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Research Article

Dirichlet and Neumann Problems Related to Nonlinear Elliptic Systems: Solvability, Multiple Solutions, Solutions with Positive Components

Luisa Toscano and Speranza Toscano

Department of Mathematics and Applications, "R. Caccioppoli.," University of Naples "Federico II", Via Claudio 21, 80125 Naples, Italy

Correspondence should be addressed to Luisa Toscano, luisatoscano@libero.it

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We study the solvability of Dirichlet and Neumann problems for different classes of nonlinear elliptic systems depending on parameters and with nonmonotone operators, using existence theorems related to a general system of variational equations in a reflexive Banach space. We also point out some regularity properties and the sign of the found solutions components. We often prove the existence of at least two different solutions with positive components.

1. Introduction

In this paper, we present some significant applications of the results got in [1] to Dirichlet problems (Section 2) of the type:

$$\begin{aligned} &-\operatorname{div}(A_i(x,u_1,\ldots,u_n,\nabla u_1,\ldots,\nabla u_n)) \\ &\qquad = \lambda_i b_i |u_i|^{p-2} u_i + d_i(x,u_1,\ldots,u_n,\nabla u_1,\ldots,\nabla u_n) + f_i \quad \text{in } \Omega, \\ &\qquad\qquad u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i = 1,\ldots,n, \end{aligned} \tag{1.1}$$

and to Neumann problems (Section 3) of the type:

$$\begin{aligned} &-\operatorname{div}(A_i(x,u_1,\ldots,u_n,\nabla u_1,\ldots,\nabla u_n)) \\ &\qquad = \lambda_i b_i |u_i|^{p-2} u_i + d_i(x,u_1,\ldots,u_n,\nabla u_1,\ldots,\nabla u_n) + f_i \quad \text{in } \Omega, \end{aligned}$$

$$A_i(x,u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n)\cdot\nu$$
$$= \mu_i\widehat{b}_i|u_i|^{p-2}u_i + \widehat{d}_i(x,u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n) + \widehat{f}_i \quad \text{on } \partial\Omega \quad \text{as } i=1,\dots,n, \tag{1.2}$$

where $n \geq 1$, λ_i, μ_i are real parameters, Ω is a bounded connected open set of R^N with regular boundary $\partial\Omega$, and ν is the outward orthogonal unitary vector to $\partial\Omega$.

The study deals with the solvability of the problems, the existence of multiple solutions with all the components not identically equal to zero and, in the homogeneous case, the existence of solutions with positive components, bounded and locally Hölderian with their first derivatives. It is suitable to recall the problem studied in [1] with some notations and hypotheses.

Let $W_1,\dots,W_n$ real reflexive Banach spaces ($n \geq 1$). Let W be the product space $X_{\ell=1}^{n} W_\ell$. Let $\|\cdot\|$ be the norm on W, $\|\cdot\|_*$ the norm on W^* (dual space of W), and $\langle\cdot,\cdot\rangle_\ell$ (resp. $\langle\langle\cdot,\cdot\rangle\rangle$) the duality between W_ℓ^* (dual space of W_ℓ) and W_ℓ (resp. W^* and W). Let us denote by "∂" Fréchet differential operator and by "∂_{u_ℓ}" Fréchet differential operator with respect to u_ℓ. Let $A \not\equiv 0$ and $D_j \not\equiv 0$ ($j=1,\dots,m; m \geq 1$) be real functionals defined in W, B_ℓ and $\widehat{B}_\ell$ ($\ell = 1,\dots,n$) real functionals defined in W_ℓ satisfying the conditions:

(i_{11}) A is lower weakly semicontinuous in W and $C^1(W \setminus \{0\})$,
B_ℓ and $\widehat{B}_\ell$ are weakly continuous in W_ℓ and $C^1(W_\ell)$,
$\exists p > 1 : A(tv) = t^p A(v)$ for all $t \geq 0$ and for all $v \in W$, $B_\ell(tv_\ell) = t^p B_\ell(v_\ell)$ and $\widehat{B}_\ell(tv_\ell) = t^p \widehat{B}_\ell(v_\ell)$ for all $t \geq 0$ and for all $v_\ell \in W_\ell$;

(i_{12}) D_j is weakly continuous in W and $C^1(W \setminus \{0\})$, $\exists q_j > 1$:
$D_j(tv) = t^{q_j} D_j(v)$ for all $t \geq 0$ and for all $v \in W, 1 < q_1 < \cdots < q_m$ if $m > 1$.

Let $F = (F_1,\dots,F_n)$ with $F_\ell \in {W_\ell}^*, \lambda_\ell$ and $\mu_\ell \in R$; let us consider the following problem.

Problem (P). Find $u = (u_1,\dots,u_n) \in W \setminus \{0\}$ such that

$$\langle \partial_{u_i} A(u), v_i\rangle_i = \lambda_i \langle \partial B_i(u_i), v_i\rangle_i + \mu_i \left\langle \partial\widehat{B}_i(u_i), v_i \right\rangle_i + \sum_{j=1}^{m} \left\langle \partial_{u_i} D_j(u), v_i\right\rangle_i + \langle F_i, v_i\rangle_i \qquad \forall i \in \{1,\dots,n\}, \quad \forall v_i \in W_i. \tag{1.3}$$

Obviously Problem (P) means to find the critical points $u \in W \setminus \{0\}$ of the Euler functional:

$$E(v) = A(v) - \sum_{\ell=1}^{n}\left[\lambda_\ell B_\ell(v_\ell) + \mu_\ell \widehat{B}_\ell(v_\ell)\right] - \sum_{j=1}^{m} D_j(v) - \langle\langle F, v\rangle\rangle \quad \forall v = (v_1,\dots,v_n) \in W, \tag{1.4}$$

where $\langle\langle F, v\rangle\rangle = \sum_{\ell=1}^{n} \langle F_\ell, v_\ell\rangle_\ell$.

Let us set

$$
\begin{aligned}
&H_{\lambda\mu}(v) = A(v) - \sum_{\ell=1}^{n}\Big[\lambda_\ell B_\ell(v_\ell) + \mu_\ell \widehat{B}_\ell(v_\ell)\Big] \\
&\forall v = (v_1,\dots,v_n) \in W, \quad \forall \lambda = (\lambda_1,\dots,\lambda_n), \quad \mu = (\mu_1,\dots,\mu_n) \in R^n, \\
&S_{\lambda\mu} = \{v \in W : H_{\lambda\mu}(v) = 1\}, \quad V^-_{\lambda\mu} = \{v \in W : H_{\lambda\mu}(v) < 0\}, \quad \text{as } m_1 = 1,\dots,m \\
&V^+(D_{m_1},\dots,D_m) = \left\{ v \in W : \sum_{j=m_1}^{m} D_j(v) > 0 \right\}, \\
&S^+(D_1,\dots,D_m) = \left\{ v \in W : \sum_{j=1}^{m} D_j(v) = 1 \right\}, \\
&S(D_j) = \{v \in W : D_j(v) = -1\}, \qquad V^+(F) = \{v \in W : \langle\langle F, v\rangle\rangle > 0\}.
\end{aligned}
\tag{1.5}
$$

About Problem (P), using Lagrange multipliers and the "fibering method," different existence theorems have been proved in [1]. They base on one of the following hypotheses:

(i_{13}) $\exists c(\lambda,\mu) > 0 : \|v\|^p \le c(\lambda,\mu) H_{\lambda\mu}(v)$ for all $v \in W$;

(i_{14}) $\exists c(\lambda,\mu) > 0 : \|v\|^p \le c(\lambda,\mu) H_{\lambda\mu}(v)$ for all $v \in V^+(D_m)$ (if $V^+(D_m) \ne \emptyset$);

(i_{15}) $\exists m_1 \in \{1,\dots,m\} : V^-_{\lambda\mu} \cap S(D_{m_1})$ is not empty and bounded in W.

Remark 1.1. In this paper, we use some existence theorems ([1], Theorems 2.1, 2.2, 3.1, and 3.2), in which as $n > 1$, in relation to a set $\mathfrak{F} \subseteq S_{\lambda\mu}$, we suppose

(i^h_{16}) for each $v = (v_1,\dots,v_n) \in \mathfrak{F}$ with $v_h = 0$, there exist $\overline{v}_h \in W_h \setminus \{0\}$ and the real functions $\phi_1,\dots,\phi_n$ such that $\phi_h \in C^0([0,1]) \cap C^1([0,1[)$ and $\phi_h(1) = 0$, $\phi_\ell \in C^1([0,1])$ and $\phi_\ell(1) = 1$ as $\ell \ne h$, $v(s) = (\phi_1(s)v_1,\dots,\phi_h(s)\overline{v}_h,\dots,\phi_n(s)v_n) \in \mathfrak{F}$ for all $s \in [s_0, 1]$ $(0 \le s_0 < 1)$, $\lim''_{s\to 1^-}(d/ds)D_j(v(s)) < +\infty$ for all $j \in \{1,\dots, m\}$, $\lim_{s\to 1^-}(d/ds)D_j(v(s)) = -\infty$ for some $j \in \{1,\dots,m\}$.

The condition (i^h_{16}) assures that for the solutions $u = (u_1,\dots,u_n)$ of Problem (P), found with the method used in the recalled theorems, we have $u_h \ne 0$ if $F_h \equiv 0$.

Before showing Dirichlet problems (including the problem studied in [2] by Drábek and Pohozaev when $n = 1$ and $m = 1$) we give Propositions 2.2–2.6 which show some cases in which hypotheses (i_{13})–(i_{15}) hold. These propositions are based on the comparison between the parameters λ_i with suitable eigenvalues connected to p-Laplacian. About Neumann problems (including the one studied in [3] by Pohozaev and Véron when $n = 1$) the same question is solved by Propositions 3.1–3.5 in which the parameters λ_i and μ_i have compared with zero. Finally, the results in Appendix are very useful: Propositions A.1 and A.2 in order to get condition (i^h_{16}), Propositions A.3 and A.4 to get qualitative properties of the solutions and the positive sign of the components of the found solutions.

2. Dirichlet Problems

Let $\Omega \subseteq R^N$ be an open, bounded, connected and $C^{2,\beta}$ set with $0 < \beta \le 1$. Let $|\cdot|_N$ the Lebesgue measure on R^N, $1 < p < \infty$, $\widetilde{p} = Np/(N-p)$ if $N > p$, $\widetilde{p} = \infty$ otherwise.

Let us assume

$$W = \left(W_0^{1,p}(\Omega)\right)^n (n \geq 1) \quad \text{with } \|v\| = \left(\sum_{\ell=1}^{n} \int_\Omega |\nabla v_\ell|^p dx\right)^{1/p} \quad \forall v = (v_1, \dots, v_n) \in W,$$
$$B_\ell(v_\ell) = p^{-1} \int_\Omega b_\ell |v_\ell|^p dx \quad \forall v_\ell \in W_0^{1,p}(\Omega) \quad \text{where } b_\ell \in L^\infty(\Omega) \setminus \{0\},\ b_\ell \geq 0, \widehat{B}_\ell \equiv 0. \tag{2.1}$$

Moreover we consider the functionals A (as in (i_{11})) such that

$$\exists \tilde{c} > 0 : A(v) \geq p^{-1}\tilde{c}\|v\|^p \quad \forall v \in W. \tag{2.2}$$

Let us use the notation H_λ (S_λ and V_λ^-, resp.) instead of $H_{\lambda\mu}$ ($S_{\lambda\mu}$ and $V_{\lambda\mu}^-$, resp.).

As $\ell = 1, \dots, n$ let λ_ℓ^* and u_ℓ^*, respectively, the first eigenvalue and the first eigenfunction of the problem:

$$u_\ell \in W_0^{1,p}(\Omega) : -\tilde{c}\ \mathrm{div}\left(|\nabla u_\ell|^{p-2}\nabla u_\ell\right) = \theta b_\ell |u_\ell|^{p-2} u_\ell \quad \text{in } \Omega. \tag{2.3}$$

Let us remember that [4]

$u_\ell^* \in C^{1,\alpha_\ell}(\overline{\Omega})$ with $0 < \alpha_\ell < 1, u_\ell^* > 0$ in Ω;

$\lambda_\ell^* = \tilde{c}\int_\Omega |\nabla u_\ell^*|^p dx / \int_\Omega b_\ell |u_\ell^*|^p dx = \min\{\tilde{c}\int_\Omega |\nabla v_\ell|^p dx / \int_\Omega b_\ell |v_\ell|^p dx : \int_\Omega b_\ell |v_\ell|^p dx > 0\}$;

λ_ℓ^* is simple, that is, each eigenfunction of (2.3) related to λ_ℓ^* is of the type $c_\ell u_\ell^*$ with $c_\ell \in R \setminus \{0\}$;

λ_ℓ^* is isolate, that is, there exists $a > 0$ such that λ_ℓ^* is the only eigenvalue of (2.3) belonging to $]0, a[$.

Remark 2.1. About the results related to problem (2.3), it is sufficient to suppose $b_\ell \in L^\infty(\Omega)$ and $b_\ell^+ = \max\{b_\ell, 0\} \not\equiv 0$ as $\ell = 1, \dots, n$. This holds also for the results of this section if we limit to consider only the parameters $\lambda_1, \dots, \lambda_n$ nonnegative.

Let us start by presenting some sufficient conditions such that (i_{13}), (i_{14}), and (i_{15}) hold.

Using the variational characterization of λ_ℓ^* it is easy to verify the following proposition.

Proposition 2.2. *If $\lambda_\ell < \lambda_\ell^*$ for all $\ell \in \{1, \dots, n\}$, then (i_{13}) holds. Consequently, (i_{14}) holds when $V^+(D_m) \neq \emptyset$.*

When $\lambda_\ell \geq \lambda_\ell^*$ for some $\ell \in \{1, \dots, n\}$, it is possible to fulfil (i_{14}) with an additional condition on D_m. Let $I = \{1, \dots, n\}$. For any $I^* \subseteq I$ let

$$\begin{aligned} V^* = \{v = (v_1, \dots, v_n) \in W : v_\ell \equiv 0 \text{ if } \ell \in I \setminus I^*, \\ v_\ell = c_\ell u_\ell^* \text{ if } \ell \in I^* \text{ with } c_\ell \in R \text{ and } c_\ell \neq 0 \text{ for some } \ell\}, \end{aligned} \tag{2.4}$$

and let us suppose

(i_{21}) There exists $I^* \subseteq I : D_m(v) < 0$ for all $v \in V^*$.

Proposition 2.3. *Let* (i_{21}) *holds with* $I^* \neq I$. *Let* $V^+(D_m) \neq \emptyset$. *If we fix the parameters set* $(\lambda_\ell)_{\ell\in I\setminus I^*}$ *with* $\lambda_\ell < \lambda_\ell^*$, *then there exists* $\delta^* > 0$ *such that* (i_{14}) *also holds for any* $(\lambda_\ell)_{\ell\in I^*} \in X_{\ell\in I^*}[\lambda_\ell^*, \lambda_\ell^* + \delta^*[$.

Proof. Arguing by contradiction, for any $k \in \mathbb{N}$ there exist $(\lambda_\ell^k)_{\ell\in I^*} \in X_{\ell\in I^*}[\lambda_\ell^*, \lambda_\ell^* + k^{-1}[$ and $v^k = (v_1^k, \dots, v_n^k) \in V^+(D_m)$ such that

$$A\left(v^k\right) - p^{-1}\sum_{\ell\in I\setminus I^*}\lambda_\ell\int_\Omega b_\ell|v_\ell|^p dx - p^{-1}\sum_{\ell\in I^*}\lambda_\ell^k\int_\Omega b_\ell\left|v_\ell^k\right|^p dx < k^{-1}\left\|v^k\right\|^p. \tag{2.5}$$

Set $w^k = \|v^k\|^{-1}v^k$, we have

$$D_m\left(w^k\right) > 0,$$
$$\tilde{c}\sum_{\ell\in I\setminus I^*}\int_\Omega\left|\nabla w_\ell^k\right|^p dx - \sum_{\ell\in I\setminus I^*}\lambda_\ell\int_\Omega b_\ell\left|w_\ell^k\right|^p dx + \tilde{c}\sum_{\ell\in I^*}\int_\Omega\left|\nabla w_\ell^k\right|^p dx - \sum_{\ell\in I^*}\lambda_\ell^k\int_\Omega b_\ell\left|w_\ell^k\right|^p dx < pk^{-1}, \tag{2.6}$$

moreover, since $\|w^k\| = 1$, there exists $w \in W$ such that (within a subsequence)

$$w^k \longrightarrow w \text{ weakly in } W, \qquad w^k \longrightarrow w \text{ strongly in } (L^p(\Omega))^n. \tag{2.7}$$

Taking into account that D_m is weakly continuous in W, from (2.6) as $k \to +\infty$ we get

$$D_m(w) \geq 0, \tag{2.8}$$

$$\sum_{\ell\in I\setminus I^*}\left[\tilde{c}\int_\Omega|\nabla w_\ell|^p dx - \lambda_\ell\int_\Omega b_\ell|w_\ell|^p dx\right] + \sum_{\ell\in I^*}\left[\tilde{c}\int_\Omega|\nabla w_\ell|^p dx - \lambda_\ell^*\int_\Omega b_\ell|w_\ell|^p dx\right] \leq 0. \tag{2.9}$$

Since

$$w_\ell \not\equiv 0 \Longrightarrow \tilde{c}\int_\Omega|\nabla w_\ell|^p dx - \lambda_\ell\int_\Omega b_\ell|w_\ell|^p dx > 0,$$
$$\tilde{c}\int_\Omega|\nabla w_\ell|^p dx - \lambda_\ell^*\int_\Omega b_\ell|w_\ell|^p dx \geq 0, \tag{2.10}$$

from (2.9), we deduce that

$$w_\ell \equiv 0 \quad \forall\ell \in I\setminus I^*, \quad \forall\ell\in I^*\exists c_\ell \in R : w_\ell = c_\ell u_\ell^*. \tag{2.11}$$

Let us add that $c_\ell \neq 0$ for some $\ell \in I^*$, since if $c_\ell = 0$ for all $\ell \in I^*$ we have the contradiction $\tilde{c} = \tilde{c}\lim_{k\to+\infty}\|w^k\|^p = 0$. Then $w \in V^*$, and consequently $D_m(w) < 0$ from (i_{21}). This last inequality contradicts (2.8). □

In the same way the following propositions can be proved.

