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Research Article

Stability Analysis for Autonomous Dynamical Switched Systems through Nonconventional Lyapunov Functions

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The stability of autonomous dynamical switched systems is analyzed by means of multiple Lyapunov functions. The stability theorems given in this paper have finite number of conditions to check. It is shown that linear functions can be used as Lyapunov functions. An example of an exponentially asymptotically stable switched system formed by four unstable systems is also given.

1. Introduction

Switched systems are present in different areas of science and technology as aeronautical and automotive control, telecommunications, traffic control, chemical process, and so forth [1–5].

The switched system is a special class of hybrid or variable structure systems [1, 3, 6–10]. Similar to variable structure systems, the dynamics of switched systems is described by different differential equations in different space regions and the change of dynamics occurs when the trajectories pass through the boundaries between two regions. Variable structure systems may have special type of solutions, the so-called sliding mode solutions. The theory and different applications of sliding mode solutions in control are investigated in many books and papers (see, e.g., [9]). In contrast to sliding mode theory, the case when the variable structure system has no sliding mode solution is not sufficiently studied. To separate these two cases, we call variable structure systems without sliding mode solutions as switched systems. In the theory of hybrid systems the change of systems dynamics may occur by action of automata or by other reasons. In this paper, the stability of switched systems is studied. The problem of stability of switched systems is not simple. First of all, we give some examples illustrating different aspects of this problem and showing that stability of all subsystems is not sufficient to ensure the stability of the whole switched system. Namely, in one example the whole switched system formed by stable subsystems is unstable or asymptotically stable depending on the structure of the switched systems, that is, depending on the regions where these subsystems are acting. If some of subsystems are unstable and others are asymptotically stable, the switched system may be unstable or asymptotically stable or it may have periodic solutions. Also, in some cases the stability of the whole switched system depends neither on the structure of the switched system nor on the order of the switching.

The papers [11, 12] were dedicated essentially to the case of linear systems. In paper [13] the stability of switched and hybrid systems is investigated. However, for each trajectory, the theorem on stability from [13] imposes certain conditions on Lyapunov functions at the moment of passing through the switching lines. These conditions are possible to check only if the trajectory is known. Furthermore, typical trajectory of switched systems has an infinite number of switchings and it would be necessary to check an infinite number of conditions. Our previous works [14, 15] investigate some stability problems in switched systems and also investigate possibilities of appearance of chaotic solutions in such systems.

In present paper, some stability theorems using multiple Lyapunov functions are established. In contrast to [13] our conditions are imposed on values of Lyapunov functions in corresponding space regions and on switching lines, allowing in this way obtaining different stability conditions without knowing trajectories, while only a finite number of conditions depending only on Lyapunov functions are verified. The Lyapunov functions used in these theorems may be different from usual Lyapunov functions defined in the whole space, which allows extending the class of functions used as Lyapunov functions. For example, it is possible to use linear Lyapunov functions to investigate the stability of switched systems. The sum of quadratic and linear functions may be used also as Lyapunov functions. Some examples of such functions are also given. The using of unusual Lyapunov functions simplifies the search for convenient Lyapunov function. We present also one example (Example 15) where all subsystems are unstable by considering the whole space, but the switched system formed by these subsystems is exponentially asymptotically stable.

Our theorems which give conditions of asymptotic stability or instability of whole switched system in case when all subsystems are only stable are especially interesting. These results may be considered as a special type of parametric excitation or parametric stabilization of stable systems.

2. Description of Autonomous Two-Dimensional Switched Systems

Suppose the phase space $\mathbb{R}^2$, $X = (x, y)^T \in \mathbb{R}^2$, with norm $\|X\|$, is divided into a finite number p of open 1-connected regions Q_i, $1 \le i \le p$, with smooth boundaries such that the origin $\{0\}$ of rectangular coordinate systems pertains to closure of any region Q_i. The boundary between two regions Q_i and Q_{i+1} is noted by $L_{i,i+1}$ and is called switching line. These switching lines are supposed to be smooth and let the normal $N_{i,i+1}(X)$ to the switching lines existing in each point $X \in L_{i,i+1}$ have direction from Q_i to Q_{i+1}. Suppose each region Q_i has points X with norms such that $\|X\| > H$, where H is any number. The topological properties of plane conduce to the conclusion that each region Q_i has only two boundaries which go from origin $\{0\}$ to infinity without intersections.

In each region Q_i the dynamics of switched systems is described by a proper autonomous equation E_i with Lipschitz continuous function $f_i(X) = f_i(x, y)$:

$$E_i\colon \dot{X} = f_i(X), \quad X \in Q_i,\ 1 \le i \le p, \tag{1}$$

with initial condition

$$X(t_0) = X_0. \tag{2}$$

The trajectory of switched system (1)-(2) may pass from region Q_i to region Q_{i+1} only crossing the switching line $L_{i,i+1}$. Suppose also the nonexistence of sliding modes in switched system (1)-(2). The sufficient condition for absence of sliding modes is the following *transversality condition*: the normal $N_{i,i+1}(X)$ which goes from Q_i to Q_{i+1} and the trajectory velocities $f_i(X)$ and $f_{i+1}(X)$ from one to another side of switching line form acute angles; that is,

$$\begin{aligned} &\operatorname{sign}(\langle N_{i,i+1}(X), f_i(X)\rangle) \\ &\quad = \operatorname{sign}(\langle N_{i,i+1}(X), f_{i+1}(X)\rangle) = 1, \quad X \in L_{i,i+1}, \end{aligned} \tag{3}$$

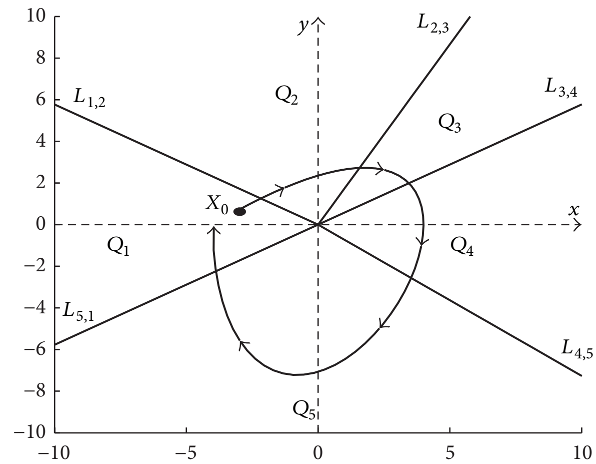

Figure 1: Autonomous two-dimensional switched system.

for every point on switching line $L_{i,i+1}$ and all lines $L_{i,i+1}$, $i = 1, 2, 3, \ldots, p$ (Figure 1). Transversality condition (3) guarantees the passage only from region Q_i to Q_{i+1} and not in opposite direction.

If at instant $t_{i,\text{fin}}$ the trajectory of switched system arrives from region Q_i on the switching line $L_{i,i+1}$, the trajectory passes across the switching line $L_{i,i+1}$ and at the following instants $t > t_{i,\text{fin}}$ the system dynamic is described by equation E_{i+1} acting in region Q_{i+1} with initial condition $X_{t_{i+1,\text{init}}}$ for the new equation E_{i+1} coinciding with the final condition $X_{t_{i,\text{fin}}}$ of the previous equation E_i on the switching line $L_{i,i+1}$, for $i = 1, 2, 3, \ldots$; that is, $X_{t_{i+1,\text{init}}} = X_{t_{i,\text{fin}}}$. Under this condition, all trajectories of switched system will be continuous for all $t > t_0$. The regions Q_i together with equations E_i completely define the switched system. The set of regions Q_i defines the geometrical structure while the set of equations E_i defines the dynamical structure of switched system.

