Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2013, Article ID 472430, 8 pages
http://dx.doi.org/10.1155/2013/472430

Research Article

Switched Convergence of Second-Order Switched Homogeneous Systems

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Received 14 June 2013; Accepted 16 August 2013

Academic Editor: Hamid Reza Karimi

This paper studies the stabilization of second-order switched homogeneous systems. We present results that solve the problem of stabilizing a switched homogeneous system; that is, we establish necessary and sufficient conditions under which the stabilization is assured. Moreover, given an initial condition, our method determines if there exists a switching law under which the solution converges to the origin and, if there exists this switching law, how it is constructed. Finally, two numerical examples are presented in order to illustrate the results.

1. Introduction

A switched system is a hybrid system which consists of several subsystems and a rule that orchestrates the switching among them. In the last two decades, there has been increasing interest in stability analysis and control design for switched systems. The motivation is from many aspects. For example, a system with several controllers can be regarded as a switched system. Both discrete- and continuous-time switched systems have been studied from different aspects. Recently, for discrete-time switched systems, the H_∞ filtering problem has been studied by including time delay [1–3]. Also, recent works [4–6] have included the stabilization problem for this kind of switched systems from different points of view.

The problem of stability has been focused in three main problems [7, 8]: first, find conditions that guarantee that the switched system is asymptotically stable under arbitrary switching; second, define a class of switching laws under which the switched system is stable; third, construct a switching law that makes the switched system asymptotically stable. Recently, the study of the stability has been extended to the case when there exists time delay [9, 10].

For the last problem, there are basically two approaches to stabilization for switched systems, one based on detailed analysis of the vector field and the other one based on Lyapunov theory. In papers such as [11, 12] or [13], Lyapunov theory is employed to construct switching laws for a class of switched systems totally composed of unstable subsystems.

With respect to the analysis of the vector field, for second-order systems, some necessary and sufficient conditions for stability/stabilizability have been obtained in [14–16] through analysis on the structure of the vector field. In [16], the problem of stabilizing two second-order switched linear systems is solved; that is, a method to define switching laws that decide when a switched system is stabilizable is given. In [15], some results about the stabilization in the nonlinear case are presented.

The switched homogeneous systems are a superclass of switched linear systems. The stability of these systems has been studied in [17–19] with the objective of finding conditions under which the switched system is asymptotically stable. Recently, in [20, 21], the problem of stability is also studied. Aleksandrov et al. [20] investigate the stability of these systems from the Lyapunov approach, and Tuna [21] considers discrete-time homogeneous systems under arbitrary switching. These papers solve the problem of stability under arbitrary switching. However, for switched homogeneous systems, the problem of stabilization, that is, the problem of constructing a switching law that stabilizes the system, is still open.

In this work, we consider this problem; that is, we search for a switching law that makes the switched homogeneous system asymptotically stable. By using the same idea described in [15, 16], we can generalize the results in order to solve the problem of stabilizing a switched homogeneous system. Hence, we establish the conditions under which the solution of a switched homogeneous system converges to the origin under two types of switching laws. It is important to remark that only the idea in [15] is used, and the results cannot be directly applied to this type of switched systems because the preliminary assumptions in [15] cannot be satisfied for switched homogeneous systems.

The remainder of this paper is organized as follows. In Section 2, the problem is formulated and the first results are obtained. In Section 3, we present the switching laws employed in order to establish the convergence. The results that assure the convergence of these switched systems are shown in Section 4. Two numerical examples are given in Section 5 in order to illustrate the results. Finally, the conclusions are provided in Section 6.

2. Preliminaries

Consider a switched nonlinear system

$$\dot{x}(t) = f_{\sigma(t)}(x(t)), \tag{1}$$

where $f_i : D \to \mathbb{R}^2$, $i = 1, 2$, is a vector field of class $\mathscr{C}^1$ in the open and connected set, D, $x \in \mathbb{R}^2$, and $\sigma : [0, \infty) \to \{1, 2\}$ is a piecewise constant function called *switching law* indicating the active subsystem at each instant. Such a function σ has a finite number of discontinuities, which we call *switching times*, on every bounded time interval and takes a constant value on every interval between two consecutive switching times.

Moreover, for $i = 1, 2$, each coordinate of f_i is a homogeneous function of grade m_i; that is, if $f_i = (f_i^1, f_i^2)$ then $f_i^1(\lambda x) = \lambda_i^m f_i^1(x)$ and $f_i^2(\lambda x) = \lambda_i^m f_i^2(x)$, for any $x \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$.

Throughout the paper, we choose the orientation in $\mathbb{R}^2$ given by the canonical basis; that is, we will say that a basis in the plane has the same orientation than the canonical basis (positive orientation) if the determinant of the change of basis matrix is strictly positive.