Proposition 2.4. *Let* (i_{21}) *holds with* $I^* = I$. *Let* $V^+(D_m) \neq \emptyset$. *Then, there exists* $\delta^* > 0$ *such that* (i_{14}) *also holds for any* $(\lambda_\ell)_{\ell\in I} \in \mathrm{X}_{\ell\in I}[\lambda^*_\ell,\ \lambda^*_\ell + \delta^*[$.

Let us pass to (i_{15}) and suppose

(i_{22}) there exist $I^* \subseteq I$ and $m_1 \in \{1,\dots,m\}$ such that $D_{m_1}(v) < 0$ and $A(v) = \widetilde{c}p^{-1}\sum_{\ell\in I^*}\int_\Omega |\nabla v_\ell|^p dx$ for any $v \in V^*$.

Proposition 2.5. *If* (i_{22}) *holds with* $I^* \neq I$, *then*

$$V^-_\lambda \cap S(D_{m_1}) \neq \emptyset \quad \forall(\lambda_\ell)_{\ell\in I} \quad \textit{with } (\lambda_\ell)_{\ell\in I^*} \in \mathop{\mathrm{X}}_{\ell\in I^*}[\lambda^*_\ell,\ +\infty[\setminus\{(\lambda^*_\ell)_{\ell\in I^*}\}. \tag{2.12}$$

Moreover, if we fix the parameters set $(\lambda_\ell)_{\ell\in I\setminus I^*}$ *with* $\lambda_\ell < \lambda^*_\ell$, *then there exists* $\delta^* > 0$ *such that*

$$V^-_\lambda \cap S(D_{m_1}) \textit{ is bounded in } W \quad \forall(\lambda_\ell)_{\ell\in I^*} \in \mathop{\mathrm{X}}_{\ell\in I^*}[\lambda^*_\ell,\ \lambda^*_\ell + \delta^*[\setminus\{(\lambda^*_\ell)_{\ell\in I^*}\}. \tag{2.13}$$

Proof. Let us prove (2.12). Let $v \in V^*$ with $v_\ell = u^*_\ell$ if $\ell \in I^*$, then $D_{m_1}(v) < 0$. Let $w = |D_{m_1}(v)|^{-1\backslash q_{m_1}} v$, we have

$$\begin{gathered} D_{m_1}(w) = |D_{m_1}(v)|^{-1} D_{m_1}(v) = -1, \\ H_\lambda(w) = p^{-1}\sum_{\ell\in I^*}\left[\widetilde{c}\int_\Omega |\nabla w_\ell|^p dx - \lambda_\ell\int_\Omega b_\ell |w_\ell|^p dx\right] < 0. \end{gathered} \tag{2.14}$$

Let us prove (2.13). Arguing by contradiction, for any $k \in \mathbb{N}$ there exist $(\lambda^k_\ell)_{\ell\in I^*} \in \mathrm{X}_{\ell\in I^*}[\lambda^*_\ell,\ \lambda^*_\ell + k^{-1}[$ with $(\lambda^k_\ell)_{\ell\in I^*} \neq (\lambda^*_\ell)_{\ell\in I^*}$ and $(v^{k,h})_{h\in\mathbb{N}} \subseteq V^-_{\lambda^k} \cap S(D_{m_1})$, where $\lambda^k_\ell = \lambda_\ell$ if $\ell \in I \setminus I^*$, such that

$$\sup_{h\in\mathbb{N}}\left\|v^{k,h}\right\| = +\infty. \tag{2.15}$$

Relation (2.15) implies that there exists $(h_k)_{k\in\mathbb{N}} \subseteq \mathbb{N}$ strictly increasing such that

$$\delta_k = \left\|v^{k,h_k}\right\| \longrightarrow +\infty \quad \text{as } k \longrightarrow +\infty. \tag{2.16}$$

Let $w^k = \delta_k^{-1} v^{k,h_k}$, we have

$$\begin{gathered} \sum_{\ell\in I\setminus I^*}\left[\widetilde{c}\int_\Omega \left|\nabla w^k_\ell\right|^p dx - \lambda_\ell\int_\Omega b_\ell\left|w^k_\ell\right|^p dx\right] + \sum_{\ell\in I^*}\left[\widetilde{c}\int_\Omega \left|\nabla w^k_\ell\right|^p dx - \lambda^k_\ell\int_\Omega b_\ell\left|w^k\right|^p dx\right] < 0, \\ D_{m_1}\left(w^k\right) = -\delta_k^{-q_{m_1}}, \\ \exists w \in W : (\text{within a subsequence})\ w^k \longrightarrow w \text{ weakly in } W, w^k \longrightarrow w \text{ strongly in } (L^p(\Omega))^n. \end{gathered} \tag{2.17}$$

Then, as $k \to +\infty$ we get

$$\sum_{\ell \in I \setminus I^*} \left[\widetilde{c} \int_{\Omega} |\nabla w_\ell|^p dx - \lambda_\ell \int_{\Omega} b_\ell |w_\ell|^p dx \right] + \sum_{\ell \in I^*} \left[\widetilde{c} \int_{\Omega} |\nabla w_\ell|^p dx - \lambda_\ell^* \int_{\Omega} b_\ell |w_\ell|^p dx \right] \leq 0, \tag{2.18}$$

$$D_{m_1}(w) = 0. \tag{2.19}$$

From (2.18), we get that $w \in V^*$. Then since (i_{22}) inequality $D_{m_1}(w) < 0$ holds, which contradicts (2.19). □

Proposition 2.6. *If (i_{22}) holds with $I^* = I$, then*

$$\begin{gathered} V_\lambda^- \cap S(D_{m_1}) \neq \emptyset \quad \forall \lambda = (\lambda_\ell)_{\ell \in I} \in \mathop{\mathsf{X}}_{\ell \in I} \left[\lambda_\ell^*, \ +\infty \right[\setminus \left\{ (\lambda_\ell^*)_{\ell \in I} \right\}, \\ \exists \delta^* > 0 : V_\lambda^- \cap S(D_{m_1}) \text{ is bounded in } W \quad \forall \lambda = (\lambda_\ell)_{\ell \in I} \in \mathop{\mathsf{X}}_{\ell \in I} \left[\lambda_\ell^*, \ \lambda_\ell^* + \delta^* \right[\setminus \left\{ (\lambda_\ell^*)_{\ell \in I} \right\}. \end{gathered} \tag{2.20}$$

The proof as in Proposition 2.5.

Remark 2.7. The applications we now show, except the first one, deal with systems with $n > 1$ equations. We consider the functionals A with $\widetilde{c} = 1$, and we suppose $b_\ell \in L^\infty(\Omega) \setminus \{0\}, b_\ell \geq 0$.

Application 2.8. Let $n = 1$. Let us consider the problem

$$-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = \lambda_1 b_1 |u|^{p-2} u + \sum_{j=1}^m d_j |u|^{q_j - 2} u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.21}$$

where

$$\begin{gathered} p < q_1 < \widetilde{p}, \quad d_1 \in L^\infty(\Omega) \setminus \{0\} \quad \text{if } m = 1, \\ p < q_1 < \cdots < q_m < \widetilde{p}, \quad d_j \in L^\infty(\Omega) \setminus \{0\} \quad \text{as } j = 1, \ldots, m, \\ d_j \leq 0 \quad \text{as } j = 1, \ldots, m-1 \quad \text{if } m > 1. \end{gathered} \tag{2.22}$$

Evidently

$$A(v) = p^{-1} \int_{\Omega} |\nabla v|^p dx, \quad D_j(v) = q_j^{-1} \int_{\Omega} d_j |v|^{q_j} dx \quad \forall v \in W. \tag{2.23}$$

Let us advance the conditions:

$$d_m^+ \not\equiv 0 \quad (\Longrightarrow V^+(D_m) \neq \emptyset), \tag{2.24}$$

$$\int_{\Omega} d_m (u_1^*)^{q_m} dx < 0 \quad (\Longrightarrow D_m(c_1 u_1^*) < 0 \; \forall c_1 \in R \setminus \{0\}). \tag{2.25}$$

Let us note that (Propositions 2.2, 2.4, and 2.6)

$$\begin{gathered}(2.24) \Longrightarrow \left((i_{14}) \text{ holds if } \lambda_1 < \lambda_1^*\right),\\ (2.24) \text{ and } (2.25) \Longrightarrow \left(\exists \delta_1^* > 0 : (i_{14}) \text{ holds if } \lambda_1 < \lambda_1^* + \delta_1^*\right),\\ (2.25) \Longrightarrow \left(\exists \delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_1 \in \left]\lambda_1^*, \lambda_1^* + \delta_2^*\right[\right).\end{gathered} \tag{2.26}$$

Proposition 2.9 (see [1], Theorems 2.1, 2.2, 4.1, and 4.2; Remarks 2.1, 2.3, 4.1, and 4.4; Proposition A.3; [5, 6]). *Under assumptions* (2.22) *we have:*

(i) *When* (2.24) *holds, with* $\lambda_1 < \lambda_1^*$ *[resp.* (2.24) *and* (2.25) *hold, with* $\lambda_1 < \lambda_1^* + \delta_1^*$*] problem* (2.21) *has at least two weak solutions* u^0 *and* $-u^0$ ($u^0 = \tau^0 v^0, \tau^0 = \text{const.} > 0, v^0 \in S_{\lambda_1} \cap V^+(D_m)$)*, and it results in* $u^0 \in L^\infty(\Omega) \cap C^{1,\alpha^0}_{\ell oc}(\Omega), u^0 > 0$;

(ii) *When* (2.25) *holds, with* $\lambda_1 \in]\lambda_1^*,\ \lambda_1^* + \delta_2^*[$ *problem* (2.21) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ ($\overline{u} = \overline{\tau}\,\overline{v}, \overline{\tau} = \text{const.} > 0,\ \overline{v} \in V^-_{\lambda_1} \cap S(D_m)$)*, and it results in* $\overline{u} \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}}_{\ell oc}(\Omega), \overline{u} > 0$.

Consequently, when (2.24) *and* (2.25) *hold, with* $\lambda_1 \in]\lambda_1^*, \lambda_1^* + \min\{\delta_1^*, \delta_2^*\}[$ *problem* (2.21) *has at least four different weak solutions.*

Remark 2.10. Our results include the ones of Drábek and Pohozaev [2] when $m = 1$.

Application 2.11. Let us consider the system:

$$\begin{gathered}-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2}u_i + \left|\sum_{\ell=1}^n d_\ell u_\ell\right|^{q_1-2}\left(\sum_{\ell=1}^n d_\ell u_\ell\right) d_i - \widetilde{d}_i |u_i|^{q_1-2}u_i \quad \text{in } \Omega,\\ u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i = 1,\ldots,n,\end{gathered} \tag{2.27}$$

where

$$1 < q_1 < \widetilde{p}, \quad q_1 \neq p, \qquad d_\ell, \widetilde{d}_\ell \in L^\infty(\Omega), \quad d_\ell, \widetilde{d}_\ell > 0. \tag{2.28}$$

System (2.27) is included among Problem (P) with:

$$\begin{gathered}A(v) = p^{-1}\sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p dx,\\ D_1(v) = q_1^{-1}\left[\int_\Omega \left|\sum_{\ell=1}^n d_\ell v_\ell\right|^{q_1} dx - \sum_{\ell=1}^n \int_\Omega \widetilde{d}_\ell |v_\ell|^{q_1} dx\right] \quad \forall v = (v_1,\ldots,v_n) \in W.\end{gathered} \tag{2.29}$$

Let us advance the conditions (compatible):

$$d_\ell^{q_1} < \widetilde{d}_\ell \quad \forall \ell \in \{1,\dots,n\} \big(\Longrightarrow D_1\big(0,\dots,c_i u_i^*,\dots,0\big) < 0 \text{ as } i = 1,\dots,n,\ c_i \in R \setminus \{0\}\big), \tag{2.30}$$

there exist $\Omega^+ \subseteq \Omega$ and a constant $\widetilde{c}_j > 0$ such that $|\Omega^+|_N > 0$ and

$$\left(\sum_{\ell \neq j} d_\ell + \widetilde{c}_j d_j\right)^{q_1} > \sum_{\ell \neq j} \widetilde{d}_\ell + \widetilde{c}_j^{q_1} \widetilde{d}_j \quad \text{in } \Omega^+ \ \big(\Longrightarrow V^+(D_1) \neq \emptyset \ (\text{Proposition A.1})\big). \tag{2.31}$$

Then (Propositions 2.2, 2.3, and 2.5)

$$(2.31) \Longrightarrow \big((i_{14}) \text{ holds if } \lambda_\ell < \lambda_\ell^* \ \forall \ell \in \{1,\dots,n\}\big), \tag{2.32}$$

and set $i \in \{1,\dots,n\}$

$$(2.30) \text{ and } (2.31) \Longrightarrow \big(\text{with } \lambda_\ell < \lambda_\ell^* \ \forall \ell \neq i \ \exists \delta_1^* > 0 : (i_{14}) \text{ holds if } \lambda_i < \lambda_i^* + \delta_1^*\big), \tag{2.33}$$

$$(2.30) \Longrightarrow \big(\text{with } \lambda_\ell < \lambda_\ell^* \ \forall \ell \neq i \ \exists \delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_i \in \big]\lambda_i^*, \lambda_i^* + \delta_2^*\big[\big). \tag{2.34}$$

Taking into account that $D_1(v_1,\dots,v_n) \leq D_1(|v_1|,\dots,|v_n|)$ and $D_1(-v) = D_1(v)$, from ([1], Theorem 2.1, Remark 2.1, and Theorem 4.1) we get the following proposition.

Proposition 2.12. *Under assumptions* (2.28) *we have:*

(i) *When* (2.31) *holds, (*(2.30) *and* (2.31) *hold resp.), choosing* $\lambda_1,\dots,\lambda_n$ *as in* (2.32) *(resp.* (2.33)*) system* (2.27) *has at least two weak solutions* u^0 *and* $-u^0$ *with* $u_\ell^0 \geq 0$ *as* $\ell = 1,\dots,n$ $(u^0 = \tau^0 v^0,\ \tau_0 = \text{const.} > 0,\ v^0 \in S_\lambda \cap V^+(D_1))$*;*

(ii) *When* (2.30) *holds, choosing* $\lambda_1,\dots,\lambda_n$ *as in* (2.34) *system* (2.27) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ $(\overline{u} = \overline{\tau}\,\overline{v}, \overline{\tau} = \text{const.} > 0, \overline{v} \in V_\lambda^- \cap S(D_1))$.

Consequently, when (2.30) *and* (2.31) *hold, with* $\lambda_\ell < \lambda_\ell^*$ *for all* $\ell \neq i$ *and* $\lambda_i \in]\lambda_i^*, \lambda_i^* + \min\{\delta_1^*, \delta_2^*\}[$ *system* (2.27) *has at least four different weak solutions.*

The following proposition is obvious.

Proposition 2.13. *The following relations hold:*

$$u_i^0 \not\equiv 0 \quad \text{as } i = 1, \ldots, n,$$
$$\exists h, k \in \{1, \ldots, n\} : \overline{u}_h \not\equiv 0, \overline{u}_k \not\equiv 0. \tag{2.35}$$

Proposition 2.14. *If $p < q_1$, then as $i = 1, \ldots, n$:*

$$u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega), \quad u_i^0 > 0. \tag{2.36}$$

Proof. It is easy to prove that

$$\sum_{i=1}^n \int_\Omega \left|\nabla u_i^0\right|^{p-2} \nabla u_i^0 \cdot \nabla v_i dx \le \int_\Omega g\left(\sum_{i=1}^n u_i^0\right)^{p-1}\left(\sum_{i=1}^n v_i\right) dx$$
$$\forall v = (v_1, \ldots, v_n) \in \left(W_0^{1,p}(\Omega) \cap L^\infty(\Omega)\right)^n \quad \text{with } v_i \ge 0, \tag{2.37}$$

where $g \in L^{q_1/(q_1-p)}(\Omega)$. Then (Proposition A.3) $u_i^0 \in L^\infty(\Omega)$ and consequently [5] $u_i^0 \in C^{1,\alpha_i^0}_{\ell oc}(\Omega)$.