Any switched system for a chosen initial condition $X(t_0) = X_0$ generates a sequence of continuous dynamical subsystems $S_{1(X_0)}, S_{2(X_0)}, \ldots, S_{i(X_0)}, \ldots$ acting in regions $Q_{1(X_0)} = Q_k$, $1 \le k \le p$, $Q_{2(X_0)} = Q_{k+1}, \ldots, Q_{i(X_0)}, \ldots$ and switching on lines $L_{1(X_0),2(X_0)} = L_{k,k+1}, L_{2(X_0),3(X_0)}, \ldots, L_{i(X_0),(i+1)(X_0)}, \ldots$; that is,

$$\mathrm{SW}(X_0) = \{S_{1(X_0)}, S_{2(X_0)}, \ldots, S_{i(X_0)}, \ldots\}. \tag{4}$$

The initial condition X_0 defines the first element in (4). Depending on X_0, the first subsystem $S_{1(X_0)}$ is acting from initial time $t_0 = t_{1(X_0),\text{init}} = t_{k,\text{init}}$ to the first switching instant $t_{1(X_0),\text{fin}} = t_{k,\text{fin}}$ in region Q_k such that $X(t_0) \in Q_k$. The second subsystem $S_{2(X_0)}$ is acting in Q_{k+1} from time $t_{2,\text{init}} = t_{1,\text{fin}}$ to the second switching instant $t_{2,\text{fin}} = t_{3,\text{init}}$ and so on. In other words, each subsystem S_i is defined by three elements $S_i = (Q_i, E_i, X(t_{i,\text{init}}))$:

(a) region Q_i where this subsystem is acting,

(b) differential equation E_i acting in Q_i,

(c) initial values $t_{i,\text{init}}$ and $X(t_{i,\text{init}})$.

This definition is slightly different from definition given in [1, 3, 13]. The sequence (4) may contain finite or infinite number of dynamical subsystems S_i although the total number of different regions Q_i is finite because trajectory can return to region Q_i after the whole rotation around origin, and therefore $Q_{p+i} = Q_i$ for all i. If condition (3) holds for all switching lines, then the sequence (4) may contain an infinite number of subsystems $S_1, S_2, \ldots, S_n, \ldots$. In case of infinite number of subsystems S_i, in sequence (4) each region Q_i has two boundary switching lines: by one of them the trajectories enter from the precedent region Q_{i-1}, and by the other, after a finite time, the trajectories go out from region Q_i to region Q_{i+1}. After the whole rotation around origin the switched system returns to initial region Q_i and $L_{p,p+1} = L_{p,1}$.

This property has, as a consequence, the following fundamental property of autonomous two-dimensional switched systems: the order of terms in sequence (4) does not depend on initial conditions.

In multidimensional phase space $\mathbb{R}^n$, $n > 2$, each region may have more than two switching surfaces. For that reason, there does not exist an analogy of announced fundamental property in multidimensional space. So, in $\mathbb{R}^n$ the order of terms in (4) may depend on initial conditions. This complicates the stability investigation of switched systems in multidimensional space $\mathbb{R}^n$, $n > 2$.

The number of dynamical subsystems $S_{i(X_0)}$ is finite, $1 \le i \le m$, if the switched system stays in the final region Q_N for all time after last switching $t_{m-1,\text{fin}}$, $t > t_{m-1,\text{fin}}$. The switched system may have more than one final region and the final region may depend on initial conditions X_0. The sufficient condition for finiteness of sequence (4) is the existence of at least one line $L_{m,m+1}$ such that from one side of the line $L_{m,m+1}$ the normal $N_{m,m+1}(X)$ and the trajectory velocities $f_m(X)$ and $f_{m+1}(X)$ form an obtuse and acute angle, respectively:

$$\begin{gathered} \operatorname{sign}\left(\left\langle N_{m,m+1}(X), f_m(X)\right\rangle\right) = -1, \\ \operatorname{sign}\left(\left\langle N_{m,m+1}(X), f_{m+1}(X)\right\rangle\right) = 1, \end{gathered} \tag{5}$$

for $X \in L_{m,m+1}$.

In this case, the subsystem S_m will be the last subsystem in the sequence (4). If there exist more than one switching line satisfying condition (5) then the switched system may have more than one final region and the final region may depend on initial conditions X_0. Let us consider two examples.

Example 1. Suppose the switched system is described by two pendulum equations E_1 and E_2 with different natural frequencies acting in regions Q_1 and Q_2 of phase plane $\mathbb{R}^2$:

$$\begin{aligned} E_1\colon \dot{x} &= y, \\ \dot{y} &= -x, \\ &\forall X \in Q_1 = \mathbb{R}^2 - Q_2, \end{aligned} \tag{6}$$

while equation E_2 is

$$\begin{aligned} E_2\colon \dot{x} &= y, \\ \dot{y} &= -\omega^2 x, \\ &\forall X \in Q_2 = \{x < 0, y > 0\}, \end{aligned} \tag{7}$$

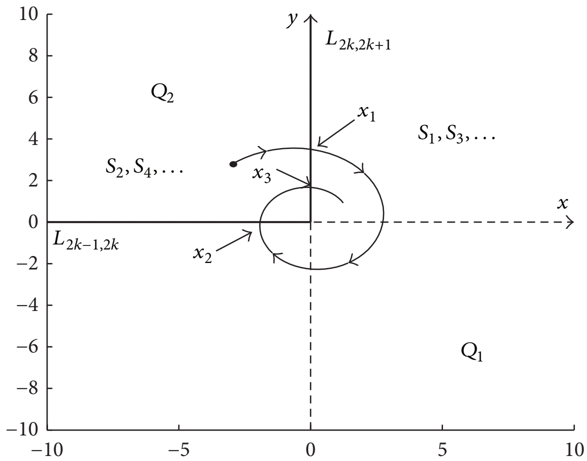

Figure 2: Switched system with infinite sequence (4) of subsystems S_i.

Every trajectory of switched system has an infinite number of switchings and it is represented by an infinite number of continuous dynamical subsystems $S_1, S_2, \ldots, S_i, \ldots$.

The subsystems $S_1, S_3, \ldots, S_{2j+1}, \ldots$ are described by the same equation E_1 in the form of (6) acting in region Q_1 and the subsystems $S_2, S_4, \ldots, S_{2j}, \ldots$ are described by equation E_2 in the form of (7) acting in region Q_2. But the initial conditions for $S_1, S_3, \ldots, S_{2j+1}, \ldots$ and $S_2, S_4, \ldots, S_{2j}, \ldots$ are different: X_1 for S_1, X_3 for S_3, and so on. Therefore all subsystems $S_1, S_2, \ldots, S_i, \ldots$ are different (see Figure 2).

Example 2. Consider now the switched system with E_1 acting in the region $Q_1 = \{x < 0\} \cap \{y > 0\} \cap \{y < (-2 + \sqrt{3})x\}$, E_2 acting in the lower half-space $Q_2 = \{y < 0\}$, and E_3 acting in $Q_3 = \mathbb{R}^2 - Q_1 - Q_2$ with

$$E_1\colon \ddot{x} + 4x + x = 0, \quad \forall X \in Q_1, \tag{8}$$

$$E_2\colon \ddot{x} + \omega^2 x = 0, \quad \forall X \in Q_2, \tag{9}$$

$$E_3\colon \ddot{x} + x = 0, \quad \forall X \in Q_3. \tag{10}$$

The equation E_1 is overdamped and its general solution is

$$\begin{aligned} x(t) = A \cdot \exp\left(\left(-2 + \sqrt{3}\right)t\right) + B \\ \cdot \exp\left(\left(-2 - \sqrt{3}\right)t\right), \quad A, B = \text{const.} \end{aligned} \tag{11}$$

The trajectories of the whole switched system are presented in Figure 3 with $\omega = 3$. The region denoted as Q_1 in Figure 3 is a final region of the switched system (8)–(10). Depending on initial conditions, the trajectory may has two, one, or zero commutations. If X_0 belongs to region Q_3 then the trajectory has two commutations, if X_0 belongs to region Q_2 then the trajectory has one commutation, and if X_0 belongs to region Q_1 then the trajectory has no commutations.