Under this orientation, given a point x in the plane, we will say that the trajectory of f_i has positive orientation (clockwise direction) at x if $\det(f_i(x), x) > 0$. Similarly, we will say that the trajectory of f_i has negative orientation (counterclockwise direction) at x if the determinant is negative. Note that if the determinant is null at x, it has no sense to consider the orientation because $\{f(x), x\}$ is not a basis for $\mathbb{R}^2$. Under this assumption, we define $G_i(x) = \det(f_i(x), x)$ for $i = 1, 2$.

We suppose that the origin is an unstable equilibrium point for each subsystem f_1 and f_2. In this work, we will use a generalization of a weaker property than the classic concept of stability. This concept was defined for the nonlinear case in [22].

Definition 1 (see [22]). A state $x_0 \in \mathbb{R}^2$ is switched convergent if there exists a switching law σ_{x_0} such that the solution of the switched system (1) that starts at $x(0) = x_0$ converges to the origin, that is, $\lim_{t\to+\infty}\phi(t; x_0, \sigma_{x_0}) = 0$, where $\phi(\cdot; x_0, \sigma_{x_0})$ is the solution of the switched system (1) for $x(0) = x_0$ and the switching law σ_{x_0}.

The switching laws under which a state is switched convergent will be called convergent switching laws.

In order to study the convergence, it is important to consider the set of points where the subsystems f_1 and f_2 are parallel. Hence, we define the function $F(x) = \det(f_1(x), f_2(x))$ and the set $\Omega = \{x \in \mathbb{R}^2 : F(x) = 0\}$.

In [15, 22], some conditions for F, G_1, and G_2 are necessary in order to apply the method of stabilization. However, it is easy to prove that, for switched homogeneous systems, these conditions are not satisfied. For this reason, we must study the properties of these functions for switched homogeneous systems and assume a new condition.

Assumption 2. For any $x_0 \in \mathbb{R}^2 \setminus \{0\}$ with $F(x_0) = 0$, it holds that $\nabla F(x_0) \neq 0$. Similarly, for any $x_0 \in \mathbb{R}^2 \setminus \{0\}$ with $G_i(x_0) = 0$ for some $i = 1, 2$, it holds that $\nabla G_i(x_0) \neq 0$.

Now, the following result lets us obtain a consequence of this assumption.

Proposition 3. *Let be g a real homogeneous function of grade m and of class $\mathscr{C}^1$ in the open and connected set $D \subset \mathbb{R}^2$. If $\{x \in \mathbb{R}^2 : g(x) = 0\} \neq \{0\}$ and $\nabla g(x_0) \neq 0$ for any $x_0 \in \mathbb{R}^2 \setminus \{0\}$ with $g(x_0) = 0$, then $\{x \in \mathbb{R}^2 : g(x) = 0\} \neq \{0\}$ is equal to the union of rays that go through the origin.*

Proof. It is deduced from the definition of homogeneous function. □

Then, under Assumption 2, from the previous proposition, it is deduced that if the sets $\{x \in \mathbb{R}^2 : F(x) = 0\}$, $\{x \in \mathbb{R}^2 : G_1(x) = 0\}$ and $\{x \in \mathbb{R}^2 : G_2(x) = 0\}$ are different from the origin, they are equal to the union of rays that go through the origin. Hence, if, for example, $\{x \in \mathbb{R}^2 : F(x) = 0\}$ is different from the origin, the complementary of this set is equal to the plane minus some rays that go through the origin. In order to identify these sets, we define the following.

Definition 4. Let l_1 and l_2 be two rays starting from the origin. The set given by

$$\left\{x \in \mathbb{R}^2 : x = \mu z_1 + (1-\mu) z_2,\ z_1 \in l_1,\ z_2 \in l_2,\ 0 \le \mu \le 1\right\} \tag{2}$$

will be called the cone delimited by l_1 and l_2 and denoted by $C(0, l_1, l_2)$.

Therefore, with this definition, if $\{x \in \mathbb{R}^2 : F(x) = 0\}$ is different from the origin, the complementary of this set is equal to the union of several cones where it is satisfied that $F(x) > 0$ or $F(x) < 0$. In the same form, we have that if, for example, $\{x \in \mathbb{R}^2 : G_i(x) = 0\} \neq \{0\}$ for some $i = 1, 2$, the complementary of this set is equal to the union of several cones where it is satisfied that $G_i(x) > 0$ or $G_i(x) < 0$.

Throughout the paper, we also assume the following.

Assumption 5. $\{x \in \mathbb{R}^2 : F(x) = 0\} \neq \{0\}$.

We can assume this condition without loss of generality because if it is not satisfied, there exists no switched convergent state (see Proposition 2 in [15]). Therefore, in that case, our study has no sense.

Under these two assumptions, two lemmas under which the main results will be established can be proved in a similar way to the lemmas presented in [15].