Let us note that u_i^0 is a weak supersolution to the equation:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2} u_i - \tilde{d}_i |u_i|^{q_1-2} u_i \quad \text{in } \Omega. \tag{2.38}$$

Then, since (2.35), it must be [6] $u_i^0 > 0$. □

Let us continue the analysis of system (2.27) under the condition:

$$\left(\sum_{\ell \ne i} d_\ell\right)^{q_1} < \min\left\{\tilde{d}_1, \ldots, \tilde{d}_n\right\} \quad \forall i \in \{1, \ldots, n\}, \tag{2.39}$$

then

$$D_1(c_1 u_1^*, \ldots, c_n u_n^*) < 0 \quad \forall (c_1, \ldots, c_n) \in R^n \setminus \{0\} \quad \text{with } c_i = 0 \text{ for at least one } i \in \{1, \ldots, n\}. \tag{2.40}$$

Hence (Proposition 2.5) if $I^* \subseteq I$ and $I^* \ne I$:

$$(2.39) \Longrightarrow \left(\text{as } \lambda_\ell < \lambda_\ell^* \ \forall \ell \in I \backslash I^* \ \exists \delta^* > 0 : (i_{15}) \text{ holds if } (\lambda_\ell)_{\ell \in I^*} \in \mathop{\mathsf{X}}_{\ell \in I^*} [\lambda_\ell^*, \lambda_\ell^* + \delta^*[\backslash (\lambda_\ell^*)_{\ell \in I^*}\right). \tag{2.41}$$

Proposition 2.15. *Under assumptions* (2.28) *and* (2.39), *choosing $\lambda_1, \ldots, \lambda_n$ as in* (2.41) *system* (2.27) *has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u} = \overline{\tau}\,\overline{v}, \overline{\tau} = const. > 0, \overline{v} \in V_\lambda^- \cap S(D_1)$) with $\overline{u}_i \not\equiv 0$ as $i = 1, \ldots, n$.*

Proof. Thanks to ([1], Theorem 4.1), there exists $\overline{v} \in V_\lambda^- \cap S(D_1)$ such that

$$H_\lambda(\overline{v}) = \inf\{H_\lambda(v) : v \in V_\lambda^- \cap S(D_1)\} = \underline{e}, \quad \overline{u} = \overline{\tau}\,\overline{v} \text{ is a weak solution of system (2.27),} \tag{2.42}$$

where $\overline{\tau} = (-pq_1^{-1}\underline{e})^{1/(q_1-p)}$.

Reasoning by contradiction, let, for example, $\overline{u}_1 \equiv 0$. Since $-1 = D_1(\overline{v}) \le D_1(0, |\overline{v}_2|, \dots, |\overline{v}_n|)$ and from (2.39) $D_1(0, |\overline{v}_2|, \dots, |\overline{v}_n|) < 0$, setting $\delta = |D_1(0, |\overline{v}_2|, \dots, |\overline{v}_n|)|^{-1/q_1}$ we have

$$D_1(0, \delta|\overline{v}_2|, \dots, \delta|\overline{v}_n|) = -1, \qquad H_\lambda(0, \delta|\overline{v}_2|, \dots, \delta|\overline{v}_n|) = \delta^p H_\lambda(\overline{v}) \le H_\lambda(\overline{v}), \tag{2.43}$$

then $H_\lambda(0, \delta|\overline{v}_2|, \dots, \delta|\overline{v}_n|) = H_\lambda(\overline{v})$. This implies that ([1], see the proof of Theorem 4.1) $(0, \overline{\tau}\delta|\overline{v}_2|, \dots, \overline{\tau}\delta|\overline{v}_n|)$ is a weak solution of system (2.27). Then $(\sum_{\ell=2}^n d_\ell |\overline{v}_\ell|)^{q_1-1} \equiv 0$ from which $\overline{u}_\ell \equiv 0$ too as $\ell = 2, \dots, n$.

Condition (2.39) holds in particular when

$$\left(\sum_{\ell=1}^n d_\ell\right)^{q_1} < \min\left\{\widetilde{d}_1, \dots, \widetilde{d}_n\right\}. \tag{2.44}$$

□

Proposition 2.16. *Replacing in Proposition 2.15* (2.39) *with* (2.44)*, it is right to say that* $\overline{u}_i \ge 0$ *and* $\overline{u}_i \not\equiv 0$ *as* $i = 1, \dots, n$*. Consequently, if* $p < q_1$

$$\overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{a}_i}_{\ell oc}(\Omega), \quad \overline{u}_i > 0 \text{ as } i = 1, \dots, n. \tag{2.45}$$

Proof. Set $\delta = |D_1(|\overline{v}_1|, \dots, |\overline{v}_n|)|^{-1/q_1}$, as in Proposition 2.15 $(\overline{\tau}\delta|\overline{v}_1|, \dots, \overline{\tau}\delta|\overline{v}_n|)$ is a weak solution to system (2.27).

Let us add that since (2.44) $\Rightarrow D_1(c_1 u_1^*, \dots, c_n u_n^*) < 0$ for all $(c_1, \dots, c_n) \in R^n \setminus \{0\}$, there exists (Proposition 2.6) $\delta^{**} > 0$ such that

$$(i_{15}) \text{ holds if } (\lambda_\ell)_{\ell \in I} \in \mathop{\times}_{\ell=1}^{n} \left[\lambda_\ell^*, \lambda_\ell^* + \delta^{**}\right[\setminus \{(\lambda_\ell^*)_{\ell\in I}\}. \tag{2.46}$$

Then the existence of $\overline{u}$ is assured also choosing $\lambda_1, \dots, \lambda_n$ as in (2.46), and the conclusions of Proposition 2.16 hold. □

Application 2.17. Let us set

$$\lambda_1 = \cdots = \lambda_n = \overline{\lambda}, \quad b_1 = \cdots = b_n = b \quad \left(\text{then } \lambda_1^* = \cdots = \lambda_n^* = \lambda^*, u_1^* = \cdots = u_n^* = u^*\right),$$
$$A(v) = p^{-1}\sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p dx, \quad D_1(v) = q_1^{-1}\int_\Omega d_1 \left(\sum_{\ell=1}^n |v_\ell|^\gamma\right)^{q_1/\gamma} dx, \quad \forall v = (v_1, \dots, v_n) \in W, \tag{2.47}$$

where

$$1<\gamma<q_1<\widetilde{p}, \quad q_1 \neq p, d_1 \in L^\infty(\Omega). \tag{2.48}$$

Let us consider the system:

$$\begin{gathered}
-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \overline{\lambda} b|u_i|^{p-2}u_i + d_1\left(\sum_{\ell=1}^{n}|u_\ell|^\gamma\right)^{(q_1/\gamma)-1}|u_i|^{\gamma-2}u_i \quad \text{in } \Omega, \\
u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i=1,\dots,n.
\end{gathered} \tag{2.49}$$

We advance the conditions

$$d_1^+ \not\equiv 0 \quad (\Longrightarrow V^+(D_1)\neq\emptyset), \tag{2.50}$$

$$\int_\Omega d_1(u^*)^{q_1}dx<0 \quad (\Longrightarrow D_1(c_1u^*,\dots,c_nu^*)<0 \;\forall(c_1,\dots,c_n)\in R^n\setminus\{0\}). \tag{2.51}$$

Therefore,

$$\begin{gathered}
(2.50) \Longrightarrow \left((i_{14}) \text{ holds if } \overline{\lambda}<\lambda^*\right) \quad (\text{Proposition 2.2}), \\
(2.50) \text{ and } (2.51) \Longrightarrow \left(\exists\delta_1^*>0 : (i_{14}) \text{ holds if } \overline{\lambda}<\lambda^*+\delta_1^*\right) \quad (\text{Proposition 2.4}), \\
(2.51) \Longrightarrow \left(\exists\delta_2^*>0 : (i_{15}) \text{ holds if } \overline{\lambda}\in\,]\lambda^*,\lambda^*+\delta_2^*[\right) \quad (\text{Proposition 2.6}).
\end{gathered} \tag{2.52}$$

Then ([1], Theorems 2.1 and 4.1, and Remarks 2.1 and 4.1).

Proposition 2.18. *Under assumption* (2.48), *we have:*

(i) *When* (2.50) *holds, ((*2.50) *and* (2.51) *hold resp.), if* $\overline{\lambda}<\lambda^*$ *(resp.* $\overline{\lambda}<\lambda^*+\delta_1^*$*) system* (2.49) *has at least two weak solutions* u^0 *and* $-u^0$ *with* $u_\ell^0 \geq 0$ *as* $\ell=1,\dots,n$ $(u^0 = \tau^0 v^0, \tau_0 = const. > 0, v^0 \in S_\lambda \cap V^+(D_1))$;

(ii) *When* (2.51) *holds, if* $\overline{\lambda}\in]\lambda^*,\lambda^*+\delta_2^*[$ *system* (2.49) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ *with* $\overline{u}_\ell \geq 0$ *as* $\ell=1,\dots,n$ $(\overline{u}=\overline{\tau}\,\overline{v}, \overline{\tau}=const.>0, \overline{v}\in V_\lambda^-\cap S(D_1))$.

Consequently, when (2.50) *and* (2.51) *hold, with* $\overline{\lambda}\in]\lambda^*,\lambda^*+\min\{\delta_1^*,\delta_2^*\}[$ *system* (2.49) *has at least four different weak solutions.*

In order to establish some properties of u^0 and $\overline{u}$ it is useful to recall that ([1], Theorems 2.1 and 4.1)

$$D_1\left(v^0\right) = \sup\{D_1(v) : v\in S_\lambda\cap V^+(D_1)\} = \overline{e}, \qquad \tau^0 = \left(q_1p^{-1}\overline{e}\right)^{1/(p-q_1)}, \tag{2.53}$$

$$H_\lambda(\overline{v}) = \inf\{H_\lambda(v) : v\in V_\lambda^-\cap S(D_1)\} = \underline{e}, \qquad \overline{\tau} = \left(-pq_1^{-1}\underline{e}\right)^{1/(q_1-p)}. \tag{2.54}$$

Proposition 2.19. *When $p < q_1$, we have*

$$u_i^0 \in L^\infty(\Omega) \cap C_{\ell oc}^{1,\alpha_i^0}(\Omega), \tag{2.55}$$

besides

$$u_i^0 \not\equiv 0 \quad \forall i \in \{1, \dots, n\} \text{ if } \gamma < p. \tag{2.56}$$

Proof. The relation $u_i^0 \in L^\infty(\Omega)$ comes from Proposition A.3. Then [5] $u_i^0 \in C_{\ell oc}^{1,\alpha_i^0}(\Omega)$.

About (2.56), it is sufficiently (Remark 1.1) to prove that

$$\left(i_{16}^h\right) \text{ holds} \quad \forall h \in \{1, \dots, n\} \text{ with } \mathfrak{F} = S_\lambda \cap V^+(D_1). \tag{2.57}$$

Let $v = (v_1, \dots, v_n) \in S_\lambda \cap V^+(D_1)$ with $v_h \equiv 0$. Since

$$v \in V^+(D_1) \Longrightarrow \left(\exists \text{ a compact set } \mathbb{K} \subseteq \Omega : |\mathbb{K}|_N > 0, d_1 > 0 \text{ and } \varphi = \sum_{\ell \neq h} |v_\ell|^\gamma > 0 \text{ in } \mathbb{K} \right), \tag{2.58}$$

let (Proposition A.1) $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega)$ with $0 \le \varphi_\varepsilon \le 1$ such that

$$\varphi_\varepsilon \longrightarrow \chi \text{ strongly in } L^s(\Omega), \quad \int_\Omega |\nabla \varphi_\varepsilon|^s dx \longrightarrow +\infty \quad \text{as } \varepsilon \longrightarrow 0^+ \ \forall s \in [1, +\infty[, \tag{2.59}$$

where χ is the characteristic function of $\mathbb{K}$. Set ε such that

$$\int_\Omega d_1 \varphi^{(q_1/\gamma)-1} \varphi_\varepsilon^\gamma dx > 0, \qquad \delta = p^{-1} \left[\int_\Omega |\nabla \varphi_\varepsilon|^p dx - \overline{\lambda} \int_\Omega b \varphi_\varepsilon^p dx \right] > 0, \tag{2.60}$$

with $v(s) = (s^{1/p} v_1, \dots, (1-s)^{1/p} \delta^{-1/p} \varphi_\varepsilon, \dots, s^{1/p} v_n)$ it results in

$$\begin{gathered} H_\lambda(v(s)) = \delta^{-1}(1-s)p^{-1} \left[\int_\Omega |\nabla \varphi_\varepsilon|^p dx - \overline{\lambda} \int_\Omega b \varphi_\varepsilon^p dx \right] + s H_\lambda(v) = 1 \quad \forall s \in [0,1], \\ \exists s_0 \in [0,1[: D_1(v(s)) > 0 \quad \forall s \in [s_0, 1], \qquad \lim_{s \to 1^-} \frac{d}{ds} D_1(v(s)) = -\infty. \end{gathered} \tag{2.61}$$

□

Proposition 2.20. *When $p < q_1$, we have*

$$\overline{u}_i \in L^\infty(\Omega) \cap C_{\ell oc}^{1,\overline{\alpha}_i}(\Omega), \tag{2.62}$$

$$\overline{u}_i > 0 \quad \forall i \in \{1, \dots, n\} \text{ if } p < \gamma. \tag{2.63}$$

Proof. We can get (2.62) from Proposition A.3 and [5].

About (2.63), it is sufficiently [6] to prove that $\overline{u}_i \not\equiv 0$ as $i = 1, \ldots, n$. Reasoning by contradiction, let, for example, $\overline{v}_1 \equiv 0$. We note that

$$\overline{v} \in V_\lambda^- \Longrightarrow \left(\exists \ell \in \{2, \ldots, n\} : \int_\Omega |\nabla \overline{v}_\ell|^p dx - \overline{\lambda} \int_\Omega b \overline{v}_\ell^p dx < 0 \right). \tag{2.64}$$

Let us suppose $\ell = 2$ and set $v(s) = ((1-s)^{1/\gamma} \overline{v}_2, s^{1/\gamma} \overline{v}_2, \overline{v}_3, \ldots, \overline{v}_n)$. Then

$$D_1(v(s)) = -1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[\, : H_\lambda(v(s)) < 0 \quad \forall s \in [s_0, 1],$$
$$\lim_{s \to 1^-} \frac{d}{ds} H_\lambda(v(s)) = +\infty. \tag{2.65}$$

Set $s_1 \in [s_0, 1[$ such that $(d/ds) H_\lambda(v(s)) > 0$ for all $s \in [s_1, 1[$ and taking into account (2.54), we get the contradiction:

$$H_\lambda(\overline{v}) \le H_\lambda(v(s)) < H_\lambda(\overline{v}) \quad \forall s \in [s_1, 1[. \tag{2.66}$$

□

Proposition 2.21. *When $\gamma = p < q_1$, we allow that as $i = 1, \ldots, n$:*

$$u_i^0 > 0, \qquad \overline{u}_i > 0. \tag{2.67}$$

Proof. The assumption $\gamma = p$ implies that

$$\forall v = (v_1, \ldots, v_n) \in W \setminus \{0\} \quad \text{with } v_h \equiv 0 \text{ for some } h \in \{1, \ldots, n\},$$
$$\exists \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_n) \in W : \tilde{v}_\ell \not\equiv 0 \quad \text{as } \ell = 1, \ldots, n, \qquad H_\lambda(\tilde{v}) = H_\lambda(v), \qquad D_1(\tilde{v}) = D_1(v). \tag{2.68}$$

Let, for example, $v_1 \equiv 0$ and $v_2 \not\equiv 0$. Set $s \in]0,1[$ and $v_1^1 = (1-s)^{1/p} v_2$, $v_2^1 = s^{1/p} v_2$, $v_\ell^1 = v_\ell$ as $\ell > 2$, with $v^1 = (v_1^1, \ldots, v_n^1)$, we have

$$H_\lambda\left(v^1\right) = H_\lambda(v), \qquad D_1\left(v^1\right) = D_1(v). \tag{2.69}$$

If $v_3 \equiv 0$, set $v_1^2 = (1-s)^{1/p} v_1^1$, $v_3^2 = s^{1/p} v_1^1$, $v_\ell^2 = v_\ell^1$ as $\ell \in \{1, \ldots, n\} \setminus \{1, 3\}$, with $v^2 = (v_1^2, \ldots, v_n^2)$, it results in

$$H_\lambda\left(v^2\right) = H_\lambda(v), \qquad D_1\left(v^2\right) = D_1(v). \tag{2.70}$$

This method let us to find $\tilde{v}$.

