Hindawi
Complexity
Volume 2017, Article ID 3720471, 7 pages
https://doi.org/10.1155/2017/3720471

Research Article

Fundamental Results of Conformable Sturm-Liouville Eigenvalue Problems

Mohammed Al-Refai[1] and Thabet Abdeljawad[2]

[1]*Department of Mathematical Sciences, UAE University, P.O. Box 15551, Al Ain, Abu Dhabi, UAE*
[2]*Department of Mathematics and General Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia*

Correspondence should be addressed to Mohammed Al-Refai; m_alrefai@uaeu.ac.ae

Received 7 May 2017; Accepted 6 August 2017; Published 14 September 2017

Academic Editor: Abdelalim Elsadany

We suggest a regular fractional generalization of the well-known Sturm-Liouville eigenvalue problems. The suggested model consists of a fractional generalization of the Sturm-Liouville operator using conformable derivative and with natural boundary conditions on bounded domains. We establish fundamental results of the suggested model. We prove that the eigenvalues are real and simple and the eigenfunctions corresponding to distinct eigenvalues are orthogonal and we establish a fractional Rayleigh Quotient result that can be used to estimate the first eigenvalue. Despite the fact that the properties of the fractional Sturm-Liouville problem with conformable derivative are very similar to the ones with the classical derivative, we find that the fractional problem does not display an infinite number of eigenfunctions for arbitrary boundary conditions. This interesting result will lead to studying the problem of completeness of eigenfunctions for fractional systems.

1. Introduction and Preliminaries

Fractional calculus is old as the Newtonian calculus [1–3]. The name fractional was given to express the integration and differentiation up to arbitrary order. Traditionally, there are two approaches to define the fractional derivative. The first approach, Riemann-Liouville approach, is to iterate the integral with respect to certain weight function and replace the iterated integral by single integral through Leibniz-Cauchy formula and then replace the factorial function by the Gamma function. In this approach, the arbitrary order Riemann-Liouville results from the integrating measure dt and the Hadamard fractional integral results from the integrating measure dt/t. The second approach, Grünwald-Letinkov approach, is to iterate the limit definition of the derivative to get a quantity with certain binomial coefficient and then fractionalize by using the Gamma function instead of the factorial in the binomial coefficient. In case of the Riemann-Liouville and Caputo fractional derivatives, a singular kernel of the form $(t-s)^{-\alpha}$ is generated for $0<\alpha<1$ to reflect the nonlocality and the memory in the fractional operator. Through history, hundreds of researchers did their best to develop the theory of fractional calculus and generalize it, either by obtaining more general fractional derivatives with different kernels or by defining the fractional operator on different time scales such as the discrete fractional difference operators (see [4–7] and the references therein) and q-fractional operators (see [8] and the references therein).

In 2014 [9], Khalil et al. introduced the so-called conformable fractional derivative by modifying the limit definition of the derivative by inserting the multiple $t^{1-\alpha}$, $0<\alpha<1$ inside the definition. The word fractional there was used to express the derivative of arbitrary order although no memory effect exists inside the corresponding integral inverse operator. This conformable (fractional) derivative seems to be kind of local derivative without memory. An interesting application of the conformable fractional derivative in Physics was discussed in [10], where it has been used to formulate an Action Principle for particles under frictional forces. Despite the many nice properties the conformable derivative has, it has the drawbacks that when α tends to zero we do not obtain the original function and the conformable integrals inverse operators are free of memory and do not have a semigroup property. It is most likely to call them conformable derivatives or local derivatives of arbitrary order. In connection with this, at the end of reference [11], the author asked whether it is possible to fractionalize the conformable (fractional) derivative by using conformable (fractional) integrals of order $0 < \alpha \leq 1$ or by iterating the conformable derivative. The first part, Riemann-Liouville approach, was answered in [12, 13], where the author iterated the (conformable) integral with weight $t^{\rho-1}$, $\rho \neq 0$ to define generalized fractional integrals and derivatives that unify Riemann-Liouville fractional integrals ($\rho = 1$) and derivatives together with Hadamard fractional integrals and derivatives. Actually, the limiting case of that generalization is when $\rho \to 0^+$ leads to Hadamard type. However, the Grünwald-Letinkov approach for conformable derivatives is still open. The conformable time-scale fractional calculus of order $0 < \alpha < 1$ is introduced in [14] and has been used to develop the fractional differentiation and fractional integration. After then, many authors got interested in this type of derivatives for their many nice behaviors [10, 15–18]. Motivated by the need of some new fractional derivatives with nice properties and that can be applied to more real world modeling, some authors introduced very recently new kinds of fractional derivatives whose kernel is nonsingular. For the fractional derivatives with exponential kernels we refer to [19]. For fractional derivatives of nonsingular Mittag-Leffler functions we refer to [20–22].

