{dp^N}}^{(\alpha)}\left(\frac{a}{dp^N}\right). \end{aligned} \quad (2.12)$$

By (1.11) and (2.12), one gets

$$\int_X d\mu_{k,q}^{(\alpha)}(x) = \tilde{\beta}_{k,q}^{(\alpha)}. \quad (2.13)$$

Therefore, we obtain the following theorem.

**Theorem 2.4.** For  $\alpha \in \mathbb{N}$  and  $k \in \mathbb{Z}_+$ , one has

$$\int_X d\mu_{k,q}^{(\alpha)}(x) = \tilde{\beta}_{k,q}^{(\alpha)}. \quad (2.14)$$

Let  $\chi$  be Dirichlet character with conductor  $d \in \mathbb{N}$ . Then one defines the generalized  $q$ -Bernoulli numbers attached to  $\chi$  as follows:

$$\begin{aligned} \tilde{\beta}_{n,\chi,q}^{(\alpha)} &= \int_X \chi(x) [x]_q^n d\mu_q(x) \\ &= \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \tilde{\beta}_{n,q^d}^{(\alpha)}\left(\frac{a}{d}\right). \end{aligned} \quad (2.15)$$

From (2.11) and (2.15), one can derive the following equation;

$$\begin{aligned}
\int_X \chi(x) d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N \rightarrow \infty} \sum_{x=0}^{dp^{N-1}} \chi(x) \mu_{k,q}^{(\alpha)}(x + dp^N \mathbb{Z}_p) \\
&= \lim_{N \rightarrow \infty} \frac{[dp^N]_q^k}{[dp^N]_q} \sum_{x=0}^{dp^{N-1}} \chi(x) q^x \tilde{\beta}_{k,q^{dp^N}}^{(\alpha)} \left( \frac{x}{dp^N} \right) \\
&= \frac{[d]_q^k}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \left\{ \lim_{N \rightarrow \infty} \frac{[p^N]_q^{kd}}{[p^N]_q^{qd}} \sum_{x=0}^{p^{N-1}} q^{dx} \tilde{\beta}_{k,q^{dp^N}}^{(\alpha)} \left( \frac{(a/d) + x}{p^N} \right) \right\} \\
&= \frac{[d]_q^k}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \tilde{\beta}_{k,q^d}^{(\alpha)} \left( \frac{a}{d} \right) = \tilde{\beta}_{k,X,q'}^{(\alpha)}, \\
\int_{pX} \chi(x) d\mu_{k,q}^{(\alpha)}(x) &= \lim_{N \rightarrow \infty} \frac{[dp^{N+1}]_q^k}{[dp^{N+1}]_q} \sum_{x=0}^{dp^{N-1}} \chi(px) q^{px} \tilde{\beta}_{k,q^{dp^{N+1}}}^{(\alpha)} \left( \frac{px}{dp^{N+1}} \right) \\
&= \frac{[p]_q^k}{[p]_q} \frac{[d]_q^k}{[d]_q^{p\alpha}} \sum_{a=0}^{d-1} \chi(pa) q^{pa} \lim_{N \rightarrow \infty} \frac{[p^N]_q^{kdp\alpha}}{[p^N]_q^{dp}} \sum_{x=0}^{p^{N-1}} q^{pdx} \tilde{\beta}_{k,q^{pdp^N}}^{(\alpha)} \left( \frac{p(xd+a)}{pdp^N} \right) \\
&= \frac{[p]_q^k}{[p]_q} \frac{[d]_q^k}{[d]_q^{p\alpha}} \sum_{a=0}^{d-1} \chi(p) \chi(a) q^{pa} \tilde{\beta}_{k,q^{pd}}^{(\alpha)} \left( \frac{a}{d} \right) = \chi(p) \frac{[p]_q^k}{[p]_q} \tilde{\beta}_{k,X,q^p}^{(\alpha)}.
\end{aligned} \tag{2.16}$$

For  $\beta (\neq 1) \in X^*$ , one has

$$\begin{aligned}
\int_{pX} \chi(x) d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{p}{\beta}\right) \frac{[p]_q^{k/\beta}}{[p]_q^{1/\beta}} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}, \\
\int_X \chi(x) d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{1}{\beta}\right) \tilde{\beta}_{k,X,q^{1/\beta}}^{(\alpha)}.
\end{aligned} \tag{2.17}$$

Therefore, we obtain the following theorem.

**Theorem 2.5.** For  $\beta (\neq 1) \in X^*$ , one has

$$\begin{aligned}
\int_X \chi(x) d\mu_{k,q}^{(\alpha)}(x) &= \tilde{\beta}_{k,X,q'}^{(\alpha)}, \\
\int_{pX} \chi(x) d\mu_{k,q}^{(\alpha)}(x) &= \chi(p) \frac{[p]_q^k}{[p]_q} \tilde{\beta}_{k,X,q^p}^{(\alpha)},
\end{aligned}$$

$$\begin{aligned} \int_{pX} \chi(x) d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{p}{\beta}\right) \frac{[p]_{q^{\alpha/\beta}}^k}{[p]_{q^{1/\beta}}} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}, \\ \int_X \chi(x) d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \chi\left(\frac{1}{\beta}\right) \tilde{\beta}_{k,X,q^{1/\beta}}^{(\alpha)}. \end{aligned} \tag{2.18}$$

Define

$$\mu_{k,\beta,q}^{(\alpha)}(U) = \mu_{k,q}^{(\alpha)}(U) - \beta^{-1} \frac{[\beta^{-1}]_{q^\alpha}^k}{[\beta^{-1}]_q} \mu_{k,q^{1/\beta}}^{(\alpha)}(\beta U). \tag{2.19}$$