Lemma 6. *Consider the switched homogeneous system* (1). *Let l_1 and l_2 be two different rays starting from the origin and $x_0 \in l_1 \setminus \{0\}$. Suppose that, for each $i = 1, 2$, the trajectory of f_i starting from x_0 is of clockwise direction (or counterclockwise) and intersects the ray l_2 at x_i. Then,*

(i) *if l_1, l_2, and the trajectory of f_i, for $i = 1, 2$, are contained in the same connected component of $\{x \in \mathbb{R}^2 : F(x) > 0\}$, then*

$$\|x_1\| < \|x_2\| \quad (\|x_2\| < \|x_1\| \text{ in counterclockwise case}); \tag{3}$$

(ii) *if l_1, l_2, and the trajectory of f_i, for $i = 1, 2$, are contained in the same connected component of $\{x \in \mathbb{R}^2 : F(x) < 0\}$, then*

$$\|x_2\| < \|x_1\| \quad (\|x_1\| < \|x_2\| \text{ in counterclockwise case}). \tag{4}$$

Before presenting the second lemma, we need to introduce some notations. We will denote by E_1 the region where the first subsystem is of clockwise direction and the second one is of counterclockwise direction. Similarly, we will denote by E_2 the region where the first subsystem is of counterclockwise direction and the second one is of clockwise direction.

Now, we know that, for each $i = 1, 2$, the set $\{x \in \mathbb{R}^2 : G_i(x) = 0\}$ is equal to the origin or the union of rays that pass through the origin. If the set $\{x \in \mathbb{R}^2 : G_i(x) = 0\}$ is equal to $\{0\}$, we have that $G_i(x) > 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$ or $G_i(x) < 0$ for all $x \in \mathbb{R}^2 \setminus \{0\}$. If, on the contrary, $\{x \in \mathbb{R}^2 : G_i(x) = 0\}$ consists of several rays that pass through the origin, $\mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 : G_i(x) = 0\}$ is equal to several cones where it is satisfied that either $G_i(x) > 0$ or $G_i(x) < 0$, then the sets E_1 and E_2 are empty, coincide with $\mathbb{R} \setminus \{0\}$, or are the union of several cones (see Figure 1).

Lemma 7. *Let* (1) *be a switched homogeneous system, and suppose that $C(0, l_1, l_2)$ is a connected component of $\{x \in \mathbb{R}^2 : F(x) < 0\} \cap E_1$. One supposes that the trajectory $\mathcal{T}_1$ of f_1 goes from $x_0 \in l_1$ to $x_1 \in l_2$ (in clockwise direction) and the trajectory $\mathcal{T}_2$ of f_2 goes from $x_1 \in l_2$ to $x_2 \in l_1$ (in counterclockwise direction); then, it holds that $\|x_0\| < \|x_2\|$.*

When $C(0, l_1, l_2)$ is a connected component of $\{x \in \mathbb{R}^2 : F(x) > 0\} \cap E_1$, it is obtained that $\|x_0\| < \|x_2\|$.

A similar result can be enunciated for E_2.

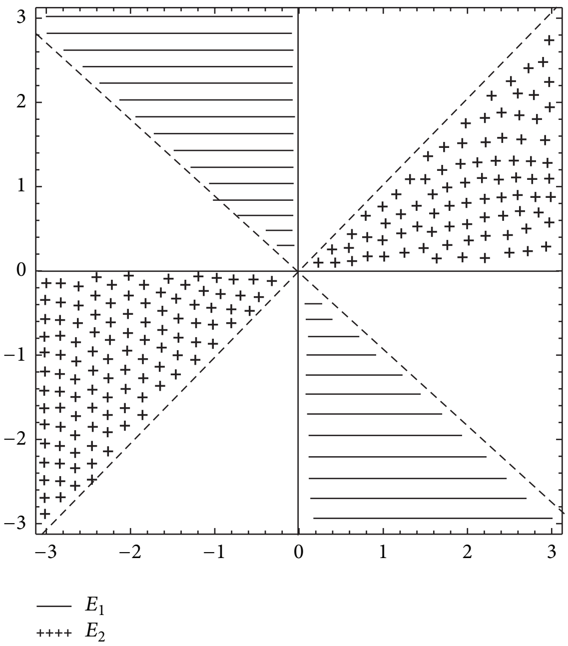

FIGURE 1: The sets E_1 and E_2 and the rays $\{x \in \mathbb{R}^2 : G_i(x) = 0\}$.

3. Convergent Switching Laws

In order to establish the convergence, two types of switching laws are presented in [15]. In this work, we use the same idea to define the switching laws although, due to the particular properties of the sets $\{x \in \mathbb{R}^2 : F(x) = 0\}$ and $\{x \in \mathbb{R}^2 : G_i(x) = 0\}$, the definition of these switching laws is different.