Motivated, as mentioned above, with the need of new fractional derivatives with nice properties we study in this article the eigenvalue problems of Sturm-Liouville into conformable (fractional) calculus. Recently, there are several analytical studies devoted to fractional Sturm-Liouville eigenvalue problems; see [23–27]. In these studies some of the well-known results of the Sturm-Liouville problems are extended to the fractional ones with left- and right-sided fractional derivatives of Riemann-Liouville and Caputo and Riesz derivatives. These results include orthogonality and completeness of eigenfunctions and countability of the real eigenvalues. Another class of fractional eigenvalue problem with Caputo fractional derivative has been studied in [28] using maximum principles and method of upper and lower solutions.

For a function $f : (0, \infty) \to \mathbb{R}$ the (conformable) fractional derivative of order $0 < \alpha \leq 1$ of f at $t > 0$ was defined by

$$D_a^\alpha f(t) = \lim_{\epsilon \to 0} \frac{f\left(t + \epsilon (t-a)^{1-\alpha}\right) - f(t)}{\epsilon}, \tag{1}$$

and the fractional derivative at a is defined as $(D_a^\alpha f)(a) = \lim_{t \to a^+} (D_a^\alpha f)(t)$. The corresponding conformable (fractional) integral of order $0 < \alpha < 1$ and starting from a is defined by

$$\left(I_a^\alpha f\right)(x) = \int_a^x f(t)\, d\alpha(t) = \int_a^x f(t)(t-a)^{\alpha-1}\, dt. \tag{2}$$

It is to be noted that the author used this modified conformable integral in order to extend it to left-right concept and confirm it by the Q-operator and obtain a left-right integration by parts version. Otherwise the integral can be given by $(I_a^\alpha f)(x) = \int_a^x f(t) t^{\alpha-1} dt$. It was shown in [9, 11] that $(I_a^\alpha D_a^\alpha f)(x) = f(x) - f(a)$ and $(D_a^\alpha I_a^\alpha f)(x) = f(x)$. For the higher order case and other details such as the product rule, chain rule, and integration by parts, we refer the reader to [9, 11].

2. Main Results

In this paper we consider the fractional extension of the Sturm-Liouville eigenvalue problem

$$D_a^\alpha \left(p(x) D_a^\alpha y\right) + q(x) y = -\lambda w(x) y, \qquad \frac{1}{2} < \alpha \leq 1, \ a < x < b, \tag{3}$$

where $p, D_a^\alpha p, q$ and the weight functions w are continuous on (a,b), $p(x) > 0$, and $w(x) > 0$, on $[a,b]$, and the fractional derivative D_a^α is the conformable fractional derivative. We discuss (3) with boundary conditions

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0, \quad c_1^2 + c_2^2 > 0, \\ r_1 y(b) + r_2 y'(b) &= 0, \quad r_1^2 + r_2^2 > 0. \end{aligned} \tag{4}$$

We say that y is 2α-continuously differentiable on $[a,b]$, if $D_a^\alpha D_a^\alpha y$ is continuous on $[a,b]$, and $y \in C^{2\alpha}[a,b]$, if $y \in C^1[a,b]$ and is 2α-continuously differentiable on $[a,b]$.