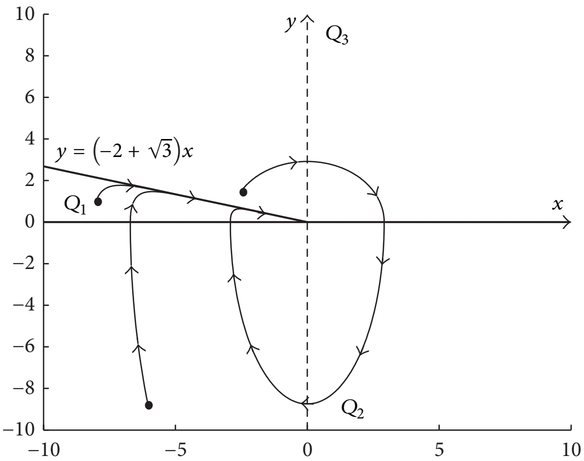

Figure 3: Switched system with finite sequence of subsystems (8)-(9).

3. Stability of Switched Systems

Suppose

$$f_i(0) = 0, \quad 1 \le i \le p. \tag{12}$$

Under condition (12) switched system (1)-(2) or (4) has the trivial solution $X(t) \equiv 0$.

In the following, the stability analysis of trivial solution (or origin) of switched system (1)-(2) will be carried out.

Let us introduce some definitions of Lyapunov stability.

Definition 3. The trivial solution $X(t) \equiv 0$ (or origin) of switched system (1)-(2) is said to be stable if for any $\epsilon > 0$ there exists a $\delta = \delta(\epsilon)$ such that the inequality $\|X(t, t_0, X_0)\| < \epsilon$ is satisfied for any time $t > t_0$ whenever $\|X_0\| < \delta(\epsilon)$.

Definition 4. The trivial solution $X(t) \equiv 0$ (or origin) of system (1)-(2) is said to be asymptotically stable if

(a) it is stable;

(b) there exists $\Delta > 0$ such that $\|X(t, t_0, X_0)\| \to 0$, $t \to \infty$ for $\|X_0\| < \Delta$.

Clearly, the stability (asymptotic stability) is uniform with respect to t_0 because the switched system is a stationary system with all elements independent of t.

Obviously, in case of finite sequence (4) the stability or instability of the whole switched system origin depends only on stability of instability of origin for final subsystem S_m.

In case of infinite sequence (4) the instability of one equation E_i does not conduce automatically to the instability of the whole switched system. Also, the stability of all equations E_i is not sufficient to conclude that the whole switched system is stable [15].

Let us give other corresponding examples.

Example 5. Consider the switched system with two equations E_1 and E_2 acting in two half-planes $Q_1 = \{\dot{x} = y > 0\}$ and $Q_2 = \{\dot{x} = y < 0\}$, respectively, with

$$E_1\colon \ddot{x} - 0.2\dot{x} + 1.01x = 0, \quad (x, \dot{x} = y) \in Q_1, \tag{13}$$

$$E_2\colon \ddot{x} + 0.4\dot{x} + 1.04x = 0, \quad (x, \dot{x} = y) \in Q_2. \tag{14}$$

Switching lines $L_{1,2}$ and $L_{2,1}$ in this case are $L_{1,2} = \{x > 0, y = 0\}$ and $L_{2,1} = \{x < 0, y = 0\}$. Equation E_1 is unstable and E_2 is asymptotically stable. The asymptotic stability of equation E_2 can be called stronger than the instability of E_1. It is easy to verify that for any $X_1 = \{x_1 > 0, y_1 = 0\}$ the subsequent points, $X_2 = \{x_2 < 0, y_2 = 0\}$, $X_3 = \{x_3 > 0, y_3 = 0\}, \ldots$, of intersection with the axis x are equal to

$$\begin{aligned} x_2 &= -x_1 \exp(-0.2\pi), \\ x_3 &= -x_2 \exp(0.1\pi) = x_1 \exp(-0.1\pi) < x_1. \end{aligned} \tag{15}$$

Therefore, the trivial solution of switched system (13)-(14) is asymptotically stable.

Replace (14) in region Q_2 by

$$E_3\colon \ddot{x} + 0.2\dot{x} + 1.01x = 0, \quad (x, \dot{x} = y) \in Q_2. \tag{16}$$

Now, the same calculations show that $x_3 = x_1$ for all x_1. Therefore all solutions of switched system formed by unstable equation (13) and by asymptotically stable equation (16) are periodic.

Example 6. Consider switched system of Example 1:

$$E_1\colon \ddot{x} + \omega^2 x = 0, \quad (x, \dot{x} = y) \in Q_1 = \{x < 0, \dot{x} > 0\}, \tag{17}$$

$$E_2\colon \ddot{x} + x = 0, \quad (x, \dot{x} = y) \in Q_2 = \mathbb{R}^2 - Q_1. \tag{18}$$

Both equations E_1 and E_2 are stable. Also, it is easy to calculate for any $X_1 = \{x_1 = 0, y_1 > 0\}$ the values of $X_2 = \{x_2 = -y_1 < 0, y_2 = 0\}$, $X_3 = \{x_3 = 0, y_3 > 0\}$ and that

$$y_3 = \omega y_1. \tag{19}$$

Therefore, if $\omega < 1$ then the switched system origin is asymptotically stable, but if $\omega > 1$ the switched system origin is unstable. Suppose now Q_1 coincides with the first quadrant; that is, $Q_1 = \{x > 0, y > 0\}$; then analogous calculations show that the switched system origin is asymptotically stable if $\omega > 1$, but if $\omega < 1$ the switched system origin is unstable.

Remark 7. These two examples show that stability of switched systems depends not only on the stability of equations E_i but also on all other elements which define the switched system; that is, it depends also on the regions where different equations E_i are acting and on the order of the switching.

Thus, the following definitions are justified.

Definition 8. The origin of switched system (4) formed by an infinite sequence of subsystems $S_{1(X_0)}, S_{2(X_0)}, \ldots, S_{i(X_0)}, \ldots$ is stable (asymptotically stable) independently of geometrical structure of switched system if the origin is stable (asymptotically stable) for any choice of equations $E_1, \ldots, E_p$ acting in arbitrary chosen regions $Q_1, \ldots, Q_p$.

Definition 9. The origin of switched system (4) formed by a sequence of subsystems $S_1, S_2, \ldots, S_i, \ldots$ is stable (asymptotically stable) for given geometrical and dynamical structure of switched system if the origin is stable (asymptotically stable) for a given choice of equations $E_1, \ldots, E_p$ acting in corresponding regions $Q_1, \ldots, Q_p$.

4. General Theorems

The previous examples show that the stability properties of switched system origin need special investigation and they do not follow directly from simple stability conditions imposed on subsystems, but also they depend on how the switching occurs within the whole system.

The only interesting case when sequence (4) which determines the switched system is infinite is considered below.

Suppose equations E_i are stable (asymptotically stable) and also suppose the existence of some smooth positive definite functions $V_i(X)$ defined for $X \in Q_i$. Each function $V_i(X)$ is called Lyapunov function in region Q_i, if it is continuously differentiable in Q_i and fulfilling $V_i(X) > 0$, $X \in Q_i$, $X \neq 0$, and $V_i(0) = 0$ [16, 17].

Denote by $\omega_k(u)$ scalar continuous nondecreasing functions (also called wedges) defined and positive for $u > 0$ such that $\omega_k(0) = 0$.