Then, if $v_h^0 \equiv 0$ (resp. $\overline{v}_h \equiv 0$) for some $h \in \{1, \dots, n\}$, with $\widetilde{v}^0$ (resp. $\widetilde{\overline{v}}$) as in (2.68) we have from (2.53) (resp. (2.54)) $D_1(\widetilde{v}^0) = \overline{e}$ (resp. $H_\lambda(\widetilde{\overline{v}}) = \underline{e}$). Consequently ([1], see the proof of Theorem 2.1 (resp. Theorem 4.1)) $\widetilde{u}^0 = \tau^0 \widetilde{v}^0$ (resp. $\widetilde{\overline{u}} = \overline{\tau}\,\widetilde{\overline{v}}$) is a weak solution of system (2.49). Therefore [6] $\widetilde{u}_i^0 > 0$ (resp. $\widetilde{\overline{u}}_i > 0$) as $i = 1, \dots, n$. □

Application 2.22. Let us assume λ_ℓ, b_ℓ, and A as in Application 2.17,

$$D_j(v) = q_j^{-1} \int_\Omega d_j \left(\sum_{\ell=1}^n |v_\ell|^{\gamma_j} \right)^{q_j/\gamma_j} dx \quad \forall v = (v_1, \dots, v_n) \in W \quad \text{as } j = 1, \dots, m, \tag{2.71}$$

where

$$\begin{gathered} p < q_1 < \cdots < q_m < \widetilde{p}, \quad 1 < \gamma_j < q_j, \quad d_m \in L^\infty(\Omega), \\ d_j \in L^\infty(\Omega) \setminus \{0\}, \quad d_j \le 0 \text{ if } j = 1, \dots, m-1. \end{gathered} \tag{2.72}$$

Let us consider the system:

$$\begin{gathered} -\operatorname{div}\left(|\nabla u_i|^{p-2} \nabla u_i \right) = \overline{\lambda} b |u_i|^{p-2} u_i + \sum_{j=1}^m d_j \left(\sum_{\ell=1}^n |u_\ell|^{\gamma_j} \right)^{(q_j/\gamma_j)-1} |u_i|^{\gamma_j - 2} u_i \quad \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i = 1, \dots, n, \end{gathered} \tag{2.73}$$

under almost one of the conditions:

$$d_m^+ \not\equiv 0, \qquad \int_\Omega d_m (u^*)^{q_m} dx < 0. \tag{2.74}$$

By using some results ([1], Theorems 2.2 and 4.2, and Remarks 2.3 and 4.4), we can advance a proposition similar to Proposition 2.18 replacing in particular $V^+(D_1)$ with $V^+(D_m)$ and $S(D_1)$ with $S(D_m)$.

Thanks to Proposition A.3 and a result of [5], for the solutions u^0 and $\overline{u}$ to system (2.73), we have

$$u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega), \qquad \overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}_i}_{\ell oc}(\Omega). \tag{2.75}$$

We continue to analyze the properties of u^0 and $\overline{u}$. To this aim we recall that ([1], Theorems 2.2 and 4.2), set for each $v \in V^+(D_m)$ (resp. $v \in V_\lambda^- \cap S(D_m)$) $\varphi(t, v) = p t^{p-1} H_\lambda(v) - \sum_{j=1}^m q_j t^{q_j - 1} D_j(v)$, we have:

$$\exists \mid t(v) > 0 : \varphi(t(v), v) = 0, \qquad \frac{\partial \varphi}{\partial t}(t(v), v) \ne 0. \tag{2.76}$$

Besides with $\tilde{\tilde{E}}(v) = (t(v))^p H_\lambda(v) - \sum_{j=1}^{m} (t(v))^{q_j} D_j(v)$, it results in

$$\tilde{\tilde{E}}\left(v^0\right) = \inf\left\{ \tilde{\tilde{E}}(v) : v \in S_\lambda \cap V^+(D_m) \right\}, \qquad \tau^0 = t\left(v^0\right), \tag{2.77}$$

$$\tilde{\tilde{E}}(\overline{v}) = \inf\left\{ \tilde{\tilde{E}}(v) : v \in V_\lambda^- \cap S(D_m) \right\}, \qquad \overline{\tau} = t(\overline{v}). \tag{2.78}$$

Proposition 2.23. *When $\gamma_m < p \le \gamma_j$ as $j = 1, \ldots, m-1$, then*

$$u_i^0 \not\equiv 0 \quad \forall i \in \{1, \ldots, n\}. \tag{2.79}$$

Proof. It is sufficiently (Remark 1.1) to prove that

$$\left(i_{16}^h\right) \text{ holds } \quad \forall h \in \{1, \ldots, n\} \quad \text{with } \mathfrak{F} = S_\lambda \cap V^+(D_m). \tag{2.80}$$

Let $v = (v_1, \ldots, v_n) \in S_\lambda \cap V^+(D_m)$ with $v_h \equiv 0$. As in Proposition 2.19, it is possible to find $\overline{v}_h \in C_0^\infty(\Omega) \setminus \{0\}$ such that with $v(s) = (s^{1/p} v_1, \ldots, (1-s)^{1/p} \overline{v}_h, \ldots, s^{1/p} v_n)$, it results in

$$\begin{gathered} H_\lambda(v(s)) = 1 \quad \forall s \in [0,1], \qquad D_m(v(s)) > 0 \quad \forall s \in [s_0, 1] \ (0 \le s_0 < 1), \\ \lim_{s \to 1^-} \frac{d}{ds} D_j(v(s)) \in R \quad \text{as } j = 1, \ldots, m-1, \qquad \lim_{s \to 1^-} \frac{d}{ds} D_m(v(s)) = -\infty. \end{gathered} \tag{2.81}$$

□

Proposition 2.24. *When $p < \gamma_m \le \gamma_j$ as $j = 1, \ldots, m-1$, then*

$$\overline{u}_i > 0 \quad \forall i \in \{1, \ldots, n\}. \tag{2.82}$$

Proof. It is sufficiently [6] to prove that $\overline{u}_i \not\equiv 0$ for all $i \in \{1, \ldots, n\}$. Reasoning by contradiction, let, for example, $\overline{v}_1 \equiv 0$ and $\overline{v}_2 \not\equiv 0$ such that

$$\int_\Omega |\nabla \overline{v}_2|^p dx - \overline{\lambda} \int_\Omega b \overline{v}_2^p dx < 0. \tag{2.83}$$

Since

$$t(\overline{v}) > 0, \qquad \varphi(t(\overline{v}), \overline{v}) = 0, \qquad \frac{\partial \varphi}{\partial t}(t(\overline{v}), \overline{v}) \ne 0, \tag{2.84}$$

there exist an open ball $\tilde{B}$ of W with centre $\overline{v}$ included in V_λ^- and a unique functional $t^*(v)$ belongs to $C^1(\tilde{B})$ such that

$$t^*(v) > 0, \quad \varphi(t^*(v), v) = 0 \quad \forall v \in \tilde{B}. \tag{2.85}$$

Then, the functional

$$E^*(v) = (t^*(v))^p H_\lambda(v) - \sum_{j=1}^{m} (t^*(v))^{q_j} D_j(v) \quad \forall v \in \widetilde{B} \tag{2.86}$$

belongs to $C^1(\widetilde{B})$, and we have

$$t(v) = t^*(v) \quad \forall v \in \widetilde{B} \cap S(D_m). \tag{2.87}$$

Then, for (2.78)

$$E^*(\overline{v}) = \inf\left\{ E^*(v) : v \in \widetilde{B} \cap S(D_m) \right\}. \tag{2.88}$$

Now, let us remark that with $v(s) = ((1-s)^{1/\gamma_m}\overline{v}_2, s^{1/\gamma_m}\overline{v}_2, \overline{v}_3, \ldots, \overline{v}_n)$, it results in

$$\begin{gathered} D_m(v(s)) = -1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[: v(s) \in \widetilde{B} \quad \forall s \in [s_0,1], \\ \lim_{s\to 1^-} \frac{d}{ds} H_\lambda(v(s)) = +\infty, \qquad \lim_{s\to 1^-} \frac{d}{ds} D_j(v(s)) \in R \quad \text{as } j = 1,\ldots,m-1. \end{gathered} \tag{2.89}$$

Then, since

$$\frac{d}{ds}E^*(v(s)) = (t^*(v(s)))^p \frac{d}{ds} H_\lambda(v(s)) - \sum_{j=1}^{m} (t^*(v(s)))^{q_j} \frac{d}{ds} D_j(v(s)) \quad \forall s \in [s_0,1[, \tag{2.90}$$

we have $\lim_{s\to 1^-}(d/ds)E^*(v(s)) = +\infty$. Consequently,

$$\exists s_1 \in [s_0,1[: \frac{d}{ds}E^*(v(s)) > 0 \quad \forall s \in [s_1,1[, \tag{2.91}$$

from which we get the contradiction:

$$E^*(\overline{v}) \le E^*(v(s)) < E^*(\overline{v}) \quad \forall s \in [s_1,1[. \tag{2.92}$$

□

Proposition 2.25. *When $p = \gamma_1 = \cdots = \gamma_m$, we allow that*

$$u_i^0 > 0, \quad \overline{u}_i > 0 \quad \forall i \in \{1,\ldots,n\}. \tag{2.93}$$

Proof. We reason as in Proposition 2.21, taking into account (2.77) and (2.78) ([1], see proofs of Theorems 2.2 and 4.2). □

Application 2.26. Let for each $v=(v_1,\dots,v_n)\in W$:

$$A(v)=p^{-1}\sum_{\ell=1}^{n}\int_\Omega|\nabla v_\ell|^p dx,\qquad D_j(v)=-\prod_{\ell=1}^{n}\int_\Omega|v_\ell|^{q_{j\ell}}dx\quad \text{as } j=1,\dots,m-1\ (m\geq 2),$$
$$D_m(v)=q_m^{-1}\left[\int_\Omega\left(\sum_{\ell=1}^{n}d_\ell|v_\ell|^\gamma\right)^{q_m/\gamma}dx-\sum_{\ell=1}^{n}\int_\Omega\tilde{d}_\ell|v_\ell|^{q_m}dx\right], \tag{2.94}$$

where

$$1<\gamma<p\leq q_{j\ell},\quad \sum_{\ell=1}^{n}q_{j\ell}=q_j<q_m<\tilde{p},\quad q_1<\cdots<q_{m-1},$$
$$d_\ell,\tilde{d}_\ell\in L^\infty(\Omega),\quad d_\ell,\ \tilde{d}_\ell>0. \tag{2.95}$$

Let us consider the system:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right)=\lambda_i b_i|u_i|^{p-2}u_i-\sum_{j=1}^{m-1}\left(q_{ji}\prod_{\ell\neq i}\int_\Omega|u_\ell|^{q_{j\ell}}dx\right)|u_i|^{q_{ji}-2}u_i$$
$$+\left(\sum_{\ell=1}^{n}d_\ell|u_\ell|^\gamma\right)^{(q_m/\gamma)-1}d_i|u_i|^{\gamma-2}u_i-\tilde{d}_i|u_i|^{q_m-2}u_i\quad \text{in } \Omega, \tag{2.96}$$
$$u_i=0\quad \text{on } \partial\Omega\quad \text{as } i=1,\dots,n.$$

Let us introduce the conditions:

$$\exists\Omega^+\subseteq\Omega:|\Omega^+|_N>0,\qquad d_{\overline{\ell}}^{q_m/\gamma}>\tilde{d}_{\overline{\ell}}\text{ in }\Omega^+\text{ for some }\overline{\ell}\in\{1,\dots,n-1\}\ (\Longrightarrow V^+(D_m)\neq\emptyset), \tag{2.97}$$

$$d_n^{q_m/\gamma}<\tilde{d}_n\quad (\Longrightarrow D_m(0,\dots,0,c_nu_n^*)<0\ \forall c_n\in R\setminus\{0\}). \tag{2.98}$$

Then (Propositions 2.2, 2.3 and 2.5)

$$(2.97)\Longrightarrow\left(\text{with }\lambda_\ell<\lambda_\ell^*\ \forall\ell\in\{1,\dots,n\}\ (i_{14})\text{ holds}\right), \tag{2.99}$$

$$(2.97)\text{ and }(2.98)\Longrightarrow\left(\text{with }\lambda_\ell<\lambda_\ell^*\ \forall\ell\in\{1,\dots,n-1\}\ \exists\delta_1^*>0:(i_{14})\text{ holds if }\lambda_n<\lambda_n^*+\delta_1^*\right), \tag{2.100}$$

$$(2.98)\Longrightarrow\left(\text{with }\lambda_\ell<\lambda_\ell^*\ \forall\ell\in\{1,\dots,n-1\}\ \exists\delta_2^*>0:(i_{15})\text{ holds if }\lambda_n\in\left]\lambda_n^*,\lambda_n^*+\delta_2^*\right[\right). \tag{2.101}$$

Since ([1], Theorems 2.2 and 4.2; Remarks 2.3 and 4.4), we get the following proposition.

Proposition 2.27. *Under assumption* (2.95) *we have:*

(i) *When* (2.97) *holds* ((2.97) *and* (2.98) *hold, resp.), set* $\lambda_1, \dots, \lambda_n$ *as in* (2.99) *(resp.* (2.100)*) system* (2.96) *has at least two weak solutions* u^0 *and* $-u^0$ *with* $u^0_\ell \geq 0$ *as* $\ell = 1, \dots, n$ ($u^0 = \tau^0 v^0, \tau^0 = const. > 0, v^0 \in S_\lambda \cap V^+(D_m)$);

(ii) *When* (2.98) *holds, set* $\lambda_1, \dots, \lambda_n$ *as in* (2.101) *system* (2.96) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ *with* $\overline{u}_\ell \geq 0$ *as* $\ell = 1, \dots, n$ ($\overline{u} = \overline{\tau}\,\overline{v}, \overline{\tau} = const. > 0, \overline{v} \in V^-_\lambda \cap S(D_m)$).

Consequently, when (2.97) *and* (2.98) *hold, with* $\lambda_\ell < \lambda^*_\ell$ *for all* $\ell \in \{1, \dots, n-1\}$ *and* $\lambda_n \in]\lambda^*_n, \lambda^*_n + \min\{\delta^*_1, \delta^*_2\}[$ *system* (2.96) *has at least four different weak solutions.*

We remark that (Proposition A.3, [5]) as $i = 1, \dots, n$:

$$u^0_i \in L^\infty(\Omega) \cap C^{1,\alpha^0_i}_{\ell oc}(\Omega), \qquad \overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}_i}_{\ell oc}(\Omega). \tag{2.102}$$

Moreover, since u^0_i (resp. $\overline{u}_i$) is a weak supersolution of the equation:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2} u_i - \sum_{j=1}^{m-1} a_j q_{ji} |u_i|^{q_{ji}-2} u_i - \widetilde{d}_i |u_i|^{q_m-2} u_i \quad \text{in } \Omega, \tag{2.103}$$

where $a_j = \prod_{\ell \neq i} \int_\Omega (u^0_\ell)^{q_{j\ell}} dx$ (resp. $a_j = \prod_{\ell \neq i} \int_\Omega (\overline{u}_\ell)^{q_{j\ell}} dx$), we have [6]

$$u^0_i > 0 \text{ if } u^0_i \not\equiv 0 \ \left[\text{resp. } \overline{u}_i > 0 \text{ if } \overline{u}_i \not\equiv 0\right]. \tag{2.104}$$

Proposition 2.28. *It results in*

$$u^0_i > 0 \quad \text{as } i = 1, \dots, n, \qquad \overline{u}_{\overline{\ell}} > 0. \tag{2.105}$$

Proof. Since (2.104), we must show that

$$u^0_i \not\equiv 0 \quad \text{as } i = 1, \dots, n, \tag{2.106}$$

$$\overline{u}_{\overline{\ell}} \not\equiv 0. \tag{2.107}$$

About (2.106), it is sufficient (Remark 1.1) to prove that

$$\left(i^h_{16}\right) \text{ holds } \forall h \in \{1, \dots, n\} \quad \text{with } \mathfrak{F} = S_\lambda \cap V^+(D_m). \tag{2.108}$$

Let $v = (v_1, \dots, v_n) \in S_\lambda \cap V^+(D_m)$ with $v_h \equiv 0$. Let $\mathbb{K} \subseteq \Omega$ be a compact set such that

$$|\mathbb{K}|_N > 0, \quad \psi = \sum_{\ell \neq h} d_\ell |v_\ell|^\gamma > 0 \quad \text{in } \mathbb{K}. \tag{2.109}$$

From Proposition A.1, there exists $\varphi \in C_0^\infty(\Omega)$, with $0 \le \varphi \le 1$, such that

$$\delta = p^{-1}\left[\int_\Omega |\nabla\varphi|^p dx - \lambda_h \int_\Omega b_h \varphi^p dx\right] > 0, \qquad \int_\Omega \varphi^{q_m/\gamma} d_h \varphi^p dx > 0. \tag{2.110}$$

Then, with $v(s) = (s^{1/p}v_1, \dots, (1-s)^{1/p}\delta^{-1/p}\varphi, \dots, s^{1/p}v_n)$, we have

$$\begin{gathered} H_\lambda(v(s)) = 1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[: D_m(v(s)) > 0 \quad \forall s \in [s_0,1], \\ \lim_{s\to 1^-} \frac{d}{ds} D_j(v(s)) \in]-\infty, 0] \quad \text{as } j = 1,\dots,m-1, \qquad \lim_{s\to 1^-}\frac{d}{ds} D_m(v(s)) = -\infty. \end{gathered} \tag{2.111}$$

Let us prove (2.107). We recall that ([1], Theorem 4.2):

$$\widetilde{\widetilde{E}}(\overline{v}) = \inf\left\{\widetilde{\widetilde{E}}(v) : v \in V_\lambda^- \cap S(D_m)\right\}, \tag{2.112}$$

where $\widetilde{\widetilde{E}}$ as in Application 2.22. Reasoning by contradiction, let $\overline{v}_{\overline{\ell}} \equiv 0$. Then, $\overline{v}_\ell \not\equiv 0$ for some $\ell \neq \overline{\ell}$ and consequently from (2.104) $\sum_{\ell\neq\overline{\ell}} d_\ell(\overline{v}_\ell)^\gamma > 0$.