Let

$$L(y, \alpha) = D_a^\alpha \left(p(x) D_a^\alpha y\right) + q(x) y; \tag{5}$$

then the fractional Sturm-Liouville eigenvalue problem (3) can be written as

$$L(y, \alpha) = -\lambda w(x) y. \tag{6}$$

The following is a generalized result of the well-known Lagrange identity.

Theorem 1 (fractional Lagrange identity). *Letting y_1, y_2 be 2α-continuously differentiable on $[a,b]$, then the following holds true:*

$$\begin{aligned} &\int_a^b \left(y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha)\right) d\alpha(x) \\ &\quad = \left[p(x)\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right)\right]\Big|_a^b. \end{aligned} \tag{7}$$

Proof. We have

$$\begin{aligned} &y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha) \\ &\quad = y_2 D_a^\alpha \left(p(x) D_a^\alpha y_1\right) + q(x) y_1 y_2 \\ &\qquad - y_1 D_a^\alpha \left(p(x) D_a^\alpha y_2\right) - q(x) y_1 y_2 \\ &\quad = y_2 D_a^\alpha \left(p(x) D_a^\alpha y_1\right) - y_1 D_a^\alpha \left(p(x) D_a^\alpha y_2\right). \end{aligned} \tag{8}$$

Using the integration by parts formula of the conformable fractional derivative [11], we have

$$\begin{aligned}\int_a^b &\left(y_2 D_a^\alpha \left(p(x) D_a^\alpha y_1\right) - y_1 D_a^\alpha \left(p(x) D_a^\alpha y_2\right)\right) d\alpha(x) \\ &= p(x) y_2 D_a^\alpha y_1 \Big|_a^b - \int_a^b p(x) D_a^\alpha y_1 D_a^\alpha y_2 d\alpha(x) \\ &\quad - p(x) y_1 D_a^\alpha y_2 \Big|_a^b + \int_a^b p(x) D_a^\alpha y_1 D_a^\alpha y_2 d\alpha(x) \\ &= \left[p(x)\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right)\right]\Big|_a^b,\end{aligned} \tag{9}$$

which proves the result. □

Proposition 2. *If $y \in C^1[0,1]$ and $y'(x_0) = 0$, for some $x_0 \in [a,b]$, then $(D_a^\alpha y)(x_0) = 0$.*

Proof. Since $y \in C^1[0,1]$, then $(D_a^\alpha y)(x) = (x-a)^{1-\alpha} y'(x)$, and the result follows for $a < x_0 \le b$. If $x_0 = a$, we have $(D_a^\alpha y)(a) = \lim_{x\to a^+}(x-a)^{1-\alpha} y'(x) = 0$. □

Proposition 3. *Let y_1 and y_2 in $C^1[a,b]$, which satisfy the boundary conditions (4); then it holds that*

$$\left[p(x)\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right)\right]\Big|_a^b = 0. \tag{10}$$

Proof. Since $y_1 \in C^1[a,b]$, then $D_a^\alpha y_1 = (x-a)^{1-\alpha} y_1'(x)$. Similarly, $D_a^\alpha y_2 = (x-a)^{1-\alpha} y_2'(x)$. We have

$$\begin{aligned}&\left[p(x)\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right)\right]\Big|_a^b \\ &\quad = p(b)\left(y_2(b)\left(D_a^\alpha y_1\right)(b) - y_1(b)\left(D_a^\alpha y_2\right)(b)\right) \\ &\qquad - p(a)\left(y_2(a)\left(D_a^\alpha y_1\right)(a) - y_1(a)\left(D_a^\alpha y_2\right)(a)\right).\end{aligned} \tag{11}$$