The symbols $\dot{V}_i(X(t))$ denote the derivatives of the functions $V_i(X(t))$ along the trajectory of equation E_i:

$$\dot{V}_i(X(t)) = \langle \nabla (V(X)), f_i(X) \rangle, \quad X \in Q_i. \tag{20}$$

The existence of a common Lyapunov function for all equations E_i simplifies the stability analysis of switched system origin. For this reason, Theorem 10 has been explicitly formulated but not proven, because this theorem is a direct consequence of the more general Theorem 12.

Theorem 10. *Suppose there exists a common Lyapunov function $V(X)$ defined in the whole space $\mathbb{R}^2$ satisfying the following:*

(a) $\omega_1(\|X\|) \le V(X) \le \omega_2(\|X\|)$, $X \in \mathbb{R}^2$,

(b) *for any equation E_i the following conditions are fulfilled*

$$\begin{aligned} &\dot{V}(X(t)) = \langle \nabla (V(X)), f_i(X) \rangle \le 0, \\ &\qquad \left(\dot{V}_i(X) \le -\omega_3(\|X\|) < 0\right), \end{aligned} \tag{21}$$

for $X \in Q_i$, $1 \le i \le p$.

Then, the trivial solution $X(t) \equiv 0$ of switched system (1)-(2) is stable (asymptotically stable) independently of geometrical structure of switched system.

Example 11. Consider three equations:

$$\begin{aligned} &E_1\colon \ddot{x} + x = 0, \\ &E_2\colon \ddot{x} + x + x^2\dot{x}^3 = 0, \\ &E_3\colon \ddot{x} + \sin(\dot{x}) + x = 0. \end{aligned} \tag{22}$$

As a common Lyapunov function $V(X)$ we can take the function

$$V(x, \dot{x}) = x^2 + \dot{x}^2. \tag{23}$$

The derivatives of these functions along the trajectories of equations E_1, E_2, and E_3 are not positive:

$$\begin{aligned} &\dot{V}_{(\text{ec21})}(x, \dot{x}) = 2x\dot{x} - 2x\dot{x} = 0, \\ &\dot{V}_{(\text{ec22})}(x, \dot{x}) = 2x\dot{x} - 2\dot{x}\left(-x - x^2\dot{x}^3\right) = -2x^2\dot{x}^4 \le 0, \\ &V_{(\text{ec23})}(x, \dot{x}) = 2x\dot{x} - 2\dot{x}(-x - \sin(\dot{x})) = -2\dot{x}\sin(\dot{x}) \\ &\qquad \le 0, \\ &\qquad\qquad |\dot{x}| < 1. \end{aligned} \tag{24}$$

Therefore, the trivial solution of the switched system formed by equations E_1, E_2, and E_3 is stable independently of geometrical structure of switched system determined by a choice of arbitrary regions Q_1, Q_2, and Q_3 such that $Q_1 \cup Q_2 \cup Q_3 = \mathbb{R}^2$.

Theorem 12. *Suppose conditions (3) hold for all switching lines $L_{i,i+1}$ and suppose that for each equation E_i acting in $Q_i \in \mathbb{R}^2$ there exists a Lyapunov function $V_i(X)$ defined in Q_i such that*

(a) $\omega_{1,i}(\|X\|) \le V_i(X) \le \omega_{2,i}(\|X\|)$, $X \in Q_i$, $i = 1, \ldots, p$,

(b) $\dot{V}_i(X(t)) = \langle \nabla(V_i(X)), f_i(X) \rangle \le 0$, $(\dot{V}_i(X(t)) \le -\omega_{3,i}(\|X(t)\|) < 0)$, $X \in Q_i$, $i = 1, \ldots, p$,

(c) *on all switching lines $L_{i,i+1}$ where trajectories pass from region Q_i to Q_{i+1}, the following inequalities hold:* $V_i(X) \ge V_{i+1}(X)$, $X \in L_{i,i+1}$.

Then the trivial solution $X(t) \equiv 0$ of switched system (4) is stable (asymptotically stable) for given switched system.

Proof. Let us proof stability of trivial solution $X(t) \equiv 0$. Denote

$$\begin{aligned} &\omega_1(u) = \min_i \left(\omega_{1,i}(u)\right), \quad u > 0, \\ &\omega_2(u) = \max_i \left(\omega_{2,i}(u)\right), \quad u > 0, \\ &\omega_3(u) = \min_i \left(\omega_{3,i}(u)\right), \quad u > 0, \\ &\omega_1(0) = \omega_2(0) = \omega_3(0) = 0. \end{aligned} \tag{25}$$

The wedge functions $\omega_i(u)$, $i = 1, 2, 3$, are scalar continuous nondecreasing functions positive for $u > 0$ and satisfy (25). For a given $\epsilon > 0$ we define a number $\delta = \delta(\epsilon)$ such that $\omega_2(\delta) \le \omega_1(\epsilon)$. Evidently, such δ does exist. Using conditions (b) and (c) we obtain that the sequence of Lyapunov functions

$V_k(X(t,t_0,X_0))$ for arbitrary k and $t_{k,\text{init}} < t < t_{k,\text{fin}}$ is nonincreasing on the trajectories of switched system:

$$\begin{aligned}\omega_1(\|X(t,t_0,X_0)\|) &\leq \omega_{1,k}(\|X(t,t_0,X_0)\|)\\ &\leq V_k(X(t,t_0,X_0))\\ &\leq V_k(X(t_{k,\text{init}},t_0,X_0))\\ &\leq V_{k-1}(X(t_{k-1,\text{fin}},t_0,X_0))\\ &\leq V_{k-1}(X(t_{k-1,\text{init}},t_0,X_0)) \leq \cdots\\ &\leq V_{1(X_0)}(X_0).\end{aligned} \tag{26}$$

Using now condition (a) of the theorem, we obtain for arbitrary k and t, $t_{k,\text{init}} < t < t_{k,\text{fin}}$ the following inequalities:

$$\begin{aligned}\omega_1(\|X(t,t_0,X_0)\|) &\leq V_{1(X_0)}(X_0) \leq \omega_{2,1}(\|X_0\|)\\ &\leq \omega_2(\|X_0\|) \leq \omega_2(\delta) \leq \omega_1(\epsilon),\end{aligned} \tag{27}$$

Therefore,

$$\|X(t,t_0,X_0)\| \leq \epsilon, \quad t \geq t_0,\ \|X_0\| \leq \delta. \tag{28}$$

Inequality (28) is equivalent to stability of trivial solution of switched system (1)-(2).

To proof asymptotic stability it is necessary to demonstrate that for any $\gamma > 0$ there exist numbers $\Delta > 0$ and $T(\gamma)$ such that $\|X(t,t_0,X_0)\| \leq \gamma$ for $t \geq t_0 + T(\gamma)$ and $\|X_0\| \leq \Delta$. Take any $\gamma > 0$ and determine $\delta > 0$ which corresponds to γ in demonstration of stability; that is, $\omega_2(\delta) \leq \omega_1(\gamma)$. Take also $T(\gamma) = 2w_2(\Delta)/\omega_3(\delta) > 0$.

Let us demonstrate that on the interval $[t_0, t_0+T(\gamma)]$ there exists an instant t_1 such that $\|X(t_1,t_0,X_0)\| \leq \delta$.