Let $\varphi \in C_0^\infty(\Omega)$, with $0 \le \varphi \le 1$, such that $\int_\Omega d_{\overline{\ell}}^{q_m/\gamma}\varphi^{q_m} dx > \int_\Omega \widetilde{d}_{\overline{\ell}}\varphi^{q_m} dx$. Let us consider the function:

$$\begin{aligned} g(s,\tau) &= D_m(\tau\overline{v}_1, \dots, s\varphi, \dots, \tau\overline{v}_n) \\ &= q_m^{-1}\left[\int_\Omega \left(s^\gamma d_{\overline{\ell}}\varphi^\gamma + \tau^\gamma \sum_{\ell\neq\overline{\ell}} d_\ell(\overline{v}_\ell)^\gamma\right)^{q_m/\gamma} dx - s^{q_m}\int_\Omega \widetilde{d}_{\overline{\ell}}\varphi^{q_m} dx - \tau^{q_m}\sum_{\ell\neq\overline{\ell}}\int_\Omega \widetilde{d}_\ell(\overline{v}_\ell)^{q_m} dx\right] \\ &\qquad \forall s \ge 0, \quad \forall \tau \ge 1. \end{aligned} \tag{2.113}$$

Since

$$\begin{gathered} g(0,1) = -1, \qquad \frac{\partial g}{\partial s}(s,\tau) > 0 \quad \forall s > 0,\ \forall \tau \ge 1, \qquad g(0,\tau) = -\tau^{q_m} < -1 \quad \forall \tau > 1, \\ \lim_{s\to+\infty} g(s,\tau) = +\infty \quad \forall \tau \ge 1, \end{gathered} \tag{2.114}$$

we have

$$\forall \tau \ge 1\ \exists \mid s(\tau) \ge 0\ \ (s(1) = 0, s(\tau) > 0 \text{ for } \tau > 1) : g(s(\tau),\tau) = -1. \tag{2.115}$$

We note that $\lim_{\tau\to 1^+} s(\tau) = 0$. In fact, if $\{\tau_n\} \subseteq]1, +\infty[$ and $\lim \tau_n = 1$, being $g(s(\tau_n),\tau_n) = -1$, $\{s(\tau_n)\}$ is bounded (else (within a subsequence) $\lim g(s(\tau_n),\tau_n) = +\infty$). Then (within a subsequence) $\lim s(\tau_n) = \omega$ with $g(\omega, 1) = 0$, from which $\omega = 0$.

We add that $s(\tau)$ belongs to $C^1(]1,+\infty[)$, and its derivative has the form:

$$s'(\tau) = -\frac{1}{(s(\tau))^{\gamma-1}}\widetilde{g}(s(\tau),\tau) \quad \forall\tau > 1 \text{ with } \lim_{\tau\to 1^+}\widetilde{g}(s(\tau),\tau) \in]-\infty,0[. \tag{2.116}$$

Hence, set $v(\tau) = (\tau\overline{v}_1,\dots,s(\tau)\varphi,\dots,\tau\overline{v}_n)$, it results in

$$\begin{gathered} D_m(v(\tau)) = -1 \quad \forall\tau \geq 1, \qquad \lim_{\tau\to 1^+}\frac{d}{d\tau}H_\lambda(v(\tau)) = pH_\lambda(\overline{v}) < 0, \\ \lim_{\tau\to 1^+}\frac{d}{d\tau}D_j(v(\tau)) = 0 \quad \text{as } j = 1,\dots,m-1. \end{gathered} \tag{2.117}$$

As in Proposition 2.24, we introduce the open ball $\widetilde{B}$ with centre $\overline{v}$ included in V_λ^- and the functionals $t^*(v)$ and $E^*(v)$ belonging to $C^1(\widetilde{B})$. Chosen $\tau_0 > 1$ such that $v(\tau) \in \widetilde{B}$ for all $\tau \in [1,\tau_0]$, we have

$$\frac{d}{d\tau}E^*(v(\tau)) = (t^*(v(\tau)))^p\frac{d}{d\tau}H_\lambda(v(\tau)) - \sum_{j=1}^{m-1}(t^*(v(\tau)))^{q_j}\frac{d}{d\tau}D_j(v(\tau)) \quad \forall\tau \in [1,\tau_0], \tag{2.118}$$

and consequently $\lim_{\tau\to 1^+}(d/d\tau)E^*(v(\tau)) < 0$. Then, taking into account (2.112), with $\tau_1 \in]1,\tau_0]$ such that $(d/d\tau)E^*(v(\tau)) < 0$ for all $\tau \in]1,\tau_1]$, we get the contradiction:

$$E^*(\overline{v}) \leq E^*(v(\tau)) < E^*(\overline{v}) \quad \forall\tau \in]1,\tau_1]. \tag{2.119}$$

□

Proposition 2.29. *If* $d_\ell^{q_m/\gamma} > \widetilde{d}_\ell$ *as* $\ell = 1,\dots,n-1$, *then*

$$\overline{u}_\ell > 0 \quad \text{as } \ell = 1,\dots,n. \tag{2.120}$$

Proof. In fact,

$$\begin{gathered} \overline{u}_\ell > 0 \quad \text{as } \ell = 1,\dots,n-1 \quad (\text{Proposition 2.23}), \\ \overline{u}_n \equiv 0 \Longrightarrow D_m(\overline{u}) > 0. \end{gathered} \tag{2.121}$$

□

Application 2.30. Let for each $v = (v_1, \dots, v_n) \in W$:

$$\begin{aligned}
A(v) &= p^{-1}\int_\Omega \left(\sum_{\ell=1}^{n}|\nabla v_\ell|^\gamma\right)^{p/\gamma} dx + \prod_{\ell=1}^{n}\int_\Omega |\nabla v_\ell|^{p_\ell} dx,\\
D_j(v) &= \int_\Omega \rho_j\left(\prod_{\ell=1}^{n}|v_\ell|^{q_{j\ell}}\right)dx \quad \text{as } j = 1,\dots,m-1,\\
D_m(v) &= q_m^{-1}\left[\int_\Omega \left|\sum_{\ell=1}^{n}d_\ell v_\ell\right|^{q_m-1}\left(\sum_{\ell=1}^{n}d_\ell v_\ell\right)dx - \int_\Omega d|v_n|^{q_m}dx\right],
\end{aligned} \tag{2.122}$$

where

$$\begin{gathered}
1 < \gamma < p,\ p_\ell > 1, \quad \sum_{\ell=1}^{n}p_\ell = p, \quad q_{j\ell} > 1, \quad \sum_{\ell=1}^{n}q_{j\ell} = q_j, \quad p < q_m,\ q_1 < \cdots < q_m < \widetilde{p},\\
\rho_j \in L^\infty(\Omega) \setminus \{0\}, \quad \rho_j \le 0, d_\ell, d \in L^\infty(\Omega), \quad d_\ell(x) \ne 0 \text{ a.e. in } \Omega,\ d > 0.
\end{gathered} \tag{2.123}$$

Let as $\ell = 1,\dots,n$ $F_\ell \in W^{-1,p'}(\Omega)$ $(p' = p/(p-1))$. Let $\langle\langle F, v\rangle\rangle = \sum_{\ell=1}^{n}\langle F_\ell, v_\ell\rangle$ for all $v \in W$. Set $\eta_i = 0$ as $i = 1,\dots,n-1$ and $\eta_n = 1$, let us consider the system:

$$\begin{gathered}
-\operatorname{div}\left(\left[\left(\sum_{\ell=1}^{n}|\nabla u_\ell|^\gamma\right)^{(p/\gamma)-1}|\nabla u_i|^{\gamma-2} + p_i\left(\prod_{\ell\ne i}\int_\Omega |\nabla u_\ell|^{p_\ell}\right)|\nabla u_i|^{p_i-2}\right]\nabla u_i\right)\\
= \lambda_i b_i|u_i|^{p-2}u_i + \sum_{j=1}^{m-1}q_{ji}\rho_j\left(\prod_{\ell\ne i}|u_\ell|^{q_{j\ell}}\right)|u_i|^{q_{ji}-2}u_i\\
+ \left|\sum_{\ell=1}^{n}d_\ell u_\ell\right|^{q_m-1}d_i - \eta_i d|u_n|^{q_m-2}u_n + F_i \quad \text{in } \Omega,\\
u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i = 1,\dots,n,
\end{gathered} \tag{2.124}$$

under at least one of the following conditions

$$\exists\Omega^+ \subseteq \Omega : |\Omega^+|_N > 0, \quad d_\ell > 0 \quad \text{in } \Omega^+ \text{ for some } \ell \in \{1,\dots,n-1\}\ (\Longrightarrow V^+(D_m) \ne \emptyset), \tag{2.125}$$

$$|d_n|^{q_m} < d \quad (\Longrightarrow D_m(0,\dots,0,c_n u_n^*) < 0\ \forall c_n \in R \setminus \{0\}). \tag{2.126}$$

Evidently, about the validity of (i_{14}) we choose $\lambda_1,\dots,\lambda_n$ as in Application 2.26.

Proposition 2.31 (see [1], Theorem 3.2). *Under assumptions* (2.123), (2.125) ((2.125) *and* (2.126), *resp.), if* $F \not\equiv 0$ *and* $\|F\|_*$ *is sufficiently small, for* $\lambda_1,\dots,\lambda_n$ *as in* (2.99) *(resp.* (2.100)*) system* (2.124) *has at least one weak solution* $\widetilde{u}$ $(\widetilde{u} = \widetilde{\tau}\,\widetilde{v}, \widetilde{\tau} = const. > 0,\ \widetilde{v} \in S_\lambda \cap V^+(D_m))$.

Let us note that

$$\widetilde{u}_h \not\equiv 0 \text{ even if } F_h \equiv 0 \text{ since } (F_h \equiv 0,\ \widetilde{u}_h \equiv 0) \Longrightarrow \sum_{\ell=1}^{n} d_\ell \widetilde{u}_\ell \equiv 0 \Longrightarrow D_m(\widetilde{u}) \leq 0. \tag{2.127}$$

Application 2.32. Let $\lambda_1 = \cdots = \lambda_n = 0$, and for each $v = (v_1, \ldots, v_n) \in W$:

$$\begin{gathered} A(v) = p^{-1} \int_\Omega \left[\sum_{\ell=1}^{n} \left(|\nabla v_\ell|^\gamma + a|v_\ell|^\gamma \right) \right]^{p/\gamma} dx, \\ D_j(v) = q_j^{-1} \int_\Omega d_j \left(\sum_{\ell=1}^{n} |v_\ell|^{\gamma_j} \right)^{q_j/\gamma_j} dx \quad \text{as } j = 1, \ldots, m, \text{ with } m > 2, \end{gathered} \tag{2.128}$$

under one of the following assumptions:

$$\begin{gathered} a \in L^\infty(\Omega), \quad a \geq 0, \qquad d_j \in L^\infty(\Omega) \setminus \{0\} \quad \text{with } d_1 \leq 0,\ d_j \geq 0 \text{ as } j \geq 2, \\ 1 < \gamma_j < \gamma < p < q_2 < \cdots < q_m < \widetilde{p} \quad \text{as } j \geq 2,\ \gamma \leq \gamma_1 < q_1 < q_2; \end{gathered} \tag{2.129}$$

$$\begin{gathered} a \in L^\infty(\Omega), \quad a \geq 0, \quad \text{as } j = 1, \ldots, m \quad d_j \in L^\infty(\Omega) \setminus \{0\}, \quad d_j \geq 0, \\ 1 < \gamma_j < \gamma < p < q_1 < \cdots < q_m < \widetilde{p}. \end{gathered} \tag{2.130}$$

Set F as in Application 2.30. Let us consider the system:

$$\begin{aligned} &-\operatorname{div}\left(\left[\sum_{\ell=1}^{n} \left(|\nabla u_\ell|^\gamma + a|u_\ell|^\gamma \right) \right]^{(p/\gamma)-1} |\nabla u_i|^{\gamma-2} \nabla u_i \right) \\ &\quad = -\left[\sum_{\ell=1}^{n} \left(|\nabla u_\ell|^\gamma + a|u_\ell|^\gamma \right) \right]^{(p/\gamma)-1} a|u_i|^{\gamma-2} u_i + \sum_{j=1}^{m} d_j \left(\sum_{\ell=1}^{n} |u_\ell|^{\gamma_j} \right)^{(q_j/\gamma_j)-1} |u_i|^{\gamma_j-2} u_i + F_i \quad \text{in } \Omega, \\ &\qquad u_i = 0 \quad \text{on } \partial\Omega \quad \text{as } i = 1, \ldots, n. \end{aligned} \tag{2.131}$$

Let us verify that

$$\begin{aligned} (2.129)\,[\text{resp. } (2.130)] \Longrightarrow \Big(\big(i_{16}^h\big) &\text{ holds } \forall h \in \{1, \ldots, n\} \text{ with } \mathfrak{F} = S_\lambda \cap V^+(D_2, \ldots, D_m) \\ &[\text{resp. } \mathfrak{F} = S_\lambda \cap V^+(D_1, \ldots, D_m)] \Big). \end{aligned} \tag{2.132}$$

Let $v = (v_1, \dots, v_n) \in \mathfrak{F}$ with, for example, $v_1 \equiv 0$. Let $j_0 \in \{2, \dots, m\}$ (resp. $j_0 \in \{1, \dots, m\}$) and $\ell_0 \in \{2, \dots, m\}$ such that $d_{j_0} v_{\ell_0} \not\equiv 0$. Let us suppose $\ell_0 = 2$ and set $v(s) = ((1-s)^{1/\gamma} v_2, s^{1/\gamma} v_2, v_3, \dots, v_n)$. Then,

$$A(v(s)) = 1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[\, : D_{j_0}(v(s)) > 0 \quad \forall s \in [s_0, 1],$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_{j_0}(v(s)) = -\infty, \quad \lim_{s \to 1^-}'' \frac{d}{ds} D_j(v(s)) < +\infty \quad \text{as } j \neq j_0. \tag{2.133}$$

Proposition 2.33. *Under assumption* (2.129) *(resp.* (2.130)*), system* (2.131) *with $F \equiv 0$ has at least two weak solutions u^0 and $-u^0$, and we have as $i = 1, \dots, n$:*

$$u_i^0 \in L^\infty(\Omega), \quad u_i^0 \geq 0, u_i^0 \not\equiv 0. \tag{2.134}$$

Consequently,

$$a \equiv 0 \Longrightarrow u_i^0 \in C^{1,\alpha_i^0}_{\ell oc}(\Omega), \quad a \equiv 0 \textit{ and } (2.129) \textit{ holds with } p \leq \gamma_1 \ [\textit{resp. } (2.130) \textit{ holds}] \Longrightarrow u_i^0 > 0. \tag{2.135}$$

Proof. The statement is due to ([1], Theorem 2.2, Remark 2.3), [5], Proposition A.3, [6]. □

Proposition 2.34 (see [1], Theorems 3.1, 3.2). *Under assumption* (2.129) *(resp.* (2.130)*), system* (2.131) *with $F \not\equiv 0$ and $\|F\|_*$ sufficiently small has at least two different weak solutions u^1 and u^2 ($u^i = \tau^i v^i$, $\tau^i = const. > 0, v^1 \in V^+(F) \cap S_\lambda$, $v^2 \in S_\lambda \cap V^+(D_2, \dots, D_m)$ [resp. $v^2 \in S_\lambda \cap V^+(D_1, \dots, D_m)$]), and we have $u_h^2 \not\equiv 0$ even if $F_h \equiv 0$.*

Remark 2.35. If $\bigcup_{j=2}^m \{x \in \Omega : d_j(x) > 0\}$ [resp. $\bigcup_{j=1}^m \{x \in \Omega : d_j(x) > 0\}$] $= \Omega$ (within a set with measure equal to zero), with the same reasoning used about (2.132), we get that

$$\left(i_{16}^h\right) \text{ holds} \quad \forall h \in \{1, \dots, n\} \quad \text{with } \mathfrak{F} = V^+(F) \cap S_\lambda, \tag{2.136}$$

hence, $u_h^1 \not\equiv 0$ even if $F_h \equiv 0$.