Firstly, under the switching laws of type I, our goal is that the solution rotates around the origin. For this reason, if the solutions must rotate around the origin in clockwise direction, it is easy to prove that the following must be verified:

$$\begin{aligned} \{x \in \mathbb{R}^2 \setminus \{0\} : F(x) \geq 0\} &\subset \{x \in \mathbb{R}^2 : G_1(x) > 0\}, \\ \{x \in \mathbb{R}^2 \setminus \{0\} : F(x) \leq 0\} &\subset \{x \in \mathbb{R}^2 : G_2(x) > 0\}. \end{aligned} \tag{5}$$

In the same form, it can be proved that if the solution must rotate around the origin in counterclockwise direction, it must be verified that

$$\begin{aligned} \{x \in \mathbb{R}^2 \setminus \{0\} : F(x) \leq 0\} &\subset \{x \in \mathbb{R}^2 : G_1(x) < 0\}, \\ \{x \in \mathbb{R}^2 \setminus \{0\} : F(x) \geq 0\} &\subset \{x \in \mathbb{R}^2 : G_2(x) < 0\}. \end{aligned} \tag{6}$$

Under these conditions, we define the switching law of type I in the following form.

Definition 8. Let (1) be a switched homogeneous system and suppose that the condition (5) (resp., (6)) is satisfied. Given an initial condition $x_0 \in \mathbb{R}^2 \setminus \{0\}$, one will say that a switching law σ is *of type I* if

(i) $\sigma(t) = 1$ (resp. $\sigma(t) = 2$) if $\phi(t; x_0, \sigma) \in \{x \in \mathbb{R}^2 : F(x) > 0\}$,

(ii) $\sigma(t) = 2$ (resp. $\sigma(t) = 1$) if $\phi(t; x_0, \sigma) \in \{x \in \mathbb{R}^2 : F(x) < 0\}$,

where ϕ is the solution of the switched system (1) under the switching law σ starting from x_0.

The initial conditions $x_0 \in \mathbb{R}^2$ for which there exists a switching law of type I are called states *of type I*, and the switched systems for which there exist switching laws of type I are called switched systems *of type I*.

Now, for the switching laws of type II, as E_1 and E_2 are equal to $\mathbb{R}^2 \setminus \{0\}$ or the union of cones, we define a switching law of type II in the following form.

Definition 9. Let (1) be a switched homogeneous system and $x_0 \in E_1$ (resp., E_2) a state. One says that a switching law is *of type II* if one switchs whenever the trajectory intersects the boundary of $\{x \in \mathbb{R}^2 : F(x) < 0\} \cap E_1$ (resp., $\{x \in \mathbb{R}^2 : F(x) > 0\} \cap E_2$).

It is important to note that the condition under which the switching law of type II can be defined is that $\{x \in \mathbb{R}^2 : F(x) < 0\} \cap E_1 \neq \emptyset$ or $\{x \in \mathbb{R}^2 : F(x) > 0\} \cap E_2 \neq \emptyset$.

4. Switched Convergence of Switched Homogeneous Systems

In this section, we solve the problem of the convergence of switched homogeneous systems by using the previous switching laws.

In [15], the hypothesis of the result that assures the convergence under switching laws of type I is not straightforward to check. However, for switched homogeneous systems, in order to solve the convergence, we only check the conditions given in the previous section.

Theorem 10. *Consider the switched homogeneous system* (1). *Suppose that* (5) *or* (6) *is satisfied. Let $x_0 \in \mathbb{R}^2 \setminus \{0\}$ be a state. It holds that x_0 is σ-convergent for any switching law σ if and only if x_0 is σ_I-convergent for any switching law σ_I of type I.*

Proof. One of the implications is obvious. Therefore, we will prove that if x_0 is convergent for any switching law, it is also convergent for a switching law of type I. Firstly, as (5) or (6) is satisfied, the solution for the switching law of type I rotates around the origin. Thus, if we choose a ray l_1 contained in $\{x \in \mathbb{R}^2 : F(x) = 0\}$, there exists $\overline{x}_0$ the point where the solution intersects, for the first time, l_1. Moreover, there exists $\{x_n : n \in \mathbb{N}\}$, the set of points where the solution of the system initialized in $\overline{x}_0$ under a switching law of type I intersects the ray l_1. Therefore, if we proceed as in the proof of Theorem 1 in [15], we can prove that the sequence $(x_n)_n$ converges to the origin and, thus, the switching law of type I is convergent. □

In [15], only a sufficient condition for the convergence of switching laws of type II is presented. However, in this case, more results are deduced from the study of this type of switching laws. Firstly, we have the following proposition.

Proposition 11. *Consider the switched homogeneous system* (1). *If $E_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\} \neq \emptyset$, then x_0 is switched convergent for any $x_0 \in E_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\}$.*

A similar result can be obtained for $E_2 \cap \{x \in \mathbb{R}^2 : F(x) > 0\}$.

Proof. Firstly, if $E_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\} \neq \emptyset$, we have that this set is equal to a cone. Therefore, there exist l_1 and l_2 such that $E_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\} = C(0, l_1, l_2)$.