If this is not so, that is, $\|X(t,t_0,X_0)\| > \delta$ for all $t \in [t_0, t_0+T(\gamma)]$, then using continuity of $X(t,t_0,X_0)$ and conditions (b) and (c) of Theorem 12 we have

$$\begin{aligned}&V_k(X(t_0+T(\gamma),t_0,X_0))\\ &\quad\leq V_{1(X_0)}(X_0) + \int_{t_0}^{t_{1,\text{fin}}} \dot V_{1(X_0)}(X(t,t_0,X_0))\,dt\\ &\qquad+ \int_{t_{1,\text{fin}}}^{t_{2,\text{fin}}} \dot V_{2(X_0)}(X(t,t_0,X_0))\,dt + \cdots\\ &\qquad+ \int_{t_{k,\text{init}}}^{t_0+T(\gamma)} \dot V_k(X(t,t_0,X_0))\,dt\\ &\quad\leq V_{1(X_0)}(X_0) - \int_{t_0}^{t_{1,\text{fin}}} \omega_{3,1}(\|X(t,t_0,X_0)\|)\,dt\\ &\qquad- \int_{t_{1,\text{fin}}}^{t_{2,\text{fin}}} \omega_{3,2}(\|X(t,t_0,X_0)\|)\,dt - \cdots\\ &\qquad- \int_{t_{k,\text{init}}}^{t_0+T(\gamma)} \omega_{3,k}(\|X(t,t_0,X_0)\|)\,dt\\ &\quad\leq V_{1(X_0)}(X_0) - \int_{t_{k,\text{init}}}^{t_0+T(\gamma)} \omega_3(\|X(t,t_0,X_0)\|)\,dt\\ &\quad\leq V_{1(X_0)}(X_0) - T(\gamma)\,\omega_3(\delta)\\ &\quad\leq \omega_2(\Delta) - T(\gamma)\,\omega_3(\delta) = -\omega_2(\Delta) < 0.\end{aligned} \tag{29}$$

Inequality (29) contradicts the positiveness of Lyapunov function V_k. Therefore, there exists an instant $T(\gamma)$ and $t_1, t_0 \leq t_1 \leq t_0 + T(\gamma)$ such that $\|X(t_1,t_0,X_0)\| \leq \delta$ for $\|X_0\| \leq \Delta$. From condition $w_2(\delta) \leq \omega_1(\gamma)$ and the stability of the trivial solution it follows that $\|X(t,t_0,X_0)\| \leq \gamma$ for all $t \geq t_1$, $t_1 \leq t_0 + T(\gamma)$. Therefore $\|X(t,t_0,X_0)\| \leq \gamma$ for all $\|X_0\| \leq \Delta$ and for all $t \geq t_0 + T(\gamma)$. The asymptotic stability of the origin is proven. □

Example 13. Consider once again first switched system of Example 6 where Q_1 coincides with second quadrant, $Q_1 = \{x < 0, y > 0\}$. Lyapunov functions for equations E_1, E_2 are

$$\begin{aligned}V_1(x,y) &= \omega^2 x^2 + y^2,\\ V_2(x,y) &= x^2 + y^2,\end{aligned} \tag{30}$$

respectively.

On switching line $L_{1,2}$ where $x = 0$ and $y > 0$, we have $V_1(x,y) = y^2 = V_2(x,y)$. On switching line $L_{2,1}$ where $y = 0$ we have $V_1(x,y) = \omega^2 x^2$ and $V_2(x,y) = x^2$. Therefore, condition (c) of Theorem 12 holds on both lines $L_{1,2}$ and $L_{2,1}$ if $\omega^2 < 1$. Under this condition, the origin of the switched system described by (17)-(18) of Example 6 is stable.

Consider now the second switched system of Example 6, where Q_1 coincides with the first quadrant $Q_1 = \{x \geq 0, y \geq 0\}$. In this case, on line $L_{2,1}$, where $x = 0$ we have $V_1(x,y) = y^2 = V_2(x,y)$ and on line $L_{1,2}$ where $y = 0$ we have $V_1(x,y) = \omega^2 x^2$ and $V_2(x,y) = x^2$. Therefore condition (c) of Theorem 12 holds if $\omega^2 > 1$. In this case, the origin of the second switched system of Example 6 is stable under this situation. This conclusion coincides with results of Example 6 obtained by direct analytical computations. Analytical calculations show asymptotic stability of switched system origin (17)-(18), while by using Theorem 12 we can only establish stability but no asymptotic stability.

Replacing (18) in region Q_2 by new equation

$$E_2\colon \ddot x + x + x^2\dot x^3 = 0, \quad X \in Q_2, \tag{31}$$

and considering the same Lyapunov function $V_2(x,y)$, we can also establish stability of the trivial solution of this new switched system (17), (31), but analytical calculations cannot be used in this case.

Example 14. Consider the switched system with two equations E_1 and E_2 acting in two half-planes $Q_1 = \{y > 0\}$ and $Q_2 = \{y < 0\}$:

$$\begin{aligned}E_1\colon \dot x &= y + x^2,\\ \dot y &= -2xy - 2x^3,\\ &\qquad \{x, y\} \in Q_1,\end{aligned}$$

$$E_2\colon\ \dot{x} = y,$$
$$\dot{y} = -x,$$
$$\{x, y\} \in Q_2. \tag{32}$$

The switching lines are $L_{1,2} = \{x > 0, y = 0\}$ and $L_{2,1} = \{x < 0, y = 0\}$. Normal $N_{1,2}$ going from Q_1 to Q_2 is $N_{1,2} = \left\{\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right\}$, and condition (3) on line $L_{1,2}$ has the form

$$\langle N_{1,2}, f_1 \rangle = 2x^3 > 0,$$
$$\langle N_{1,2}, f_2 \rangle = x > 0. \tag{33}$$

On the other hand, normal $N_{2,1}$ going from Q_2 to Q_1 is $N_{2,1} = \left\{\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right\}$, and condition (3) on line $L_{2,1}$ has the form

$$\langle N_{2,1}, f_1 \rangle = -2x^3 > 0,$$
$$\langle N_{2,1}, f_2 \rangle = -x > 0. \tag{34}$$

As Lyapunov functions for equations E_1 and E_2 in regions $y > 0$ and $y < 0$ take

$$V_1(x, y) = x^2 + y > 0, \quad y > 0, \tag{35}$$

$$V_2(x, y) = x^2 + y^2, \quad y < 0. \tag{36}$$

The strange function (35) satisfies condition (a) of Theorem 12 for unusual norm in $\mathbb{R}^2$ of the form $\|X\| = x^2 + |y|$. The switching lines are $L_{1,2} = \{y = 0, x > 0\}$ and $L_{2,1} = \{y = 0, x < 0\}$ and on these switching lines $V_1(x, y) = x^2 = V_2(x, y)$. The derivatives of functions (35) and (36) in regions $y > 0$ and $y < 0$ are equal to

$$\dot{V}_1(x, y) = 2x\dot{x} + \dot{y} = 2x\left(y + x^2\right) + \left(-2xy - 2x^3\right)$$
$$= 0, \quad y > 0; \tag{37}$$
$$\dot{V}_2(x, y) = 2x\dot{x} + 2y\dot{y} = 2xy - 2xy = 0, \quad y < 0.$$

All conditions of Theorem 12 hold and therefore, the trivial solution of switched system (32) is stable.

Example 15. Consider a switched system with four equations acting in four quadrants of the plane as shown in Figure 4:

$$E_1\colon\ \dot{x} = 3x + 2y,$$
$$\dot{y} = -4x - 3y,$$
$$(x, y) \in Q_1 = \{x > 0, y > 0\},$$
$$E_2\colon\ \dot{x} = -3x + 5y,$$
$$\dot{y} = -2x + 4y,$$
$$(x, y) \in Q_2 = \{x > 0, y < 0\},$$

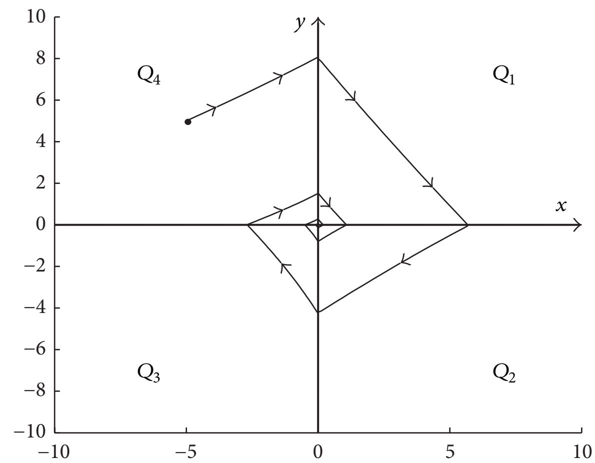

Figure 4: Trajectory of switched system (38).

$$E_3\colon\ \dot{x} = 3x + y,$$
$$\dot{y} = -4x - 2y,$$
$$(x, y) \in Q_3 = \{x < 0, y < 0\},$$
$$E_4\colon\ \dot{x} = -4x + 3y,$$
$$\dot{y} = -2x + 2y,$$
$$(x, y) \in Q_4 = \{x < 0, y > 0\}, \tag{38}$$

where $Q_1 \cup Q_2 \cup Q_3 \cup Q_4 = \mathbb{R}^2$.