Since $c_1^2 + c_2^2 > 0$, and $r_1^2 + r_2^2 > 0$, we first assume that, without loss of generality, $c_1 \neq 0$ and $r_1 \neq 0$, and the proof of other cases will be obtained analogously. We have

$$\begin{aligned} y(a) &= -\frac{c_2}{c_1} y'(a), \\ y(b) &= -\frac{r_2}{r_1} y'(b).\end{aligned} \tag{12}$$

Thus,

$$\begin{aligned}&y_2(b)\left(D_a^\alpha y_1\right)(b) - y_1(b)\left(D_a^\alpha y_2\right)(b) = -\frac{r_2}{r_1} y_2'(b) \\ &\quad \cdot \left(D_a^\alpha y_1\right)(b) + \frac{r_2}{r_1} y_1'(b)\left(D_a^\alpha y_2(b)\right) \\ &\quad = -\frac{r_2}{r_1}\left(y_2'(b)(b-a)^{1-\alpha} y_1(b)\right. \\ &\quad \left. - y_1'(b)(b-a)^{1-\alpha} y_2'(b)\right) = 0.\end{aligned} \tag{13}$$

Analogously,

$$y_2(a)\left(D_a^\alpha y_1\right)(a) - y_1(a)\left(D_a^\alpha y_2\right)(a) = 0, \tag{14}$$

which proves the result. □

Definition 4. We say that f and g are α-orthogonal with respect to the weight function $\mu(x) \ge 0$, if

$$\int_a^b \mu(x) f(x) g(x) d\alpha(x) = 0. \tag{15}$$

Theorem 5. *The eigenfunctions of the fractional eigenvalue problem (3)-(4) corresponding to distinct eigenvalues are α-orthogonal with respect to the weight function $w(x)$.*

Proof. Let λ_1 and λ_2 be two distinct eigenvalues and y_1 and y_2 are the corresponding eigenfunctions. We have

$$L(y_1, \alpha) = -\lambda_1 w(x) y_1, \tag{16}$$

$$L(y_2, \alpha) = -\lambda_2 w(x) y_2. \tag{17}$$

Multiplying (16) by y_2 and (17) by y_1 and subtracting the two equations yield

$$y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha) = -\left(\lambda_1 - \lambda_2\right) w(x) y_1 y_2. \tag{18}$$

Performing the fractional integral I_a^α and using the fractional Lagrange identity we have

$$\begin{aligned}&-\left(\lambda_1 - \lambda_2\right)\int_a^b w(x) y_1 y_2 d\alpha(x) \\ &\quad = \int_a^b \left(y_2 L(y_1, \alpha) - y_1 L(y_2, \alpha)\right) d\alpha(x) \\ &\quad = \left[p(x)\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right)\right]\Big|_a^b = 0,\end{aligned} \tag{19}$$

by virtue of Proposition 3. Since $\lambda_1 \neq \lambda_2$, we have $\int_a^b w(x) y_1 y_2 d\alpha(x) = 0$, and the result is obtained. □

Theorem 6. *The eigenvalues of the fractional eigenvalue problem (3)-(4) are real.*

Proof. Let y be a solution to the fractional Sturm-Liouville eigenvalue problem (3)-(4). Taking the complex conjugate of (3)-(4) and using the fact that $p(x), q(x)$ and $w(x)$ are real valued functions, we have

$$\begin{aligned}L(\overline{y}, \alpha) &= D_a^\alpha\left(p(x)\overline{D_a^\alpha y}\right) + q(x)\overline{y} \\ &= -\lambda w(x)\overline{y}, \\ c_1\overline{y}(a) &+ c_2\overline{y'}(a) = 0, \\ r_1\overline{y}(b) &+ r_2\overline{y'}(b) = 0.\end{aligned} \tag{20}$$