Moreover, if $x_0 \in E_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\}$, then it is possible to define for x_0 a switching law of type II. Hence, the solution under this switching law intersects the rays l_1 and l_2. If we denote by $\{x_n : n \in \mathbb{N}\}$ the set of points where the solution intersects, for example, the ray l_1, from Lemma 7 it is deduced that the sequence $(x_n)_n$ converges to the origin. As the solution is bounded, we have that the solution also converges to the origin. □

Now, it is possible to obtain, in this case, a necessary and sufficient condition for the switched convergence.

Theorem 12. *Consider the switched homogeneous system* (1). *Suppose that E_1 (resp., E_2) is not empty. Let S_1 (resp., S_2) be a connected component of E_1 (resp., E_2), $x_0 \in S_1$ (resp., S_2) and σ a switching law such that the solution under the switching law σ starting from x_0 is contained in S_1. Then, x_0 is σ-convergent if and only if*

$$S_1 \cap \left\{x \in \mathbb{R}^2 : F(x) < 0\right\} \neq \emptyset, \quad \left(resp.\ S_2 \cap \left\{x \in \mathbb{R}^2 : F(x) > 0\right\} \neq \emptyset\right). \tag{7}$$

Proof. Suppose that $S_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\} \neq \emptyset$ (the proof is similar with S_2). Then, from Proposition 11, it is deduced that $x_0 \in S_1$ is σ_{II}-convergent under a switching law σ_{II} of type II. Moreover, by definition of the switching law of type II, the solution under this switching law is contained in S_1.

Suppose now that there exists a switching law σ such that x_0 is σ-convergent and the solution is contained in S_1 but $S_1 \cap \{x \in \mathbb{R}^2 : F(x) < 0\}$ is empty. Then,

$$S_1 \subseteq \left\{x \in \mathbb{R}^2 : F(x) \geq 0\right\}. \tag{8}$$

Therefore, S_1 is a cone so it is contained in $\{x \in \mathbb{R}^2 : F(x) \geq 0\}$, which is equal to the union of cones. Thus, $S_1 = C(0, l_1, l_2)$, and in l_1 and l_2, it is satisfied that $G_1(x) = 0$ or $G_2(x) = 0$. Now, we can distinguish the following cases.

(i) For each $x \in l_1 \cup l_2$, $G_1(x) = 0$ (see Figure 2(a)).

(ii) For each $x \in l_1 \cup l_2$, $G_2(x) = 0$ (see Figure 2(b)).

(iii) For each $x \in l_1$, $G_1(x) = 0$ and for each $x \in l_2$, $G_2(x) = 0$ (see Figure 2(c)).

(iv) For each $x \in l_1$, $G_2(x) = 0$ and for each $x \in l_2$, $G_1(x) = 0$ (see Figure 2(d)).

Before studying each case, we can suppose, without loss of generality, that $\sigma(0) = 1$ and x_1 is the state where the first switching is produced.

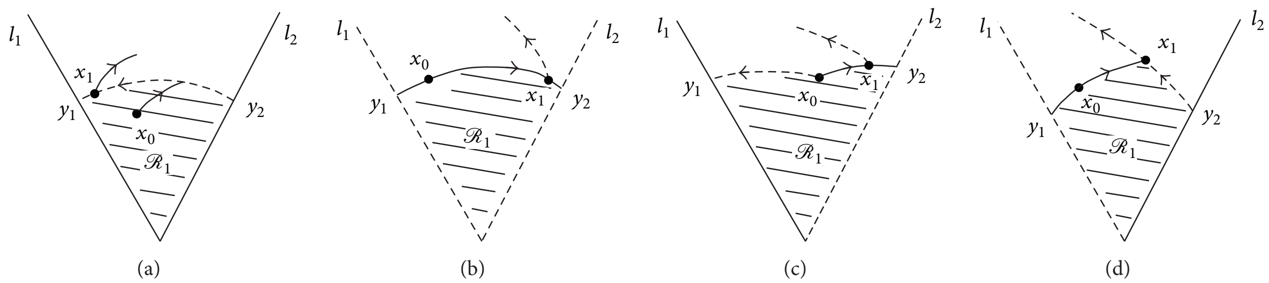

FIGURE 2: Figures for the proof of Theorem 12.

If $G_1(x) = 0$ if $x \in l_1 \cup l_2$, let y_1 and y_2 be the states where the trajectory of f_2 that goes through x_1 intersects l_1 and l_2, respectively (see Figure 2(a)). Hence, we consider the closed curve consisting of the union of this trajectory, the arc $[y_1, 0]$ of l_1, and the arc $[0, y_2]$ of l_2. Then, by the Jordan curve theorem, there exist two regions that are open, connected, and disjoint and whose union is the complementary set of the union of the curves. Let $\mathscr{R}_1$ denote the bounded region whose boundaries are the considered curves. When, according to the switching law σ, the second switching is produced in the trajectory of f_2, we switch to the first subsystem and the solution leaves $\mathscr{R}_1$ so $F(x) \geq 0$.