It is easy to observe that equations E_1, E_2, E_3, and E_4 are unstable in the whole plane $\mathbb{R}^2$. To analyze the stability of the switched system origin consider the following four linear Lyapunov functions defined in corresponding regions:

$$V_1(x, y) = x + y,$$
$$V_1(x, y) > 0, \ (x, y) \in Q_1 = \{x > 0, y > 0\},$$
$$V_2(x, y) = x - y,$$
$$V_2(x, y) > 0, \ (x, y) \in Q_2 = \{x > 0, y < 0\},$$
$$V_3(x, y) = -x - y, \tag{39}$$
$$V_3(x, y) > 0, \ (x, y) \in Q_3 = \{x < 0, y < 0\},$$
$$V_4(x, y) = -x + y,$$
$$V_4(x, y) > 0, \ (x, y) \in Q_4 = \{x < 0, y > 0\}.$$

Clearly, functions (39) satisfy conditions (a) of Theorem 12 if the norm in $\mathbb{R}^2$ is equal to $\|X\|_1 = |x| + |y|$. The derivatives of functions (39) are

$$\dot{V}_1(x, y) = \dot{x} + \dot{y} = 3x + 2y - 4x - 3y = -x - y$$
$$= -V_1, \quad (x, y) \in Q_1,$$

$$\begin{aligned}\dot{V}_2(x,y) &= \dot{x} - \dot{y} = -3x + 5y + 2x - 4y = -x + y \\ &= -V_2, \quad (x,y) \in Q_2, \\ \dot{V}_3(x,y) &= -\dot{x} - \dot{y} = -3x - y + 4x + 2y = x + y \\ &= -V_3, \quad (x,y) \in Q_3, \\ \dot{V}_4(x,y) &= -\dot{x} + \dot{y} = 4x - 3y - 2x + 2y = 2x - y \\ &< -V_4, \quad (x,y) \in Q_4.\end{aligned} \tag{40}$$

The derivatives of Lyapunov functions V_1, V_2, V_3, and V_4 are negative definite with respect to norm $\|X\|_1$ in the corresponding regions Q_i.

On switching lines $L_{1,2}, \ldots, L_{4,1}$, where $x = 0$ or $y = 0$, the corresponding Lyapunov functions are

$$\begin{aligned}
L_{1,2}&: V_1(x,y) = V_2(x,y) = x, \\
&N_{1,2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \\
&\langle N_{1,2}, f_1 \rangle = 4x > 0, \\
&\langle N_{1,2}, f_2 \rangle = 2x > 0, \\
&\qquad\qquad y = 0, \\
L_{2,3}&: V_2(x,y) = V_3(x,y) = -y, \\
&N_{2,3} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \\
&\langle N_{2,3}, f_2 \rangle = -5y > 0, \\
&\langle N_{2,3}, f_3 \rangle = -y > 0, \\
&\qquad\qquad x = 0, \\
L_{3,4}&: V_3(x,y) = V_4(x,y) = -x, \\
&N_{3,4} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\
&\langle N_{3,4}, f_3 \rangle = -4x > 0, \\
&\langle N_{3,4}, f_4 \rangle = -2x > 0, \\
&\qquad\qquad y = 0, \\
L_{4,1}&: V_4(x,y) = V_1(x,y) = y, \\
&N_{4,1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
&\langle N_{4,1}, f_1 \rangle = 3y > 0, \\
&\langle N_{4,1}, f_1 \rangle = 2y > 0, \\
&\qquad\qquad x = 0.
\end{aligned} \tag{41}$$

Furthermore, all conditions of Theorem 12 are fulfilled; therefore, the trivial solution of the switched system is asymptotically stable (Figure 4). Moreover, as $\dot{V}_i \le -V_i$ the trivial solution is exponentially asymptotically stable.

Example 16. Consider now the switched systems formed by the following nonlinear subsystems:

$$\begin{aligned}
E_1&: \dot{x} = 3x + 2y - xy, \\
&\quad \dot{y} = -4x - 3y - x^2 y^2, \\
&\qquad\qquad (x,y) \in Q_1, \\
E_2&: \dot{x} = -3x + 5y + xy, \\
&\quad \dot{y} = -2x + 4y + x^2 y, \\
&\qquad\qquad (x,y) \in Q_2, \\
E_3&: \dot{x} = 3x + y, \\
&\quad \dot{y} = -4x - 2y + xy, \\
&\qquad\qquad (x,y) \in Q_3, \\
E_4&: \dot{x} = -4x + 3y - xy, \\
&\quad \dot{y} = -2x + 2y + x^3 y, \\
&\qquad\qquad (x,y) \in Q_4,
\end{aligned} \tag{42}$$

with Q_1, Q_2, Q_3, and Q_4 as in Example 15. As before, functions (39) fulfill conditions of Theorem 12 if the norm in $\mathbb{R}^2$ is equal to $\|X\|_1 = |x| + |y|$. The derivatives of functions (39) are

$$\begin{aligned}
\dot{V}_1(x,y) &= \dot{x} + \dot{y} = -x - y - xy - x^2 y \le -V_1(x,y) \\
&< 0, \quad (x,y) \in Q_1, \\
\dot{V}_2(x,y) &= \dot{x} - \dot{y} = -x + y + xy - x^2 y \le -V_2(x,y) \\
&< 0, \quad (x,y) \in Q_2, \\
\dot{V}_3(x,y) &= -\dot{x} - \dot{y} = x + y - xy \le -V_3(x,y) < 0, \\
&\qquad (x,y) \in Q_3, \\
\dot{V}_4(x,y) &= -\dot{x} + \dot{y} = 2x - y + xy + x^3 y \le -V_4(x,y) \\
&< 0, \quad (x,y) \in Q_4.
\end{aligned} \tag{43}$$

The derivatives of Lyapunov functions V_1, V_2, V_3, and V_4 are negative definite with respect to norm $\|X\|_1$ in the corresponding regions Q_i, and on switching lines $L_{1,2}, \ldots, L_{4,1}$, where $x = 0$ or $y = 0$, the corresponding Lyapunov functions coincide with (41). Therefore, the trivial solution of the switched system (42) is exponentially asymptotically stable. The simulation results are shown in Figure 5.

Theorem 17. *Suppose that sequence (4) is infinite and that for each equation E_i, $i = 1, \ldots, p$, there exists a Lyapunov function $V_i(X)$ such that*

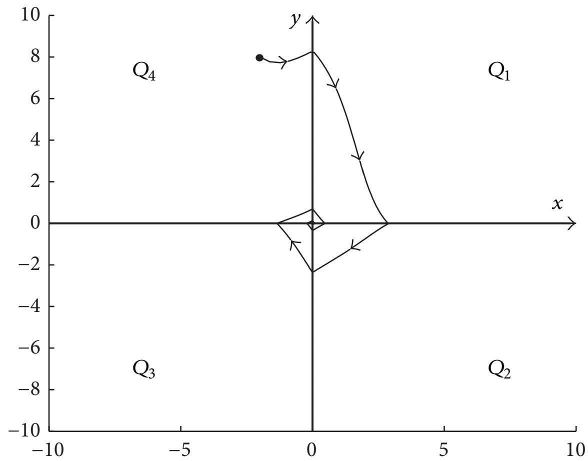

Figure 5: Trajectory of switched system (42).