3. Neumann Problems

Let $\Omega \subseteq R^N$ be an open, bounded, and connected $C^{0,1}$ set. Let $|\cdot|_N, p$ and $\tilde{p}$ as in Section 2, σ the measure on $\partial\Omega$, ν the outward unit normal to $\partial\Omega$, $\hat{p} = (N-1)p/(N-p)$ if $p < N$, $\hat{p} = \infty$ if

$p \geq N$. Let us assume

$$W = \left(W^{1,p}(\Omega)\right)^n (n \geq 1) \quad \text{with } \|v\| = \left(\sum_{\ell=1}^{n} \int_{\Omega} \left[|\nabla v_\ell|^p + |v_\ell|^p\right] dx\right)^{1/p} \quad \forall v = (v_1, \dots, v_n) \in W,$$
$$B_\ell(v_\ell) = p^{-1} \int_{\Omega} b_\ell |v_\ell|^p dx \quad \forall v_\ell \in W^{1,p}(\Omega), \quad \text{where } b_\ell \in L^\infty(\Omega) \setminus \{0\},$$
$$\widehat{B}_\ell(v_\ell) = p^{-1} \int_{\partial\Omega} \widehat{b}_\ell |v_\ell|^p d\sigma \quad \forall v_\ell \in W^{1,p}(\Omega), \quad \text{where } \widehat{b}_\ell \in L^\infty(\partial\Omega) \setminus \{0\}. \tag{3.1}$$

We note that for each $v_\ell \in W^{1,p}(\Omega)$ we set $\gamma_0(v_\ell) = v_\ell$ where γ_0 is the trace operator from $W^{1,p}(\Omega)$ into $W^{1-(1/p),p}(\partial\Omega)$. Morever we consider the functionals A (as in (i_{11})) such that

$$\exists \widetilde{c} > 0 : A(v) \geq p^{-1} \widetilde{c} \sum_{\ell=1}^{n} \int_{\Omega} |\nabla v_\ell|^p dx \quad \forall v \in W. \tag{3.2}$$

It is easy to verify the following.

Proposition 3.1. *Let $b_\ell, \widehat{b}_\ell \geq 0$ as $\ell = 1, \dots, n$. Then,*

$$(i_{13}) \text{ holds if } \lambda_\ell, \mu_\ell \leq 0, \quad \lambda_\ell + \mu_\ell < 0 \quad \text{as } \ell = 1, \dots, n. \tag{3.3}$$

Let us set $I = \{1, \dots, n\}$ and for each $I^* \subseteq I$

$$C^* = \{c = (c_1, \dots, c_n) \in R^n : c_\ell = 0 \text{ if } \ell \in I \setminus I^*, \ c_\ell \neq 0 \text{ for some } \ell \in I^*\}. \tag{3.4}$$

Let us introduce the conditions:

(i_{31}) there exists $I^* \subseteq I : D_m(c) < 0$ for all $c \in C^*$;

(i_{32}) there exist $I^* \subseteq I$ and $m_1 \in \{1, \dots, m\} : D_{m_1}(c) < 0$ and $A(c) = 0$ for all $c \in C^*$.

Proposition 3.2. *Let (i_{31}) holds with $I^* \neq I$. Let $V^+(D_m) \neq \emptyset$. Let b_ℓ, $\widehat{b}_\ell \geq 0$ as $\ell \in I \setminus I^*$. Then with λ_ℓ, $\mu_\ell \leq 0$ and $\lambda_\ell + \mu_\ell < 0$ as $\ell \in I \setminus I^*$ $\exists \delta^* > 0$: (i_{14}) holds if $|\lambda_\ell|, |\mu_\ell| \leq \delta^*$ as $\ell \in I^*$.*

Proof. Reasoning by contradiction, for each $k \in \mathbb{N}$ there exist $\lambda_\ell^k, \mu_\ell^k \in [-k^{-1}, k^{-1}]$, with $\ell \in I^*$, and $v^k = (v_1^k, \dots, v_n^k) \in V^+(D_m)$ such that

$$\begin{aligned} \left\|v^k\right\|^p > k \Bigg\{ & A\left(v^k\right) - \sum_{\ell \in I \setminus I^*} p^{-1} \left[\lambda_\ell \int_{\Omega} b_\ell \left|v_\ell^k\right|^p dx + \mu_\ell \int_{\partial\Omega} \widehat{b}_\ell \left|v_\ell^k\right|^p d\sigma\right] \\ & - \sum_{\ell \in I^*} p^{-1} \left[\lambda_\ell^k \int_{\Omega} b_\ell \left|v_\ell^k\right|^p dx + \mu_\ell^k \int_{\partial\Omega} \widehat{b}_\ell \left|v_\ell^k\right|^p d\sigma\right] \Bigg\}, \end{aligned} \tag{3.5}$$

then, set $w^k = \|v^k\|^{-1} v^k$, we have

$$D_m\left(w^k\right) > 0, \quad p^{-1}\left\{\tilde{c}\sum_{\ell=1}^{n}\int_\Omega \left|\nabla w_\ell^k\right|^p dx - \sum_{\ell\in I\setminus I^*}\left[\lambda_\ell \int_\Omega b_\ell \left|w_\ell^k\right|^p dx + \mu_\ell \int_{\partial\Omega} \widehat{b}_\ell \left|w_\ell^k\right|^p d\sigma\right]\right\}$$
$$< k^{-1} + \sum_{\ell\in I^*} p^{-1}\left[\lambda_\ell^k \int_\Omega b_\ell \left|w_\ell^k\right|^p dx + \mu_\ell^k \int_{\partial\Omega} \widehat{b}_\ell \left|w_\ell^k\right|^p d\sigma\right]. \tag{3.6}$$

Since $\|w^k\| = 1$, there exists $w \in W$ such that (within a subsequence)

$$w^k \longrightarrow w \text{ weakly in } W, \quad w^k \longrightarrow w \text{ strongly in } (L^p(\Omega))^n, \quad w^k \longrightarrow w \text{ strongly in } (L^p(\partial\Omega))^n. \tag{3.7}$$

Consequently, from (3.6), passing to limit as $k \to +\infty$, we get

$$D_m(w) \geq 0, \qquad \sum_{\ell=1}^{n}\int_\Omega |\nabla w_\ell|^p dx = 0, \qquad \sum_{\ell\in I\setminus I^*}\left[\lambda_\ell \int_\Omega b_\ell |w_\ell|^p dx + \mu_\ell \int_{\partial\Omega} \widehat{b}_\ell |w_\ell|^p d\sigma\right] = 0, \tag{3.8}$$

from which $w = 0$, and then the contradiction $0 = \lim_{k\to+\infty}\|w^k\| = 1$. □

Proposition 3.3. *Let (i_{31}) holds with $I^* = I$. Let $V^+(D_m) \neq \emptyset$. Then,*

$$\exists \delta^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, |\mu_\ell| \leq \delta^* \quad \text{as } \ell = 1, \ldots, n. \tag{3.9}$$

The proof as in Proposition 3.2.

Proposition 3.4. *Let (i_{32}) holds with $I^* \neq I$. Let $\int_\Omega b_\ell dx, \int_{\partial\Omega} \widehat{b}_\ell d\sigma > 0$ as $\ell \in I^*$. Then,*

$$V_{\lambda\mu}^- \cap S(D_{m_1}) \neq \emptyset \quad \forall(\lambda_\ell,\ \mu_\ell)_{\ell\in I} \text{ with } \lambda_\ell,\ \mu_\ell \geq 0\ \forall \ell \in I^*, \quad \lambda_\ell + \mu_\ell > 0 \quad \text{for some } \ell \in I^*. \tag{3.10}$$

Moreover, if $b_\ell, \widehat{b}_\ell \geq 0$ as $\ell \in I \setminus I^*$, we have

with $\lambda_\ell,\ \mu_\ell \leq 0$ and $\lambda_\ell + \mu_\ell < 0$ as $\ell \in I \setminus I^*$ $\exists \delta^* > 0 : (i_{15})$ holds
if $\lambda_\ell, \mu_\ell \in [0, \delta^*]$ $\forall \ell \in I^*$ and $\lambda_\ell + \mu_\ell > 0$ for some $\ell \in I^*$.

Proof. The first statement is evident. Let us prove the second one. Reasoning by contradiction, for each $k \in \mathbb{N}$ there exist $\lambda_\ell^k, \mu_\ell^k \in [0, k^{-1}]$, with $\ell \in I^*$ and $\lambda_\ell^k + \mu_\ell^k > 0$ for some $\ell \in I^*$, and a sequence $(v^{k,h})_{h\in\mathbb{N}}$ such that

$$\left(v^{k,h}\right)_{h\in\mathbb{N}} \subseteq V_{\lambda^k\mu^k}^- \cap S(D_{m_1}) \quad \left(\lambda_\ell^k = \lambda_\ell,\ \mu_\ell^k = \mu_\ell \text{ as } \ell \in I \setminus I^*\right), \quad \sup_h \left\|v^{k,h}\right\| = +\infty. \tag{3.11}$$

Let $\{h_k\} \subseteq \mathbb{N}$ be a strictly increasing sequence such that $\|v^{k,h_k}\| \to +\infty$ as $k \to +\infty$. Let $w^k = \|v^{k,h_k}\|^{-1} v^{k,h_k}$. Then, $D_{m_1}(w^k) = -\|v^{k,h_k}\|^{-q_{m_1}}$ and

$$p^{-1}\left\{ \tilde{c}\sum_{\ell=1}^{n}\int_{\Omega}\left|\nabla w_\ell^k\right|^p dx - \sum_{\ell\in I\setminus I^*}\left[\lambda_\ell\int_\Omega b_\ell\left|w_\ell^k\right|^p dx + \mu_\ell\int_{\partial\Omega}\widehat{b}_\ell\left|w_\ell^k\right|^p d\sigma\right]\right\} < p^{-1}\sum_{\ell\in I^*}\left[\lambda_\ell^k\int_\Omega b_\ell\left|w_\ell^k\right|^p dx + \mu_\ell^k\int_{\partial\Omega}\widehat{b}_\ell\left|w_\ell^k\right|^p d\sigma\right], \tag{3.12}$$

moreover, there exists $w \in W$ such that (within a subsequence)

$$w^k \longrightarrow w \text{ weakly in } W, \qquad w^k \longrightarrow w \text{ strongly in } (L^p(\Omega))^n, \qquad w^k \longrightarrow w \text{ strongly in } (L^p(\partial\Omega))^n. \tag{3.13}$$

Consequently,

$$D_{m_1}(w) = 0, \qquad \sum_{\ell=1}^{n}\int_\Omega |\nabla w_\ell|^p dx = 0, \qquad \sum_{\ell\in I\setminus I^*}\left[\lambda_\ell\int_\Omega b_\ell|w_\ell|^p dx + \mu_\ell\int_{\partial\Omega}\widehat{b}_\ell|w_\ell|^p d\sigma\right] = 0, \tag{3.14}$$

then $w = 0$, and the contradiction $0 = \lim_{k\to+\infty}\|w^k\| = 1$. □

Proposition 3.5. *Let (i_{32}) holds with $I^* = I$. Let $\int_\Omega b_\ell dx, \int_{\partial\Omega}\widehat{b}_\ell d\sigma > 0$ as $\ell = 1, \ldots, n$. Then,*

$$V_{\lambda\mu}^- \cap S(D_{m_1}) \neq \emptyset \quad \text{if } \lambda_\ell,\ \mu_\ell \geq 0 \quad \forall \ell \in I \quad \text{and} \quad \lambda_\ell + \mu_\ell > 0 \quad \text{for some } \ell \in I,$$
$$\exists \delta^* > 0 : (i_{15}) \text{ holds} \quad \text{if } \lambda_\ell, \mu_\ell \in [0, \delta^*] \quad \forall \ell \in I \quad \text{and} \quad \lambda_\ell + \mu_\ell > 0 \quad \text{for some } \ell \in I. \tag{3.15}$$

The proof as in Proposition 3.4.

Remark 3.6. It is suitable to make some clarifications.

(i) The assumption "$b_\ell, \widehat{b}_\ell \geq 0$" (see Propositions 3.1, 3.2, and 3.4) can be replaced by "$b_\ell, \widehat{b}_\ell$ do not change sign." In this case we can choose λ_ℓ and μ_ℓ such that $\lambda_\ell b_\ell \leq 0, \mu_\ell \widehat{b}_\ell \leq 0$ and $|\lambda_\ell| + |\mu_\ell| > 0$.

(ii) The assumption "$\int_\Omega b_\ell dx, \int_{\partial\Omega}\widehat{b}_\ell d\sigma > 0$" (see Propositions 3.4 and 3.5) can be replaced by "$\int_\Omega b_\ell dx, \int_{\partial\Omega}\widehat{b}_\ell d\sigma \neq 0$". In this case, we can choose λ_ℓ and μ_ℓ such that $\lambda_\ell\int_\Omega b_\ell dx, \mu_\ell\int_{\partial\Omega}\widehat{b}_\ell d\sigma \geq 0$ and $|\lambda_\ell|+|\mu_\ell| > 0$ for some ℓ, with $|\lambda_\ell|, |\mu_\ell| \leq \delta^*$ instead of $\lambda_\ell, \mu_\ell \in [0, \delta^*]$.

(iii) When for each $\ell \in \{1, \ldots, n\}$ $b_\ell, \widehat{b}_\ell$ do not change sign, then the conclusion of the Proposition 3.2 [resp. Proposition 3.3] holds even if $\lambda_\ell b_\ell, \mu_\ell \widehat{b}_\ell \leq 0$ and $|\lambda_\ell|+|\mu_\ell| > \delta^*$ as $\ell \in I^*$ (resp. as $\ell = 1, \ldots, n$).

In order to simplify the presentation of the applications, we suppose in the next $b_\ell \in L^\infty(\Omega) \setminus \{0\}$ and $\widehat{b}_\ell \in L^\infty(\partial\Omega) \setminus \{0\}$, while the additional assumptions on $b_\ell, \widehat{b}_\ell$ and the assumptions on $\int_\Omega b_\ell dx, \int_{\partial\Omega} \widehat{b}_\ell d\sigma$ (the same of Propositions 3.1, 3.2, 3.4, and 3.5) will be pointed out case by case.

Passing to the applications (with $n > 1$), we recall that in [3] Pohozaev and Véron in the case $n = 1$ have studied the Neumann problem:

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = \lambda b(x)|u|^{p-2}u + c(x)|u|^{s-2}u + a(x)|u|^{q-2}u \quad \text{in } \Omega,$$
$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = k(x)|u|^{r-2}u \quad \text{on } \partial\Omega. \tag{3.16}$$

The existence theorems proved by these authors can be got by using some results of ([1], Theorems 2.1, 2.2, 4.1, and 4.2; Remarks 2.1, 2.3, 4.1, and 4.4), Propositions 3.3 and 3.5.