If S_1 is delimited by the rays given by $G_2(x) = 0$, we consider the trajectory of f_1 that goes through x_0 and intersects l_1 and l_2 at y_1 and y_2, respectively (see Figure 2(b)). Again, we define the closed curve consisting of this trajectory, the arc $[y_2, 0]$ of l_2, and the arc $[0, y_1]$ of l_1. Then, by the Jordan curve theorem, there exist two regions that are open, connected, and disjoint and whose union is the complementary set of the union of the curves. Let $\mathscr{R}_1$ denote the bounded region whose boundaries are the considered curves. When, according to the switching law σ, the second switching is produced in the trajectory of f_2, we switch to the first subsystem and the solution leaves $\mathscr{R}_1$ so $F(x) \geq 0$.

When $G_1(x) = 0$ if $x \in l_1$ and $G_2(x) = 0$ if $x \in l_2$, we choose the trajectory of f_1 starting from x_0 that intersects l_2 at y_2 and the trajectory of f_2 starting from x_0 that intersects l_1 at y_1 (see Figure 2(c)). Now, the closed curve considered is the union of the trajectories of f_1 and f_2 and the arcs $[0, y_1]$ and $[y_2, 0]$ of l_1 and l_2, respectively. By the Jordan curve theorem, there exist two regions that are open, connected, and disjoint and whose union is the complementary set of the union of the curves. Let $\mathscr{R}_1$ denote the bounded region whose boundaries are the considered curves. Again, when the next switching is produced, the solution goes into the unbounded region, so $F(x) \geq 0$.

Finally, suppose that $G_2(x) = 0$ if $x \in l_1$ and $G_1(x) = 0$ if $x \in l_2$. Then, let y_1 be the point where the trajectory of $-f_1$ starting from x_0 intersects l_1, and let y_2 be the point where the trajectory of $-f_2$ starting from x_1 intersects l_2 (see Figure 2(d)). Consider the closed curve equal to the union of the trajectories of f_1 and f_2 and the arcs $[0, y_1]$ and $[0, y_2]$ of l_1 and l_2, respectively. By the Jordan Curve Theorem, we obtain a bounded region $\mathscr{R}_1$. In this case, it is clear that when the next switching is produced at x_1, the solution goes into the unbounded region.

Hence, in all cases, we have obtained a closed curve and two regions (bounded and unbounded), and the solution, in the next switching, goes into the unbounded region. As, by hypothesis, the solution converges to the origin and is contained in S_1, this solution must go into the bounded region $\mathscr{R}_1$ and, thus, must intersect the closed curve. In particular, it must intersect the arc that goes from y_1 to y_2 and is given by the trajectories of f_1 and/or f_2. We will prove that this is not possible.

If the solution is given by the subsystem f_1, it is clear that it cannot intersect the arc of trajectory given by f_1 so they are solutions of the same subsystem. And it cannot intersect the arc of trajectory of f_2 so, in that case, we could find a state $\overline{x} \in S_1$ such that $F(\overline{x}) < 0$.

In the same form, if the solution is given by the subsystem f_2, it is clear that it cannot intersect the arc of trajectory of f_2 so they are solutions of the same subsystem, and it cannot intersect the trajectory of f_1 so, in that case, we could find a state $\overline{x} \in S_1$ such that $F(\overline{x}) < 0$.

Therefore, as the solution cannot intersect the closed curve, it cannot go into $\mathscr{R}_1$, but this is a contradiction; so we have supposed that the solution converges to the origin in S_1. □

In the previous result, we have proved the convergence when the solution cannot leave a connected component of E_1 or E_2. Our goal is proving the convergence without assuming that the solution is in a cone. For this reason, the following definition is needed.

Definition 13. Let S_1 be a connected component of E_1. One will say that S_1 is a *sealed cone* if it is a cone equal to $C(0, l_1, l_2)$ such that $G_1(x) = 0$ if $x \in l_2$ and $G_2(x) = 0$ if $x \in l_1$.

Similarly, given S_2 a connected component of E_2, one will say that S_2 is a *sealed cone* if it is a cone equal to $C(0, l_1, l_2)$ such that $G_1(x) = 0$ if $x \in l_1$ and $G_2(x) = 0$ if $x \in l_2$.

Given an initial condition x_0 in the interior of a sealed cone S_1, if we choose the trajectory of f_1, we know that this trajectory is of clockwise direction and cannot intersect l_2; so in this ray, it holds that $G_1(x) = 0$. In the same manner, if we choose the trajectory of f_2, this trajectory cannot intersect l_1 so, in this ray, it holds that $G_2(x) = 0$. Therefore, the following result can be deduced.