Applying analogous steps to the proofs of Theorem 5 and Proposition 3 with $y_1 = y$ and $y_2 = \overline{y}$, we have

$$\begin{aligned}&-\left(\lambda - \overline{\lambda}\right)\int_a^b w(x)\left|y(x)\right|^2 d\alpha(x) \\ &\quad = \int_a^b \left(\overline{y} L(y, \alpha) - y L(\overline{y}, \alpha)\right) dx(\alpha) \\ &\quad = \left[p(x)\left(\overline{y} D_a^\alpha y - y\overline{D_a^\alpha y}\right)\right]\Big|_a^b = 0,\end{aligned} \tag{21}$$

and thus $\lambda = \overline{\lambda}$ which completes the proof. □

Definition 7. Let f and g be α-differentiable; the fractional Wronskian function is defined by

$$W_\alpha(f,g) = f D_a^\alpha g - g D_a^\alpha f. \tag{22}$$

Theorem 8. *Let y_1 and y_2 be 2α-continuously differentiable on $[a,b]$, and they are linearly independent solutions of (3); then*

$$W_\alpha(y_1,y_2) = \frac{W_\alpha(y_1,y_2)(a)\,p(a)}{p(x)}. \tag{23}$$

Proof. Applying the product rule one can easily verify that

$$D_a^\alpha W_\alpha(y_1,y_2) = y_1 D_a^\alpha D_a^\alpha y_2 - y_2 D_a^\alpha D_a^\alpha y_1. \tag{24}$$

Analogously, applying the product rule to (3) yields

$$D_a^\alpha D_a^\alpha y = -\frac{1}{p}\left(D_a^\alpha p D_a^\alpha y + (q+\lambda w)\,y\right). \tag{25}$$

Substituting the last equation in (24) yields

$$\begin{aligned} D_a^\alpha W_\alpha(y_1,y_2) &= -\frac{y_1}{p}\left(D_a^\alpha p D_a^\alpha y_2 + (q+\lambda w)\,y_2\right) \\ &\quad + \frac{y_2}{p}\left(D_a^\alpha p D_a^\alpha y_1 + (q+\lambda w)\,y_1\right) \\ &= \frac{D_a^\alpha p}{p}\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right) \\ &= -\frac{D_a^\alpha p}{p} W_\alpha(y_1,y_2). \end{aligned} \tag{26}$$

One can easily verify that the solution of the above fractional differential equation is

$$W_\alpha(y_1,y_2) = \frac{c}{p}, \tag{27}$$

where c is constant. Now, $W_\alpha(y_1,y_2)(a) = c/p(a)$, and thus $c = W_\alpha(y_1,y_2)(a)p(a)$, and hence the result. □

Theorem 9. *The eigenvalues of the fractional eigenvalue problem (3)-(4) are simple.*

Proof. Let y_1 and y_2 be two eigenfunctions for the same eigenvalue λ. From (18) we have

$$\begin{aligned} 0 &= y_2 L(y_1,\alpha) - y_1 L(y_2,\alpha) \\ &= y_2 D_a^\alpha\left(p(x) D_a^\alpha y_1\right) - y_1 D_a^\alpha\left(p(x) D_a^\alpha y_2\right) \\ &= y_2\left(D_a^\alpha p D_a^\alpha y_1 + p D_a^\alpha D_a^\alpha y_1\right) \\ &\quad - y_1\left(D_a^\alpha p D_a^\alpha y_2 + p D_a^\alpha D_a^\alpha y_2\right) \\ &= p\left(y_2 D_a^\alpha D_a^\alpha y_1 - y_1 D_a^\alpha D_a^\alpha y_2\right) \\ &\quad + D_a^\alpha p\left(y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right) \\ &= D_a^\alpha\left(p\left[y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right]\right). \end{aligned} \tag{28}$$

Thus

$$p\left[y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2\right] = c, \tag{29}$$

and since y_1 and y_2 satisfy the same boundary conditions, we have $c = 0$ and

$$y_2 D_a^\alpha y_1 - y_1 D_a^\alpha y_2 = 0. \tag{30}$$

Since $W_\alpha(y_1,y_2) = 0$, and y_1 and y_2 are both solutions to the fractional eigenvalue problem (3)-(4), then they are linearly dependent. □