Application 3.7. Let for each $v = (v_1, \ldots, v_n) \in W$:

$$A(v) = p^{-1}\sum_{\ell=1}^{n}\int_\Omega |\nabla v_\ell|^p dx, \qquad D_1(v) = q_1^{-1}\left[\int_{\partial\Omega}\left(\sum_{\ell=1}^{n} d_\ell |v_\ell|^\gamma\right)^{q_1/\gamma} d\sigma - \sum_{\ell=1}^{n}\int_{\partial\Omega}\widehat{d}_\ell |v_\ell|^{q_1} d\sigma\right], \tag{3.17}$$

where

$$1 < \gamma < q_1 < \widehat{p}, \quad q_1 \neq p, \qquad d_\ell, \widehat{d}_\ell \in L^\infty(\partial\Omega), \quad d_\ell, \widehat{d}_\ell > 0. \tag{3.18}$$

Let us consider the system:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2}u_i \quad \text{in } \Omega,$$
$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} = \mu_i \widehat{b}_i |u_i|^{p-2}u_i + \left(\sum_{\ell=1}^{n} d_\ell |u_\ell|^\gamma\right)^{(q_1/\gamma)-1} d_i |u_i|^{\gamma-2}u_i$$
$$-\widehat{d}_i |u_i|^{q_1-2}u_i \quad \text{on } \partial\Omega \text{ as } i = 1, \ldots, n. \tag{3.19}$$

Let us introduce the conditions:

$$\int_{\partial\Omega}\left(\sum_{\ell=1}^{n} d_\ell\right)^{q_1/\gamma} d\sigma < \int_{\partial\Omega}\widehat{d}\, d\sigma \quad \left(\widehat{d} = \min\left\{\widehat{d}_1, \ldots, \widehat{d}_n\right\}\right), \tag{3.20}$$

$$\exists \Gamma \subseteq \partial\Omega : \sigma(\Gamma) > 0, \qquad \left(\sum_{\ell=1}^{n} d_\ell\right)^{q_1/\gamma} > \sum_{\ell=1}^{n}\widehat{d}_\ell \quad \text{on } \Gamma, \tag{3.21}$$

$$\int_\Omega b_\ell dx > 0, \quad \int_{\partial\Omega}\widehat{b}_\ell d\sigma > 0 \quad \text{as } \ell = 1, \ldots, n. \tag{3.22}$$

Evidently (3.20) $\Rightarrow$ $D_1(c) < 0$ for all $c \in R^n \setminus \{0\}$. Moreover (3.21) $\Rightarrow$ $V^+(D_1) \neq \emptyset$ (Proposition A.2). Hence (Propositions 3.3 and 3.5)

$$\begin{aligned}&(3.20) \text{ and } (3.21)\\ &\quad\Longrightarrow \left(\exists\delta_1^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, |\mu_\ell| \le \delta_1^* \ \forall\ell \in \{1,\dots,n\}\right),\end{aligned} \tag{3.23}$$

$$\begin{aligned}&(3.20) \text{ and } (3.22)\\ &\quad\Longrightarrow \left(\exists\delta_2^* > 0\ (i_{15}) \text{ holds if } \lambda_\ell, \mu_\ell \in [0,\ \delta_2^*]\ \forall\ell \in \{1,\dots,n\},\ \lambda_\ell + \mu_\ell > 0 \text{ for some } \ell\right).\end{aligned} \tag{3.24}$$

Proposition 3.8 (see ([1], Theorems 2.1 and 4.1; Remarks 2.1 and 4.1); Proposition A.4; [5, 6]). *Under assumption* (3.18), *we have:*

(i) *When* (3.20) *and* (3.21) *hold, with* λ_ℓ, μ_ℓ *as in* (3.23) *system* (3.19) *has at least two weak solutions* u^0 *and* $-u^0$ ($u^0 = \tau^0 v^0$, $\tau^0 = const. > 0$, $v^0 \in S_{\lambda\mu} \cap V^+(D_1)$)*, and it results in*

$$u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega), \quad u_i^0 \ge 0 \text{ as } i = 1,\dots,n,\ u_i^0 > 0 \text{ if } u_i^0 \not\equiv 0; \tag{3.25}$$

(ii) *When* (3.20) *and* (3.22) *hold, with* λ_ℓ, μ_ℓ *as in* (3.24) *system* (3.19) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ ($\overline{u} = \overline{\tau}\,\overline{v}$, $\overline{\tau} = const. > 0$, $\overline{v} \in V^-_{\lambda\mu} \cap S(D_1)$)*, and it results in*

$$\overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}_i}_{\ell oc}(\Omega), \quad \overline{u}_i \ge 0 \text{ as } i = 1,\dots,n,\ \overline{u}_i > 0 \text{ if } \overline{u}_i \not\equiv 0. \tag{3.26}$$

Consequently, when (3.20)–(3.22) *hold, with* λ_ℓ, μ_ℓ *as in* (3.24) *and* $\min\{\delta_1^*, \delta_2^*\}$ *instead of* δ_2^* *system* (3.19) *has at least four different weak solutions.*

Proposition 3.9. *If* $\gamma < p < q_1$, *then* $u_i^0 > 0$ *as* $i = 1,\dots n$.

Proof. It is sufficient (Remark 1.1) to verify that

$$\left(i_{16}^h\right) \text{ holds as } h = 1,\dots,n \quad \text{with } \mathfrak{F} = S_{\lambda\mu} \cap V^+(D_1). \tag{3.27}$$

Let $v = (v_1,\dots,v_n) \in V^+(D_1) \cap S_{\lambda\mu}$. Let, for example, $v_1 \equiv 0$. Since $\int_{\partial\Omega} \left(\sum_{\ell\neq 1} d_\ell |v_\ell|^\gamma\right)^{q_1/\gamma} d\sigma > 0$, there exists $\Gamma^+ \subseteq \partial\Omega$ such that

$$\sigma(\Gamma^+) > 0, \qquad \sum_{\ell\neq 1} d_\ell |v_\ell|^\gamma > 0 \quad \text{on } \Gamma^+. \tag{3.28}$$

Let $\mathbb{K} \subseteq \Omega$ a compact set and Ω' an open set such that

$$|\mathbb{K}|_N > 0, \quad \mathbb{K} \subseteq \Omega', \quad \overline{\Omega'} \subseteq \Omega. \tag{3.29}$$

Since Propositions A.1 and A.2, there exist a compact set $\widehat{\Gamma}^+ \subseteq \Gamma^+$, with $\sigma(\widehat{\Gamma}^+) > 0$, and $(\varphi_{1\varepsilon})_{0<\varepsilon<\varepsilon_0}, (\varphi_{2\varepsilon})_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(R^N)$ such that

$$
\begin{gathered}
0 \le \varphi_{1\varepsilon} \le 1, \quad \operatorname{supp} \varphi_{1\varepsilon} \subseteq \Omega', \ \varphi_{1\varepsilon} \longrightarrow \chi \text{ strongly in } L^s(\Omega) \\
\int_\Omega |\nabla \varphi_{1\varepsilon}|^s dx \longrightarrow +\infty \quad \text{as } \varepsilon \longrightarrow 0^+ \ \forall s \in [1, +\infty[, \\
0 \le \varphi_{2\varepsilon} \le 1, \quad \operatorname{supp} \varphi_{2\varepsilon} \subseteq R^N \setminus \overline{\Omega\prime}, \ \varphi_{2\varepsilon} \longrightarrow \widehat{\chi} \text{ strongly in } L^s(\partial\Omega), \\
\int_\Omega \varphi_{2\varepsilon}^s dx \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+ \ \forall s \in [1, +\infty[,
\end{gathered} \tag{3.30}
$$

where χ (resp. $\widehat{\chi}$) is the characteristic function of $\mathbb{K}$ (resp. $\widehat{\Gamma}^+$). Let us choose ε such that

$$
\begin{gathered}
\delta = p^{-1}\left[\int_\Omega |\nabla \varphi_\varepsilon|^p dx - \lambda_1 \int_\Omega b_1 \varphi_\varepsilon^p dx - \mu_1 \int_{\partial\Omega} \widehat{b}_1 \varphi_\varepsilon^p d\sigma\right] > 0, \\
\int_{\partial\Omega} \left(\sum_{\ell \ne 1} d_\ell |v_\ell|^\gamma\right)^{(q_1/\gamma)-1} d_1 \varphi_\varepsilon^\gamma d\sigma > 0 \quad (\varphi_\varepsilon = \varphi_{1\varepsilon} + \varphi_{2\varepsilon}),
\end{gathered} \tag{3.31}
$$

and we set $v(s) = ((1-s)^{1/p}\delta^{-1/p}\varphi_\varepsilon,\ s^{1/p}v_2, \dots, s^{1/p}v_n)$. Then,

$$
\begin{gathered}
H_{\lambda\mu}(v(s)) = 1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[: D_1(v(s)) > 0 \quad \forall s \in [s_0, 1], \\
\lim_{s \to 1^-} \frac{d}{ds} D_1(v(s)) = -\infty.
\end{gathered} \tag{3.32}
$$

□

Proposition 3.10. *If*

$$
d_\ell^{q_1/\gamma} < \widehat{d}_\ell \quad \text{as } \ell = 1, \ \dots, n, \tag{3.33}
$$

$$
\lambda_\ell + \mu_\ell > 0 \quad \text{as } \ell = 1, \dots, n, \tag{3.34}
$$

then $\overline{u}_i > 0$ *as* $i = 1, \dots, n$.

Proof. We recall that ([1], Theorem 4.1)

$$
H_{\lambda\mu}(\overline{v}) = \inf\left\{ H_{\lambda\mu}(v) : \ v \in V_{\lambda\mu}^- \cap S(D_1) \right\}. \tag{3.35}
$$

Reasoning by contradiction let, for example, $\overline{v}_1 \equiv 0$. As $c_1 = \text{const.} > 0$ and $g(s,\tau) = D_1(sc_1, \tau\overline{v}_2, \dots, \tau\overline{v}_n) = q_1^{-1}[[\int_{\partial\Omega} (d_1 s^\gamma c_1^\gamma + \tau^\gamma \sum_{\ell \ne 1} d_\ell (\overline{v}_\ell)^\gamma)^{q_1 \setminus \gamma} d\sigma - s^{q_1} c_1^{q_1} \int_{\partial\Omega} \widehat{d}_1 d\sigma - \tau^{q_1} \sum_{\ell \ne 1} \int_{\partial\Omega} \widehat{d}_\ell (\overline{v}_\ell)^{q_1} d\sigma]$ for all $s, \tau \ge 0$, we have $g(0,\tau) = -\tau^{q_1} > -1$ for all $\tau \in]0,1[$ and since (3.33) $\lim_{s \to +\infty} g(s,\tau) = -\infty$ for all $\tau \ge 0$. Then for all $\tau \in]0,1[$, it is possible to choose $s(\tau) > 0$ such that $g(s(\tau),\tau) = -1$. Let us add that there exist $s_0 > 0$ and $\tau_0 \in]0,1[$ such that $(\partial g/\partial s)(s,\tau) > 0$ for all $(s,\tau) \in]0, s_0[\mathrm{x}]\tau_0, 1[$.

Let now $\{\tau_n\} \subseteq]\tau_0, 1[$ and $\lim \tau_n = 1$. Since $g(s(\tau_n), \tau_n) = -1$, $\{s(\tau_n)\}$ is necessarily bounded. Then (within a subsequence) $\lim s(\tau_n) = \omega \geq s_0$. Consequently, from the inequality:

$$H_{\lambda\mu}(\overline{v}) \leq H_{\lambda\mu}(v(\tau_n)), \quad \text{where } v(\tau_n) = (s(\tau_n)c_1, \tau_n\overline{v}_2, \dots, \tau_n\overline{v}_n) \in V_{\lambda\mu}^- \cap S(D_1), \tag{3.36}$$

as $n \to +\infty$ and from (3.34), we get the contradiction:

$$H_{\lambda\mu}(\overline{v}) \leq -p^{-1}\omega^p c_1^p \left(\lambda_1 \int_\Omega b_1 dx + \mu_1 \int_{\partial\Omega} \widehat{b}_1 d\sigma\right) + H_{\lambda\mu}(\overline{v}) < H_{\lambda\mu}(\overline{v}). \tag{3.37}$$

□

Remark 3.11. Let us note that the conditions (3.20), (3.21), and (3.33) are compatible.

Application 3.12. Let for each $v = (v_1, \dots, v_n) \in W$:

$$\begin{aligned} A(v) &= p^{-1}\left[\sum_{\ell=1}^{n-1}\int_\Omega |\nabla v_\ell|^p dx + \int_\Omega \left(|\nabla v_n|^\gamma + \int_{\partial\Omega} |v_n|^\gamma d\sigma\right)^{p/\gamma} dx\right], \\ D_1(v) &= q_1^{-1}\left[\sum_{\ell=1}^{n-1}\int_\Omega \rho_\ell |v_\ell + v_n|^{q_1-1}(v_\ell + v_n)dx - \sum_{\ell=1}^{n}\int_{\partial\Omega} \widehat{d}_\ell |v_\ell|^{q_1} d\sigma\right], \end{aligned} \tag{3.38}$$

where

$$1 < \gamma < p, \quad 1 < q_1 < \widehat{p}, \quad q_1 \neq p, \quad \rho_\ell \in L^\infty(\Omega), \quad \rho_\ell > 0, \quad \widehat{d}_\ell \in L^\infty(\partial\Omega), \quad \widehat{d}_\ell > 0. \tag{3.39}$$

Let us consider the system:

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2}u_i + \rho_i |u_i + u_n|^{q_1-1} \quad \text{in } \Omega \text{ as } i = 1, \dots, n-1, \\ &-\operatorname{div}\left[\left(|\nabla u_n|^\gamma + \int_{\partial\Omega}|u_n|^\gamma d\sigma\right)^{(p/\gamma)-1}|\nabla u_n|^{\gamma-2}\nabla u_n\right] = \lambda_n b_n |u_n|^{p-2}u_n + \sum_{\ell=1}^{n-1}\rho_\ell |u_\ell + u_n|^{q_1-1} \quad \text{in } \Omega, \\ &|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} = \mu_i \widehat{b}_i |u_i|^{p-2}u_i - \widehat{d}_i |u_i|^{q_1-2}u_i \quad \text{on } \partial\Omega \text{ as } i = 1, \dots, n-1, \\ &\left(|\nabla u_n|^\gamma + \int_{\partial\Omega}|u_n|^\gamma d\sigma\right)^{(p/\gamma)-1}|\nabla u_n|^{\gamma-2}\frac{\partial u_n}{\partial \nu} \\ &\qquad = \mu_n \widehat{b}_n |u_n|^{p-2}u_n - \left[\int_\Omega \left(|\nabla u_n|^\gamma + \int_{\partial\Omega}|u_n|^\gamma d\sigma\right)^{(p/\gamma)-1} dx\right]|u_n|^{\gamma-2}u_n \\ &\qquad\quad - \widehat{d}_n |u_n|^{q_1-2}u_n \quad \text{on } \partial\Omega. \end{aligned} \tag{3.40}$$

Pointing out that $V^+(D_1) \neq \emptyset$, we advance the conditions

$$\int_\Omega \left(\sum_{\ell=1}^{n-1} \rho_\ell \right) dx < \int_{\partial\Omega} \hat{d}\, d\sigma \quad \left(\hat{d} = \min\left\{ \hat{d}_1, \ldots, \hat{d}_n \right\} \right), \tag{3.41}$$

$$\int_\Omega b_\ell dx > 0, \quad \int_{\partial\Omega} \hat{b}_\ell d\sigma > 0 \quad \text{as } \ell = 1, \ldots, n-1, \tag{3.42}$$

$$b_n \geq 0, \qquad \hat{b}_n \geq 0. \tag{3.43}$$

Taking into account that

$$(3.41) \Longrightarrow D_1(c_1, \ldots, c_{n-1}, 0) < 0 \quad \forall (c_1, \ldots, c_{n-1}) \in R^{n-1} \setminus \{0\}, \tag{3.44}$$

we have (Propositions 3.2 and 3.4)

$$\begin{aligned} &(3.41) \text{ and } (3.43) \\ &\quad \Longrightarrow \left(\text{with } \lambda_n, \mu_n \leq 0,\ \lambda_n + \mu_n < 0\ \exists \delta_1^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, \left|\mu_\ell\right| \leq \delta_1^* \text{ as } \ell = 1, \ldots, n-1 \right), \end{aligned} \tag{3.45}$$

$$\begin{aligned} &(3.41)\text{–}(3.43) \\ &\quad \Longrightarrow \left(\text{with } \lambda_n, \mu_n \leq 0,\ \lambda_n + \mu_n < 0\ \exists \delta_2^* > 0 : (i_{15}) \text{ holds} \right. \\ &\qquad \left. \text{if } \lambda_\ell, \mu_\ell \in \left[0, \delta_2^*\right] \text{ as } \ell = 1, \ldots, n-1 \text{ and } \lambda_\ell + \mu_\ell > 0 \text{ for some } \ell \right). \end{aligned} \tag{3.46}$$

Proposition 3.13 (see ([1], Theorems 2.1 and 4.1; Remark 2.1); Proposition A.4; [5, 6]). *Under assumption* (3.39)*, we have*

(i) *When* (3.41) *and* (3.43) *hold, with* λ_ℓ, μ_ℓ *as in* (3.45) *system* (3.40) *has at least one weak solution* u^0 ($u^0 = \tau^0 v^0$, $\tau^0 = \text{const.} > 0$, $v^0 \in S_{\lambda\mu} \cap V^+(D_1)$)*, and it results in*

$$\begin{aligned} u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega), \quad u_i^0 > 0 \text{ as } i = 1, \ldots, n-1, \\ u_n^0 \in L^\infty(\Omega), \quad u_n^0 \geq 0,\ u_n^0 \not\equiv 0; \end{aligned} \tag{3.47}$$

(ii) *When* (3.41)–(3.43) *hold, with* λ_ℓ, μ_ℓ *as in* (3.46) *system* (3.40) *has at least one weak solution* $\overline{u}$ ($\overline{u} = \overline{\tau}\,\overline{v}$, $\overline{\tau} = \text{const.} > 0$, $\overline{v} \in V^-_{\lambda\mu} \cap S(D_1)$)*, and it results in* $\overline{u}_i \not\equiv 0$ *as* $i = 1, \ldots, n$.