Proposition 14. *Consider the switched homogeneous system* (1). *Suppose that E_1 (resp., E_2) is not empty. If S_1 (resp., S_2) is*

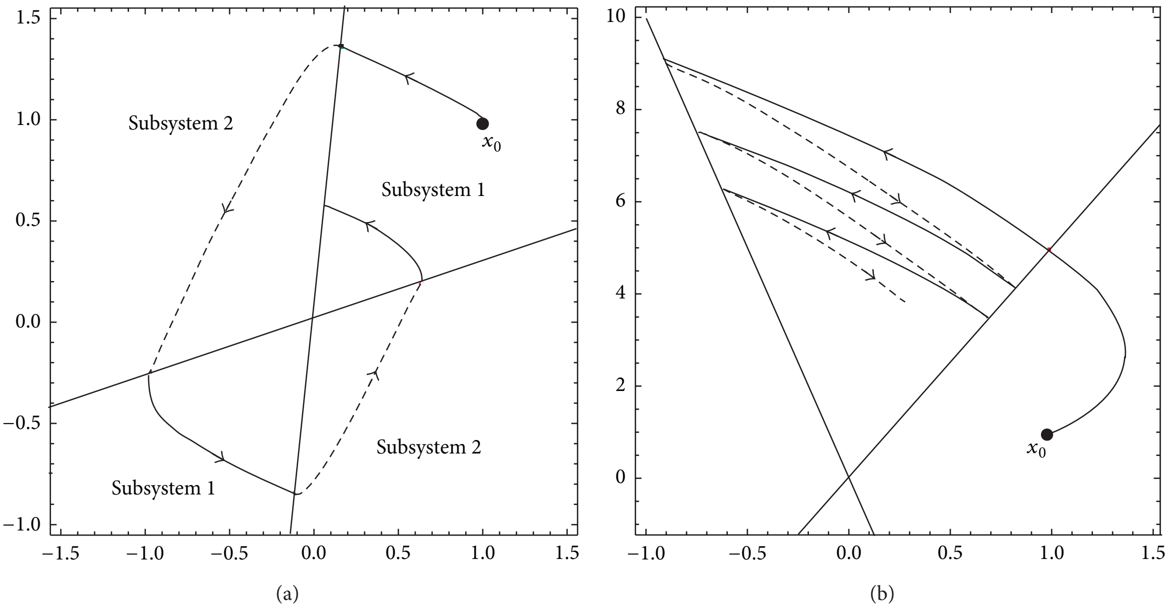

FIGURE 3: Solution and rays given by $F(x) = 0$ in Example 1 (a) and Example 2 (b).

a sealed cone, for any $x_0 \in S_1$ (resp., S_2) and any switching law σ, the solution of the switched system (1) *is in S_1 (resp., S_2).*

This proposition proves that if the solution starts in a sealed cone, it cannot leave this cone. Hence, by using the previous theorem and this proposition, we have the following result.

Corollary 15. *Consider the switched homogeneous system* (1). *Suppose that E_1 (resp., E_2) is not empty. If S_1 (resp., S_2) is a sealed cone of E_1 (resp., E_2) and $x_0 \in S_1$ (resp., S_2), then x_0 is σ-convergent for some switching law σ if and only if*

$$\begin{aligned} &S_1 \cap \left\{x \in \mathbb{R}^2 : \det\left(A_1 x, A_2 x\right) > 0\right\} \neq \emptyset, \\ &\left(\textit{resp. } S_2 \cap \left\{x \in \mathbb{R}^2 : \det\left(A_1 x, A_2 x\right) < 0\right\} \neq \emptyset\right). \end{aligned} \tag{9}$$

5. Numerical Examples

Now, in order to illustrate the results, two numerical examples are presented.

Example 1. Consider a switched nonlinear system given by (1), where the subsystems are defined as follows:

$$\begin{aligned} f_1\left(x_1, x_2\right) &= \left(\frac{x_1^3}{6} - \frac{x_1^2 x_2}{2} + \frac{x_1 x_2^2}{3} - x_2^3, \frac{x_1^2 x_2}{6} + \frac{x_2^3}{3}\right), \\ f_2\left(x_1, x_2\right) &= \left(\frac{5x_1^3}{2} - \frac{x_1^2 x_2}{2} + 5x_1 x_2^2 - x_2^3,\right. \\ &\qquad \left.\frac{9x_1^3}{2} - \frac{x_1^2 x_2}{2} + 9x_1 x_2^2 - x_2^3\right). \end{aligned} \tag{10}$$

Firstly, this switched system is homogeneous because each coordinate is homogeneous of grade 3. In order to apply the results in the paper, we study the functions F, G_1, and G_2:

$$\begin{aligned} F\left(x_1, x_2\right) &= \frac{1}{12}\left(x_1^2 + 2x_2^2\right)^2 \left(9x_1^2 - 33x_1 x_2 + 4x_2^2\right), \\ G_1\left(x_1, x_2\right) &= -\frac{1}{2} x_2^2 \left(x_1^2 + x_2^2\right), \\ G_2\left(x_1, x_2\right) &= -\frac{1}{2}\left(-3x_1 + x_2\right)^2 \left(x_1^2 + 2x_2^2\right). \end{aligned} \tag{11}$$