Theorem 10 (fractional Rayleigh Quotient). *The eigenvalues λ of problem (3) satisfy*

$$\lambda = \frac{\int_a^b p\left(D_a^\alpha y\right)^2 d\alpha(x) - \int_a^b q y^2 d\alpha(x) - p y D_a^\alpha y\big|_a^b}{\int_a^b w y^2 d\alpha(x)} \tag{31}$$

Proof. Multiplying (3) by y and integrating yields

$$\begin{aligned} &\int_a^b y D_a^\alpha\left(p(x) D_a^\alpha y\right) d\alpha(x) + \int_0^1 q(x)\,y^2 d\alpha(x) \\ &\quad = -\lambda \int_a^b w(x)\,y^2 d\alpha(x). \end{aligned} \tag{32}$$

Integrating the first integral by parts we have

$$\begin{aligned} &p y D_a^\alpha y\big|_a^b - \int_a^b p\left(D_a^\alpha y\right)^2 d\alpha(x) + \int_a^b q(x)\,y^2 d\alpha(x) \\ &\quad = -\lambda \int_a^b w(x)\,y^2 d\alpha(x) \end{aligned} \tag{33}$$

which proves the result. □

Corollary 11. *Letting $y \in C^1[a,b]$ and $q(x) \le 0$, then the eigenvalues of (3) associated with homogeneous boundary conditions of Dirichlet or Neumann type are nonnegative.*

Proof. Since the boundary conditions are of Dirichlet or Neumann type then it holds that

$$y D_a^\alpha y\big|_a^b = 0. \tag{34}$$

Then the result is directly obtained from the fractional Rayleigh Quotient as $q(x) \le 0$. □

Now if y is a stationary function for

$$\begin{aligned} J_a^\alpha(y) &= \int_a^b F\left(y, D_a^\alpha y, x\right) dx(\alpha) \\ &= \int_a^b F\left(y, D_a^\alpha y, x\right)(x-a)^{\alpha-1}\,dx, \end{aligned} \tag{35}$$

then it holds that, see [10],

$$\frac{\partial F}{\partial y}\left(y, D_a^\alpha y, x\right) - D_a^\alpha\left(\frac{\partial F}{\partial y^\alpha}\left(y, D_a^\alpha y, x\right)\right) = 0, \tag{36}$$

the fractional Euler equation. We remark here that the above equation is a necessary condition for a stationary point and not sufficient. In the following we show that the fractional Sturm-Liouville eigenvalue problem (3)-(4) is equivalent to the following:

(i) Finding the stationary function $y(x)$ of

$$F[y]=\int_a^b\left(p\left(D_a^\alpha y\right)^2-qy^2\right)(x-a)^{\alpha-1}\,dx, \quad (37)$$

subject to $G[y]=1$, where

$$G[y]=\int_a^b wy^2(x-a)^{\alpha-1}\,dx. \quad (38)$$

To find the stationary of $F[y]$ subject to $G[y]=1$, we first find the stationary value y of $K[y]=F[y]-\lambda G[y]$ and then eliminate λ using $G[y]=1$. Applying the fractional Euler Equation (36) to $K[y]$ yields

$$-2qy-2\lambda wy-D_a^\alpha\left(2pD_a^\alpha y\right)=0, \quad (39)$$

or

$$D_a^\alpha\left(pD_a^\alpha y\right)+qy=\lambda wy, \quad (40)$$

which is the fractional Sturm-Liouville problem. Moreover, multiplying (3) by y and integrating yields

$$\begin{aligned}&\int_a^b yD_a^\alpha\left(pD_a^\alpha y\right)(x-a)^{\alpha-1}\,dx+\int_a^b qy^2(x-a)^{\alpha-1}\,dx\\&=-\lambda\int_a^b wy^2(x-a)^{\alpha-1}\,dx.\end{aligned} \quad (41)$$