(a) $\omega_{1,i}(\|X\|) < V_i(X) < \omega_{2,i}(\|X\|)$, $X \in Q_i$, $i = 1, \ldots, p$,

(b) $\dot{V}_i(X(t)) = \langle \nabla(V_i), f_i(X) \rangle \le 0$, $X \in Q_i$, $i = 1, \ldots, p$,

(c) *on all switched lines $L_{i,i+1}$, $1 < i < p$, where the trajectories pass from region Q_i to Q_{i+1}, the following inequalities hold:*

$$V_i(X) \ge V_{i+1}(X), \quad X \in L_{i,i+1}, \; 1 < i < p. \tag{44}$$

Furthermore, there exists at least one line $L_{k,k+1}$, $1 < k < p$ such that

$$V_k(X) \ge V_{k+1}(X) + \omega_{3,k}(\|X\|), \quad X \in L_{k,k+1}. \tag{45}$$

Then, the trivial solution $X(t) \equiv 0$ of the switched system (1)-(2) or (4) is asymptotically stable for a given switched system.

Proof. The trivial solution is stable because conditions of Theorem 12 are satisfied. It means that any Lyapunov function $V_k(X(t))$ in region Q_k as function of t does not increase. Moreover, the sequence of successive Lyapunov functions as function of t does not increase either. More exactly, for any numbers j, k and any instants t_j, t_k such that $X(t_j) \in Q_j$, $X(t_k) \in Q_k$, $t_j < t_k$, the following inequality holds: $V_k(X(t_k)) \le V_j(X(t_j))$.

Suppose the trivial solution $X(t) \equiv 0$ of switched system (4) is not asymptotically stable. It means that there exists the solution $X(t, t_0, X_0)$ with sufficiently small X_0, $\|X_0\| < \delta$, such that it is bounded $\|X(t, t_0, X_0)\| < H$ and does not tend to zero as $t \to \infty$. In this case there exists $\gamma > 0$ and a sequence $t_j \to \infty$ such that $\|X(t_j, t_0, X_0)\| > \gamma$, $\gamma < H$. The solution $X(t, t_0, X_0)$ must satisfy condition

$$\|X(t, t_0, X_0)\| \ge \mu, \quad t \ge t_1, \tag{46}$$

where μ is a number corresponding to γ in definition of stability for solution $X(t) \equiv 0$; that is, $\omega_2(\mu) \le \omega_1(\gamma)$. If (46) does not hold then there exists an instant $T > t_1$ such that

$$\|X(T, t_0, X_0)\| < \mu. \tag{47}$$

Taking now T as a new initial moment and using stability of $X(t, t_0, X_0) = X(t, T, X(T, t_0, X_0))$ this solution must satisfy condition

$$\|X(t, t_0, X_0)\| < \gamma, \quad t > T. \tag{48}$$

Condition (48) contradicts condition (46) and this means that condition (46) holds. It follows from (46) that for an infinite number of moments $t_{i(k)} > T$, $i(k) \to \infty$, when solution $X(t, t_0, X_0)$ crosses line $L_{k,k+1}$ we have $H > \|X(t_{i(k)}, t_0, X_0)\| \ge \mu > 0$, $\omega_{3,k}(\|X(t_{i(k)}, t_0, X_0)\|) \ge \omega_{3,k}(\mu) > 0$, and

$$\begin{aligned}
\omega_2(H) &\ge V_k(X(t_{i(k)}, t_0, X_0)) \\
&\ge V_{k+1}(X(t_{i(k)}, t_0, X_0)) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)}, t_0, X_0)\|) \\
&\ge V_k(X(t_{i(k)-1}, t_0, X_0)) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)-1}, t_0, X_0)\|) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)}, t_0, X_0)\|) \\
&\ge V_{k+1}(X(t_{i(k)-1}, t_0, X_0)) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)-1}, t_0, X_0)\|) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)}, t_0, X_0)\|) \ge \cdots \\
&\ge V_{k+1}(X(t_1, t_0, X_0)) \\
&\quad + \omega_{3,k}(\|X(t_1, t_0, X_0)\|) + \cdots \\
&\quad + \omega_{3,k}(\|X(t_{i(k)-1}, t_0, X_0)\|) \\
&\quad + \omega_{3,k}(\|X(t_{i(k)}, t_0, X_0)\|) \ge i(k)\, \omega_{3,k}(\mu) \\
&\longrightarrow \infty, \quad i(k) \longrightarrow \infty.
\end{aligned} \tag{49}$$

Inequality (49) contradicts condition $\omega_2(H) < \infty$ and our supposition that $X(t, t_0, X_0)$ does not tend to zero conducts to a contradiction. Furthermore, the asymptotic stability of trivial solution $X(t) \equiv 0$ of switched system (4) is proven. □

Theorem 18. *Suppose that sequence (4) is infinite and that for all switching lines $L_{i,i+1}$ and that for each E_i, $i = 1, \ldots, p$, there exists a Lyapunov function $V_i(X)$ such that*

(a) $\omega_{1,i}(\|X\|) < V_i(X) < \omega_{2,i}(\|X\|)$, $X \in Q_i$, $i = 1, \ldots, p$,

(b) $\dot{V}_i(X(t)) = \langle \nabla(V_i), f_i(X) \rangle \ge 0$, $X \in Q_i$, $i = 1, \ldots, p$,

(c) *on all switched lines $L_{i,i+1}$, $1 \le i < p$, where the trajectories pass from region Q_i to Q_{i+1}, the following inequalities hold:*

$$V_i(X) \le V_{i+1}(X), \quad X \in L_{i,i+1}, \; 1 \le i < p, \tag{50}$$

furthermore, there exists at least one line $L_{k,k+1}$, $1 \le k < p$, such that

$$V_k(X) \le V_{k+1}(X) + \omega_{3,k}(\|X\|), \quad X \in L_{k,k+1}. \tag{51}$$

Then, the trivial solution $X(t) \equiv 0$ of the switched system (4) is unstable for a given switched system.

Proof. Theorem 18 is a clear modification of Theorem 17. Therefore its proof is omitted. □

Example 19. Consider the switched system formed by two nonlinear pendulums:

$$E_1\colon \ddot{x} + \sin x = 0, \quad (x, \dot{x} = y) \in Q_1 = \{x < 0, y > 0\}, \tag{52}$$
$$E_2\colon \ddot{x} + \omega^2 \sin x = 0, \quad (x, \dot{x} = y) \in \mathbb{R}^2 - Q_1.$$

Take as Lyapunov functions for (52) the following functions:

$$V_1(x, y) = (1 - \cos x) + \frac{1}{2}y^2, \quad (x, \dot{x} = y) \in Q_1 = \{x < 0, y > 0\},$$
$$V_2(x, y) = \omega^2 (1 - \cos x) + \frac{1}{2}y^2, \quad (x, \dot{x} = y) \in \mathbb{R}^2 - Q_1. \tag{53}$$

The derivatives of functions (53) along the trajectories of (52) are equal to zero: $\dot{V}_1 = \dot{V}_2 = 0$. On switching line $L_{1,2}$ where $x = 0$, $y > 0$ we have $V_1(x, y) = (1/2)y^2 = V_2(x, y)$. On switching line $L_{2,1}$ where $x < 0$ and $y = 0$ we have $V_1(x, y) = (1 - \cos x)$ and $V_2(x, y) = \omega^2(1 - \cos x)$. Therefore, condition (c) of Theorem 18 holds on both lines $L_{1,2}$ and $L_{2,1}$ if

$$\omega^2 < 1. \tag{54}$$

Condition (54) is a condition of instability of the trivial solution of switched system (52). Also, this condition may be considered as a condition of parametric discontinuous excitations for the nonlinear pendulum.