By a simple calculation, one gets

$$\begin{aligned} \int_{X^*} \chi(x) d\mu_{k,\beta,q}^{(\alpha)}(x) &= \int_X \chi(x) d\mu_{k,q}^{(\alpha)}(x) - \beta^{-1} \frac{[\beta^{-1}]_{q^\alpha}^k}{[\beta^{-1}]_q} \int_{pX} \chi(x) \mu_{k,q^{1/\beta}}^{(\alpha)}(x) \\ &= \tilde{\beta}_{k,X,q}^{(\alpha)} - \chi(p) \frac{[p]_{q^\alpha}^k}{[p]_q} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}, \\ \frac{[\beta^{-1}]_{q^\alpha}^k}{[\beta^{-1}]_q} \int_{X^*} \chi(x) d\mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) &= \frac{[1/\beta]_{q^\alpha}^k}{[1/\beta]_q} \chi\left(\frac{1}{\beta}\right) \tilde{\beta}_{k,X,q^{1/\beta}}^{(\alpha)} \\ &\quad - \chi\left(\frac{p}{\beta}\right) \frac{[p/\beta]_{q^\alpha}^k}{[p/\beta]_q} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}. \end{aligned} \tag{2.20}$$

By (2.19) and (2.20), one gets

$$\begin{aligned} \int_{X^*} \chi(x) d\mu_{k,\beta,q}^{(\alpha)}(\beta x) &= \int_X \chi(x) d\mu_{k,q}^{(\alpha)}(x) - \beta^{-1} \frac{[\beta^{-1}]_{q^\alpha}^k}{[\beta^{-1}]_q} \int_{pX} \chi(x) \mu_{k,q^{1/\beta}}^{(\alpha)}(\beta x) \\ &= \tilde{\beta}_{k,X,q}^{(\alpha)} - \chi(p) \frac{[p]_{q^\alpha}^k}{[p]_q} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)} - \frac{1}{\beta} \frac{[1/\beta]_{q^\alpha}^k}{[1/\beta]_q} \chi\left(\frac{1}{\beta}\right) \tilde{\beta}_{k,X,q^{1/\beta}}^{(\alpha)} \\ &\quad + \chi\left(\frac{p}{\beta}\right) \frac{[p/\beta]_{q^\alpha}^k}{[p/\beta]_q} \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}. \end{aligned} \tag{2.21}$$

Now one defines the operator  $\chi^y = \chi^{y,k,\alpha;q}$  on  $f(q)$  by

$$\chi^y f(q) = \chi^{y,k,\alpha;q} f(q) = \frac{[y]_{q^\alpha}^k}{[y]_q} \chi(y) f(q^y). \tag{2.22}$$

Thus, by (2.22), one gets

$$\begin{aligned}
 \chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q} f(q) &= \chi^{x,k,\alpha;q} \frac{[y]_q^k}{[y]_q} \chi(y) f(q^y) \\
 &= \frac{[y]_q^k}{[y]_q} \chi(y) \chi(x) \frac{[y]_q^k}{[y]_q} \chi(y) f(q^{xy}) \\
 &= \frac{[xy]_q^k}{[xy]_q} \chi(xy) f(q^{xy}) \\
 &= \chi^{xy,k,\alpha;q} f(q) \\
 &= \chi^{xy} f(q).
 \end{aligned} \tag{2.23}$$

Let us define  $\chi^x \chi^y = \chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q}$ . Then one has

$$\chi^x \chi^y = \chi^{xy}. \tag{2.24}$$

From the definition of  $\chi^x$ , one can easily derive the following equation;

$$(1 - x^p) \left( 1 - \frac{1}{\beta} x^{1/\beta} \right) = 1 - \frac{1}{\beta} x^{1/\beta} - x^p + \frac{1}{\beta} x^{p/\beta}. \tag{2.25}$$

Let  $f(q) = \tilde{\beta}_{k,X,q}^{(\alpha)}$ . Then one gets

$$\begin{aligned}
 (1 - x^p) \left( 1 - \frac{1}{\beta} x^{1/\beta} \right) \tilde{\beta}_{k,X,q}^{(\alpha)} &= \tilde{\beta}_{k,X,q}^{(\alpha)} - \frac{1}{\beta} \frac{[1/\beta]_q^k}{[1/\beta]_q} \chi \left( \frac{1}{\beta} \right) \tilde{\beta}_{k,X,q}^{(\alpha)} - \frac{[p]_q^k}{[p]_q} \chi(p) \tilde{\beta}_{k,X,q^p}^{(\alpha)} \\
 &\quad + \frac{1}{\beta} \frac{[p/\beta]_q^k}{[p/\beta]_q} \chi \left( \frac{p}{\beta} \right) \tilde{\beta}_{k,X,q^{p/\beta}}^{(\alpha)}.
 \end{aligned} \tag{2.26}$$

By (2.21) and (2.26), one obtains the following equation:

$$\int_{X^*} \chi(x) d\mu_{k,\beta,q}^{(\alpha)}(\beta x) = (1 - x^p) \left( 1 - \frac{1}{\beta} x^{1/\beta} \right) \tilde{\beta}_{k,X,q}^{(\alpha)}, \tag{2.27}$$

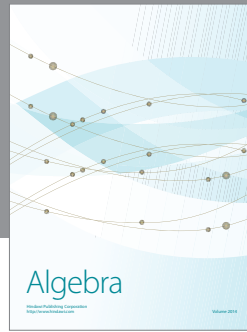
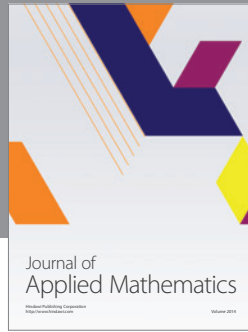
where  $\beta (\neq 1) \in X^*$ .

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