Performing integration by parts of the first integral yields

$$\begin{aligned}&pyD_a^\alpha y\big|_a^b-\int_a^b p\left(D_a^\alpha y\right)^2(x-a)^{\alpha-1}\,dx\\&\quad+\int_a^b qy^2(x-a)^{\alpha-1}\,dx\\&=-\lambda\int_a^b wy^2(x-a)^{\alpha-1}\,dx.\end{aligned} \quad (42)$$

Since

$$yD_a^\alpha y\big|_a^b=0, \quad (43)$$

we have

$$\begin{aligned}&\lambda\int_a^b wy^2(x-a)^{\alpha-1}\,dx\\&=\int_a^b\left(p\left(D_a^\alpha y\right)^2-qy^2\right)(x-a)^{\alpha-1}\,dx.\end{aligned} \quad (44)$$

Since $\int_a^b wy^2(x-a)^{\alpha-1}dx=1$, we have

$$\lambda=\int_a^b\left(p\left(D_a^\alpha y\right)^2-qy^2\right)(x-a)^{\alpha-1}\,dx. \quad (45)$$

That is, λ is determined by $F[y]$ in (37).

The problem in (i) is equivalent to the problem of finding the stationary function of (ii) $A[y]=F[y]/G[y]$. Thus the eigenvalues of the fractional Sturm-Liouville eigenvalue problem are the values given by $A[y]$. The proof of (i) being equivalent to (ii) is well-known in the literature and we present it here for the sake of completeness.

We have

$$\delta A=\frac{G\delta F-F\delta G}{G^2}, \quad (46)$$

and $\delta A=0$ if and only if $G\delta F-F\delta G=0$, or

$$\delta F-\frac{F}{G}\delta G=\delta F-AG=0, \quad (47)$$

which is the same as δK.

Using the above results and the fractional Rayleigh Quotient result we have the following.

Lemma 12. *For the fractional eigenvalue problem (3)-(4) it holds that*

$$\lambda=\frac{\int_a^b p\left(D_a^\alpha y\right)^2d\alpha(x)-\int_a^b qy^2d\alpha(x)}{\int_a^b wy^2d\alpha(x)} \quad (48)$$

and the eigenfunction y is a stationary (minimum) value of the above ratio.

Remark 13. Assuming that the eigenvalues of (3)-(4) are ordered, $\lambda_1<\lambda_2\cdots<\lambda_n\cdots$, then the above result can be used to give an upper estimate value of the first eigenvalue λ_1, by choosing arbitrary function ψ that satisfies the same boundary conditions, and computing the ratio in (48) for ψ.

3. Illustrative Examples

Example 1. Consider the fractional eigenvalue problem (3)-(4) with $p=1, q=0, w=1,\ 0<x<1$ and with Dirichlet boundary condition $y(0)=y(1)=0$. The eigenfunctions are $\phi_n=\sin(n\pi x^\alpha)$ and the corresponding eigenvalues are $\lambda_n=n^2\alpha^2\pi^2$.

In the following we apply the fractional Rayleigh Quotient to obtain lower estimates of the first eigenvalue. We start with the atrial function $\psi(x)=x^\alpha-x^{2\alpha}$, which satisfies the homogenous boundary conditions $\psi(0)=\psi(1)=0$. We have $D_0^\alpha\psi=\alpha(1-2x^\alpha)$, and thus

$$\begin{aligned}\lambda_1&\le\frac{\int_0^1\left(D_0^\alpha\psi\right)^2x^{\alpha-1}dx}{\int_0^1\psi^2x^{\alpha-1}dx}=\frac{\int_0^1\alpha^2\left(1-2x^\alpha\right)^2x^{\alpha-1}dx}{\int_0^1\left(x^\alpha-x^{2\alpha}\right)^2x^{\alpha-1}dx}\\&=10\alpha^2.\end{aligned} \quad (49)$$