5. Autonomous Multidimensional Switched Systems

Consider multidimensional switched system acting in $\mathbb{R}^n$ which is divided into a finite number p of open 1-connected regions Q_k, $1 \le k \le p$, $Q_1 + Q_2 + \cdots + Q_p = \mathbb{R}^n$, such that the origin $\{0\}$ of rectangular coordinates systems pertains to closure of any region Q_k. The boundaries of all regions Q_k are supposed to be smooth. Suppose, also, that each region Q_l has points X fulfilling condition $\|X\| > H$, where H is any number. An equation E_k is defined in each region Q_k satisfying conditions of existence and uniqueness of solutions. In multidimensional case, the switched system may have much more complicated behavior because region Q_k may have more than two boundaries separating it from other regions. To eliminate topological complications, consider only the case when all boundaries (switching surfaces) $B_{k,k+l}$ separating regions Q_k and Q_{k+l} are planes in $\mathbb{R}^n$. Suppose the trajectories of switched system pass through the boundary $B_{k,k+l}$ in all points only in direction from Q_k to Q_{k+l} and on all lines $L_{k,k+l}$ where Q_k touches other regions all trajectories pass from Q_k only to one determined region Q_{k+j}. Suppose also the existence of some Lyapunov functions $V_k(X)$ for all equation E_k, $1 \le k \le p$. In this case theorems similar to Theorems 10, 12, 17, and 18 may be established. The definitions of stability for switched systems in $\mathbb{R}^n$ are the same as for the two-dimensional case. Now, consider the case of a switched system with infinite number of switchings. Let us formulate one of the theorems on stability.

Theorem 20. *Suppose that for each equation E_k acting in $Q_k \in \mathbb{R}^n$ there exists a Lyapunov function $V_k(X)$ defined in Q_k such that*

(a) $\omega_{1,k}(\|X\|) < V_k(X) < \omega_{2,k}(\|X\|)$, $X \in Q_k$, $k = 1, \ldots, p$,

(b) $\dot{V}_k(X(t)) = \langle \nabla(V_k), f_k(X) \rangle < 0$, $X \in Q_k$, $k = 1, \ldots, p$,

(c) *on all switched surfaces $B_{k,k+j}$ and lines $L_{k,k+j}$ where the trajectories pass from region Q_k to Q_{k+j}, the following inequalities hold:*

$$V_k(X) > V_{k+j}(X), \quad X \in B_{k,k+j},$$
$$V_k(X) > V_{k+l}(X), \quad X \in L_{k,k+l}. \tag{55}$$

Then, the trivial solution $X(t) \equiv 0$ of the switched system acting in $\mathbb{R}^n$ is stable (asymptotically stable) for a given switched structure.

Proof. The demonstration of this theorem is similar to proof of Theorem 12. Therefore it is omitted. □

Example 21. In space $\mathbb{R}^3$ consider two regions $Q_1 = \{z > 0\}$ and $Q_2 = \{z < 0\}$. Suppose equations E_1 and E_2 are defined in Q_1 and Q_2:

$$\begin{aligned} E_1\colon \dot{x} &= -x + y + y^2, \\ \dot{y} &= -x - y, \\ \dot{z} &= -2z - zx^2 - 2y^2 x, \quad (x, y, z) \in Q_1, \\ E_2\colon \dot{x} &= -x + y - y^2, \\ \dot{y} &= -x - y - xy, \\ \dot{z} &= -2z - 4y^2 x, \quad (x, y, z) \in Q_2. \end{aligned} \tag{56}$$

Normals $N_{1,2}$ and $N_{2,1}$ to the surface $B = z = 0$ separating Q_1 and Q_2 are $N_{1,2}^T = [0, 0, -1]^T$ and $N_{2,1}^T = [0, 0, 1]^T$. Therefore, condition (3) holds on surface $B = z = 0$:

$$\begin{gathered} \left\langle N_{1,2}, f_{E_1}\right\rangle = 2y^2x > 0, \\ \left\langle N_{1,2}, f_{E_2}\right\rangle = 4y^2x > 0, \\ x > 0, \\ \left\langle N_{2,1}, f_{E_1}\right\rangle = -2y^2x > 0, \\ \left\langle N_{2,1}, f_{E_2}\right\rangle = -4y^2x > 0, \\ x < 0. \end{gathered} \tag{57}$$

Therefore, surfaces $B_{1,2}$ and $B_{2,1}$ where the trajectories pass from E_1 to E_2 or vice versa are $B_{1,2} = \{z = 0, x > 0, y = \text{arbitrary}\}$ and $B_{2,1} = \{z = 0, x < 0, y = \text{arbitrary}\}$. On line $\{z = 0, x = 0, y = \text{arbitrary}\}$ the derivatives $\dot{x}$ from one and another side of this line are positive: $\dot{x} = y^2 > 0$. Therefore, the trajectory which arrives on this line passes throughout it and then enters into the region Q_2. Consider two Lyapunov functions V_1 and V_2 defined in Q_1 and Q_2, respectively:

$$\begin{gathered} V_1(x, y, z) = x^2 + y^2 + z, \quad (x, y, z) \in Q_1, \\ V_2(x, y, z) = x^2 + y^2 - z, \quad (x, y, z) \in Q_2. \end{gathered} \tag{58}$$

Derivatives $\dot{V}_{1,E_1}$ and $\dot{V}_{2,E_2}$ computed on the trajectories of equations E_1 and E_2, respectively, are

$$\begin{aligned} \dot{V}_{1,E_1} &= 2x\dot{x} + 2y\dot{y} + \dot{z} \\ &= 2x\left(-x + y + y^2\right) + 2y\left(-x - y\right) \\ &\quad + \left(-2z - zx^2 - 2y^2x\right) \\ &= -2x^2 - 2y^2 - 2z - zx^2 < -2V_1, \\ &\qquad (x, y, z) \in Q_1, \\ \dot{V}_{2,E_2} &= 2x\dot{x} + 2y\dot{y} - \dot{z} \\ &= 2x\left(-x + y - y^2\right) + 2y\left(-x - y - xy\right) \\ &\quad + \left(-2z - 4y^2x\right) = -2x^2 - 2y^2 + 2z = -2V_2, \\ &\qquad (x, y, z) \in Q_2. \end{aligned} \tag{59}$$

All conditions of Theorem 20 are fulfilled and therefore the considered switching system (56) is asymptotically stable and, moreover, exponentially asymptotically stable (see Figure 6).

6. Conclusions

The stability of autonomous switched systems is investigated. Some theorems giving sufficient conditions of stability, asymptotic stability, or instability of switched systems are established. Some results are similar to conditions of parametric stabilization or excitation of stable systems.

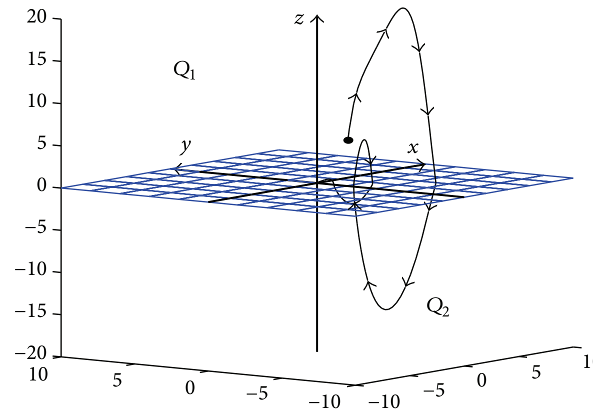

Figure 6: Trajectory of switched system (56).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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