From the expressions of F, G_1, and G_2, it is deduced that Assumptions 2 and 5 are satisfied. Furthermore, we have that $\{x \in \mathbb{R}^2 : G_1(x) = 0\}$ is equal to the ray given by $x_2 = 0$ and $\{x \in \mathbb{R}^2 : G_2(x) = 0\}$ is equal to the ray given by $x_2 = 3x_1$. Moreover, the trajectories of both subsystems are of counterclockwise direction for any initial condition. Therefore, $E_1 = E_2 = \emptyset$ and the switching laws of type II cannot be defined.

Also, $\{x \in \mathbb{R}^2 : F(x) = 0\}$ is equal to two rays that go through the origin. Moreover, it is easy to prove that (6) holds; thus, we can define switching laws of type I and, by Theorem 10, the switching laws determine the convergence of the system.

Let see what happens for the initial condition $x_0 = (1, 1)$. First, $F(x_0) < 0$; thus, according to the definition of switching law of type I, we choose the first subsystem and switch when the solution intersects the rays in $\{x \in \mathbb{R}^2 : F(x) = 0\}$. In this case, the solution converges to the origin (see Figure 3(a)).

Example 2. Consider a switched nonlinear system given by (1), where the subsystems are defined as follows:

$$f_1(x_1,x_2) = \left(x_1^3 - \frac{x_1^2x_2}{2} + 2x_1x_2^2 - x_2^3, x_1^2x_2 + 2x_2^3\right),$$
$$f_2(x_1,x_2) = \left(\frac{7x_1^3}{2} + \frac{x_1^2x_2}{2} + 7x_1x_2^2 + x_2^3, -\frac{25x_1^3}{2} - \frac{3x_1^2x_2}{2} - 25x_1x_2^2 - 3x_2^3\right). \tag{12}$$

Firstly, this switched system is homogeneous because each coordinate is homogeneous of grade 3. In order to apply the results in the paper, we again study the functions F, G_1, and G_2:

$$F(x_1,x_2) = -\frac{1}{4}\left(50x_2^2 - 5x_1x_2 - x_2^2\right)\left(x_1^2 + 2x_2^2\right)^2,$$
$$G_1(x_1,x_2) = -\frac{1}{2}x_2^2\left(x_1^2 + 2x_2^2\right), \tag{13}$$
$$G_2(x_1,x_2) = \frac{1}{2}(5x_1 + x_2)^2\left(x_1^2 + 2x_2^2\right).$$

From the expressions of F, G_1, and G_2, it is deduced that Assumptions 2 and 5 are satisfied. Furthermore, we have that $\{x \in \mathbb{R}^2 : G_1(x) = 0\}$ is equal to the ray given by $x_2 = 0$ and $\{x \in \mathbb{R}^2 : G_2(x) = 0\}$ is equal to the ray given by $x_2 = -5x_1$. Moreover, the trajectory of f_1 is of counterclockwise direction and the trajectory of f_2 is of clockwise direction. Therefore, E_2 is the complementary of these rays, that is, equal to the union of two cones.

Furthermore, as the set $\{x \in \mathbb{R}^2 : F(x) = 0\}$ is equal to two rays and $E_2 \cap \{x \in \mathbb{R}^2 : F(x) > 0\}$ is not empty, the switching laws of type II can be defined and are convergent by Theorem 12. It is important to note that the cones in E_2 are not sealed; thus, we cannot apply in this case Corollary 15.

In particular, if the initial condition is $x_0 = (1,1)$, then $x_0 \in E_2$ but $F(x_0) < 0$. Thus, we first choose the subsystem f_1 until going into $\{x \in \mathbb{R}^2 : F(x) > 0\}$. Then, we define a switching law of type II, and, by Proposition 11, the switching law of type II is convergent (see Figure 3(b)).

6. Conclusions

In this work, we have solved the problem of stabilization of a second-order switched homogeneous system. Although the idea in [16] for the linear case is used and the results for nonlinear case in [15] are mentioned, it has been necessary to study this kind of switched systems properly so it is not possible to apply directly these results. Furthermore, the study of this kind of switched systems generalizes the linear case by reducing the number of cases to only two switching laws.

As for future work, these results can be applied to study the stabilization of switched nonlinear systems where the conditions in [15] cannot be satisfied, for example, in switched polynomial systems. Hence, it could be possible to solve completely the problem of convergence of second-order switched nonlinear systems. Furthermore, projections may be employed to apply this research to the convergence of higher order switched homogeneous systems. Finally, the method could be employed in real process that can be modelled by homogeneous systems.

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