So, we obtain an upper estimate $\overline{\lambda_1} = 10\alpha^2$, which is comparable with the exact eigenvalue $\lambda_1 = \pi^2\alpha^2$. However, this upper bound can be improved by choosing a trial function

$$\psi(x) = x^\alpha (1 - x^\alpha) + a(x^\alpha (1 - x^\alpha))^2, \tag{50}$$

with parameter a and then choosing a to minimize the fractional Rayleigh Quotient. Direct calculations show that

$$\begin{aligned} \int_0^1 (D_0^\alpha \psi)^2 x^{\alpha-1} dx &= \frac{\alpha}{105}(35 + 2a(a+7)), \\ \int_0^1 \psi^2 x^{\alpha-1} dx &= \frac{21 + a(a+9)}{630a}. \end{aligned} \tag{51}$$

Thus, the fractional Rayleigh Quotient will produce

$$\mathrm{FR}(a,\alpha) = \alpha^2 \frac{630}{105}\frac{35 + a(a+7)}{21 + a(a+9)}. \tag{52}$$

The minimum value of

$$R(a) = \frac{630}{105}\frac{35 + a(a+7)}{21 + a(a+9)} \tag{53}$$

is 9.86975 and occurs at $a = 1.13314\cdots$. Hence, an upper estimate $\overline{\lambda_1} = 9.86975\alpha^2$ is obtained which is very close to the exact one.

Example 2. Consider the fractional eigenvalue problem (3)-(4) with $p = 1, q = 0, w = 1,\ 0 < x < 1$ and with boundary condition $y(0) - y'(0) = 0$, $y'(1) = 0$. The eigenfunctions are

$$\phi_n = a_n \sin(\lambda_n x^\alpha) + b_n \cos(\lambda_n x^\alpha). \tag{54}$$

We choose $a_n = 0$, so that $\phi_n' = \lambda_n \alpha x^{\alpha-1} a_n \cos(\lambda_n x^\alpha) - \lambda_n \alpha x^{\alpha-1} b_n \sin(\lambda_n x^\alpha)$ is defined at $x = 0$. Thus, $\phi_n = b_n \cos(\lambda_n x^\alpha)$, and applying the boundary conditions we have $\phi_n = 0$. That is, the problem possesses no eigenfunctions for $1/2 < \alpha < 1$.

Remark 3. It is well-known that the regular Sturm-Liouville eigenvalue problem with integer derivative possesses an infinite number of eigenvalues. This result is not valid for the fractional one as shown in the previous example. However, the fractional Sturm-Liouville equation in (3) can be discussed with fractional boundary conditions of the type

$$\begin{aligned} c_1 y(a) + c_2 (D_a^\alpha y)(a) = 0, &\quad c_1^2 + c_2^2 > 0, \\ r_1 y(b) + r_2 (D_a^\alpha y)(b) = 0, &\quad r_1^2 + r_2^2 > 0. \end{aligned} \tag{55}$$

We believe that the above fractional eigenvalue problem possesses an infinite number of eigenvalues and we left it for a future work.

4. Conclusion

We have considered a regular conformable fractional Sturm-Liouville eigenvalue problem. We proved that the eigenvalues are real and simple and the eigenfunctions are orthogonal. We also established the fractional Wronskian result for any two linearly independent solutions of the problem. We obtained a fractional Rayleigh Quotient and applied a fractional variational principle to show that the minimum value of the Quotient is obtained at an eigenfunction. This result is used to estimate the first eigenvalue and the presented example illustrates the efficiency of the result. We illustrated by an example that the existence of eigenfunctions is not guaranteed unlike the result for the regular Sturm-Liouville eigenvalue problem. Most of the obtained results are analogous for the ones of regular Sturm-Liouville eigenvalue problems and they open the door for establishing other results such as the countability of eigenfunctions and completeness of eigenfunctions which are essential in solving fractional differential equations by fractional eigenfunction expansion.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The first author gratefully acknowledges the support of the United Arab Emirates University under the Grant 31S239-UPAR(1) 2016.

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