Consequently, when (3.41)–(3.43) *hold, with* λ_ℓ, μ_ℓ *as in* (3.46) *and* $\min\{\delta_1^*, \delta_2^*\}$ *instead of* δ_2^* *system* (3.40) *has at least two different weak solutions.*

About the properties of u_i^0 and $\overline{u}_i$ expressed by Proposition 3.13, it is necessary to remark that if $u = (u_1, \ldots, u_n)$ is a nontrivial weak solution to system (3.40), then $u_i \not\equiv 0$ as $i = 1, \ldots, n$. In fact,

$$u_n \equiv 0 \Longrightarrow u_i \equiv 0 \quad \text{as } i = 1, \ldots, n-1, \qquad u_i \equiv 0 \quad \text{for some } i \in \{1, \ldots, n-1\} \Longrightarrow u_n \equiv 0. \tag{3.48}$$

Application 3.14. Let $n = 2$ and for any $v = (v_1, v_2) \in W$:

$$A(v) = p^{-1}\sum_{\ell=1}^{2}\int_{\Omega}|\nabla v_\ell|^p dx, \qquad D_j(v) = q_j^{-1}\int_{\Omega}\rho_j\left|\sum_{\ell=1}^{2}d_{j\ell}|v_\ell|^{\gamma_j}\right|^{q_j/\gamma_j}dx \quad \text{as } j = 1,\dots,m-1,$$
$$D_m(v) = q_m^{-1}\int_{\partial\Omega}\rho_m\left(\sum_{\ell=1}^{2}|v_\ell|^{\gamma_m}\right)^{q_m/\gamma_m}d\sigma, \tag{3.49}$$

where

$$1 < \gamma_j < q_j \quad \text{as } j = 1,\dots,m, \qquad p < q_1 < \cdots < q_m < \hat{p}, \qquad \rho_j \in L^\infty(\Omega), \quad \rho_j < 0,$$
$$d_{j\ell} \in L^\infty(\Omega)\setminus\{0\}, \qquad \rho_m \in L^\infty(\partial\Omega). \tag{3.50}$$

Let us consider the system:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i|u_i|^{p-2}u_i$$
$$+\sum_{j=1}^{m-1}\rho_j\left|\sum_{\ell=1}^{2}d_{j\ell}|u_\ell|^{\gamma_j}\right|^{(q_j/\gamma_j)-2}\left(\sum_{\ell=1}^{2}d_{j\ell}|u_\ell|^{\gamma_j}\right)d_{ji}|u_i|^{\gamma_j-2}u_i \quad \text{in } \Omega,$$
$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} = \mu_i\hat{b}_i|u_i|^{p-2}u_i + \rho_m\left(\sum_{\ell=1}^{2}|u_\ell|^{\gamma_m}\right)^{(q_m/\gamma_m)-1}|u_i|^{\gamma_m-2}u_i \quad \text{on } \partial\Omega \text{ as } i = 1,2. \tag{3.51}$$

Let us introduce the conditions:

$$\rho_m^+ \not\equiv 0 \quad (\Longrightarrow V^+(D_m) \neq \emptyset \ (\text{Proposition A.2})), \tag{3.52}$$

$$\int_{\partial\Omega}\rho_m d\sigma < 0 \quad \left(\Longrightarrow D_m(c_1,c_2) < 0 \ \forall (c_1,c_2) \in R^2\setminus\{0\}\right), \tag{3.53}$$

$$\int_{\Omega}b_\ell dx > 0, \quad \int_{\partial\Omega}\hat{b}_\ell d\sigma > 0 \quad \text{as } \ell = 1,2, \tag{3.54}$$

we have (Propositions 3.3 and 3.5)

$$(3.52) \text{ and } (3.53) \Longrightarrow \left(\exists \delta_1^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, \left|\mu_\ell\right| \le \delta_1^* \text{ as } \ell = 1,2\right), \tag{3.55}$$

$$(3.53) \text{ and } (3.54)$$
$$\Longrightarrow \left(\exists \delta_2^* > 0 : (i_{15}) \text{ holds if } \lambda_\ell, \mu_\ell \in \left[0,\ \delta_2^*\right] \text{ as } \ell = 1,2,\ \lambda_\ell + \mu_\ell > 0 \text{ for some } \ell\right). \tag{3.56}$$

Proposition 3.15 (see ([1], Theorems 2.2 and 4.2; Remarks 2.3 and 4.4); Proposition A.4; [5]). *Under assumption* (3.50), *we have*

(i) *When* (3.52) *and* (3.53) *hold, with* λ_ℓ, μ_ℓ *as in* (3.55) *system* (3.51) *has at least two weak solutions* u^0 *and* $-u^0$ $(u^0 = \tau^0 v^0,\ \tau^0 = const. > 0,\ v^0 \in\ S_{\lambda\mu} \cap V^+(D_m))$, *and it results in*

$$u_i^0 \in L^\infty(\Omega) \cap C^{1,\alpha_i^0}_{\ell oc}(\Omega), \qquad u_i^0 \geq 0 \quad as\ i = 1,2; \tag{3.57}$$

(ii) *When* (3.53) *and* (3.54) *hold, with* λ_ℓ, μ_ℓ *as in* (3.56) *system* (3.51) *has at least two weak solutions* $\overline{u}$ *and* $-\overline{u}$ $(\overline{u} = \overline{\tau}\,\overline{v},\ \overline{\tau} = const. > 0,\ \overline{v} \in\ V^-_{\lambda\mu} \cap S(D_m))$, *and it results in*

$$\overline{u}_i \in L^\infty(\Omega) \cap C^{1,\overline{\alpha}_i}_{\ell oc}(\Omega), \quad \overline{u}_i \geq 0 \quad as\ i = 1,2. \tag{3.58}$$

Consequently, when (3.52)–(3.54) *hold, with* λ_ℓ, μ_ℓ *as in* (3.56), *and* $\min\{\delta_1^*, \delta_2^*\}$ *instead of* δ_2^* *system* (3.51) *has at least four different weak solutions.*

Proposition 3.16. *Under the assumption* $p \leq 2\gamma_j$ *and* $d_{j1} \cdot d_{j2} < 0$ *as* $j = 1, \ldots, m-1$, *we have*

(i) *if* $\gamma_{j_0} < p$ *for some* $j_0 \in \{1, \ldots, m\}$, *then* $u_i^0 > 0$ *as* $i = 1,2$;

(ii) *if* $\gamma_{j_0} < \gamma_m \leq p$ *for some* $j_0 \in \{1, \ldots, m-1\}$, *then* $\overline{u}_i > 0$ *as* $i = 1,2$.

Proof. First of all u_i^0 is a weak supersolution to the equation:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) = \lambda_i b_i |u_i|^{p-2}u_i + \sum_{j=1}^{m-1}\rho_j\left|d_{j1}\left(u_1^0\right)^{\gamma_j} + d_{j2}\left(u_2^0\right)^{\gamma_j}\right|^{(q_j/\gamma_j)-2} d_{ji}^2 |u_i|^{2\gamma_j - 2}u_i \quad \text{in } \Omega. \tag{3.59}$$

Also, $\overline{u}_i$ has a similar property. Then [6] it is sufficient to verify that

$$u_i^0 \not\equiv 0, \tag{3.60}$$

$$\overline{u}_i \not\equiv 0. \tag{3.61}$$

About (3.60), let us prove (Remark 1.1) that

$$\left(i_{16}^h\right) \text{ holds as } h = 1,2 \quad \text{with } \mathfrak{F} = S_{\lambda\mu} \cap V^+(D_m). \tag{3.62}$$

Let $v = (v_1, v_2) \in V^+(D_m) \cap S_{\lambda\mu}$. Let, for example, $v_1 \equiv 0$. Let

$$\begin{gathered} \mathbb{K} \subseteq \Omega \text{ a compact set} : |\mathbb{K}|_N > 0,\ v_2 \not\equiv 0 \text{ in } \mathbb{K}, \\ \Omega' \text{ an open set} : \mathbb{K} \subseteq \Omega', \quad \overline{\Omega'} \subseteq \Omega, \\ \Gamma \subseteq \partial\Omega : \sigma(\Gamma) > 0, \quad \rho_m |v_2| > 0 \quad \text{on } \Gamma. \end{gathered} \tag{3.63}$$

Since Propositions A.1 and A.2, there exists $\varphi \in C_0^\infty(R^N)$, with $0 \le \varphi \le 1$ and supp $\varphi \subseteq \Omega' \cup (R^N \setminus \overline{\Omega'})$, such that

$$\int_\Omega \rho_j \left| d_{j2} |v_2|^{\gamma_j} \right|^{(q_j/\gamma_j)-2} |v_2|^{\gamma_j} \varphi^{\gamma_j} d_{j1} d_{j2}\, dx > 0 \quad \text{as } j = 1, \dots, m-1, \qquad \int_{\partial\Omega} \rho_m |v_2|^{q_m - \gamma_m} \varphi^{\gamma_m} d\sigma > 0,$$
$$\delta = p^{-1} \left[\int_\Omega |\nabla \varphi|^p dx - \lambda_1 \int_\Omega b_1 \varphi^p dx - \mu_1 \int_{\partial\Omega} \hat{b}_1 \varphi^p d\sigma \right] > 0. \tag{3.64}$$

Then with $v(s) = ((1-s)^{1/p} \delta^{-1/p} \varphi,\ s^{1/p} v_2)$, we have

$$H_{\lambda\mu}(v(s)) = 1 \quad \forall s \in [0,1], \quad \exists s_0 \in [0,1[: D_m(v(s)) > 0 \quad \forall s \in [s_0, 1],$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_{j_0}(v(s)) = -\infty, \quad \lim_{s \to 1^-} \frac{d}{ds} D_j(v(s)) < +\infty \quad \text{as } j \ne j_0. \tag{3.65}$$

Passing to (3.61), let us introduce the function $\psi(t,v) = pt^{p-1} H_{\lambda\mu}(v) - \sum_{j=1}^m q_j t^{q_j - 1} D_j(v)$, and let us remember that ([1], Theorem 4.2)

$$\forall v \in V_{\lambda\mu}^- \cap S(D_m) \exists \mid t(v) > 0 : \psi(t(v), v) = 0,$$
$$\tilde{\tilde{E}}(\overline{v}) = \inf\left\{ \tilde{\tilde{E}}(v) : v \in V_{\lambda\mu}^- \cap S(D_m) \right\}, \tag{3.66}$$

where $\tilde{\tilde{E}}(v) = (t(v))^p H_{\lambda\mu}(v) - \sum_{j=1}^m (t(v))^{q_j} D_j(v)$.

Reasoning by contradiction, let us set, for example, $\overline{v}_1 \equiv 0$ and set $v(s) = ((1-s)^{1/\gamma_m} \overline{v}_2,\ s^{1/\gamma_m} \overline{v}_2)$. Since

$$D_m(v(s)) = -1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[:\ H_{\lambda\mu}(v(s)) < 0 \quad \forall s \in [s_0, 1],$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_{j_0}(v(s)) = -\infty, \quad \lim_{s \to 1^-} \frac{d}{ds} D_j(v(s)) < +\infty \quad \text{as } j \in \{1, \dots, m-1\} \setminus \{j_0\}, \tag{3.67}$$

as in Proposition 2.24, we get the contradiction:

$$\tilde{\tilde{E}}(\overline{v}) \le \tilde{\tilde{E}}(v(s)) < \tilde{\tilde{E}}(\overline{v}) \quad \forall s \in [s_1, 1[\ (s_0 \le s_1 < 1). \tag{3.68}$$

□

Application 3.17. Let $n > 2$ and set for each $v = (v_1, \dots, v_n) \in W$:

$$A(v) = p^{-1} \sum_{\ell=1}^n \int_\Omega |\nabla v_\ell|^p dx,$$
$$D_j(v) = q_j^{-1} \sum_{\substack{\ell=1 \\ \ell \ne j}}^n \int_\Omega \rho_j \left| d_{jj} |v_j|^{\gamma_j} + d_{j\ell} |v_\ell|^{\gamma_j} \right|^{q_j/\gamma_j} dx \quad \text{as } j = 1, \dots, n, \tag{3.69}$$
$$D_{n+1}(v) = q_{n+1}^{-1} \int_{\partial\Omega} \rho_{n+1} \left(\sum_{\ell=1}^n |v_\ell|^{\gamma_{n+1}} \right)^{q_{n+1}/\gamma_{n+1}} d\sigma,$$

where

$$1<\gamma_j<q_j \quad \text{as } j=1,\dots,n+1, \qquad p<q_1<\cdots<q_{n+1}<\widehat{p}, \qquad \rho_j \in L^{\infty}(\Omega), \quad \rho_j<0,$$
$$d_{j\ell} \in L^{\infty}(\Omega)\setminus\{0\}, \qquad \rho_{n+1}\in L^{\infty}(\partial\Omega). \tag{3.70}$$

Let us consider the system:

$$-\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right)=\lambda_i b_i|u_i|^{p-2}u_i+\sum_{\ell\neq i}\rho_i\left|d_{ii}|u_i|^{\gamma_i}+d_{i\ell}|u_\ell|^{\gamma_i}\right|^{(q_i/\gamma_i)-2}\left(d_{ii}|u_i|^{\gamma_i}+d_{i\ell}|u_\ell|^{\gamma_i}\right)d_{ii}|u_i|^{\gamma_i-2}u_i$$
$$+\sum_{j\neq i}\rho_j\left|d_{jj}\left|u_j\right|^{\gamma_j}+d_{ji}|u_i|^{\gamma_j}\right|^{(q_j/\gamma_j)-2}\left(d_{jj}\left|u_j\right|^{\gamma_j}+d_{ji}|u_i|^{\gamma_j}\right)d_{ji}|u_i|^{\gamma_j-2}u_i \quad \text{in } \Omega,$$
$$|\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu}=\mu_i\widehat{b}_i|u_i|^{p-2}u_i+\rho_{n+1}\left(\sum_{\ell=1}^{n}|u_\ell|^{\gamma_{n+1}}\right)^{(q_{n+1}/\gamma_{n+1})-1}|u_i|^{\gamma_{n+1}-2}u_i \quad \text{on } \partial\Omega \text{ as } i=1,\dots,n. \tag{3.71}$$

Let us make the assumptions:

$$\rho_{n+1}^{+}\not\equiv 0, \qquad \int_{\partial\Omega}\rho_{n+1}d\sigma<0, \qquad \int_{\Omega}b_\ell dx>0, \quad \int_{\partial\Omega}\widehat{b}_\ell d\sigma>0 \quad \text{as } \ell=1,\dots,n. \tag{3.72}$$

About Neumann's problem (3.71), we have an existence result similar to the one of Proposition 3.15 related to system (3.51). About the positive sign of the components of the weak solutions u^0 and $\overline{u}$ to system (3.71), as in Proposition 3.16, we show.

Proposition 3.18. *Under the assumption $p\leq 2\gamma_j$ as $j=1,\dots,n$ and $d_{jj}\cdot d_{j\ell}<0$ as $j,\ell\in\{1,\dots,n\}$ with $\ell\neq j$, we have*

(i) *if either $\gamma_{n+1}<p$ or $\gamma_j<p$ for all $j\in\{1,\dots,n\}\setminus\{j_0\}$ for some j_0, then $u_i^0>0$ as $i=1,\dots,n$;*

(ii) *if $\gamma_j<\gamma_{n+1}\leq p$ for all $j\in\{1,\dots,n\}\setminus\{j_0\}$ for some j_0, then $\overline{u}_i>0$ as $i=1,\dots,n$.*

The following remark deals also with Application 3.14.

Remark 3.19. Making in (3.50) (resp. (3.70)) the change

$$q_1<\cdots<q_m<p \quad \left[\text{resp. } q_1<\cdots<q_{n+1}<p\right], \tag{3.73}$$

system (3.51) (resp. (3.71)) has at least the two weak solutions $\overline{u}$ and $-\overline{u}$ ([1], Theorem 4.2; Remark 4.4). The components of $\overline{u}$ keep the properties that Propositions 3.15 and 3.16 (Proposition 3.15 and Proposition 3.18 resp.) underline.

Application 3.20. Let for each $v = (v_1, \dots, v_n) \in W$:

$$\begin{aligned} A(v) &= p^{-1}\sum_{\ell=1}^{n}\int_{\Omega}|\nabla v_\ell|^p dx, \\ D_j(v) &= q_j^{-1}\int_{\partial\Omega}\rho_j\left|d_{j1}\left|v_j\right|^{\gamma_j} + d_{j2}|v_n|^{\gamma_j}\right|^{q_j/\gamma_j}d\sigma \quad \text{as } j = 1, \dots, n-1, \\ D_n(v) &= \left(\int_{\partial\Omega}|v_n|^{\widehat{\gamma}_n}d\sigma\right)\left(\int_{\Omega}\rho_n|v_n|^{\gamma_n}dx\right), \end{aligned} \tag{3.74}$$

where

$$\begin{gathered} 1 < \gamma_j < q_j < \widehat{p} \quad \text{as } j = 1, \dots, n-1, \qquad 1 < \gamma_n < \widetilde{p}, \qquad 1 < \widehat{\gamma}_n < \widehat{p}, \\ p < q_1 < \cdots < q_{n-1} < q_n = \gamma_n + \widehat{\gamma}_n, \\ \rho_j \in L^\infty(\partial\Omega), \quad \rho_j < 0, \qquad d_{j1}, d_{j2} \in L^\infty(\partial\Omega) \setminus \{0\}, \qquad \rho_n \in L^\infty(\Omega). \end{gathered} \tag{3.75}$$

Let us consider the system:

$$\begin{aligned} -\operatorname{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) &= \lambda_i b_i |u_i|^{p-2} \quad \text{in } \Omega \text{ as } i = 1, \dots, n-1, \\ -\operatorname{div}\left(|\nabla u_n|^{p-2}\nabla u_n\right) &= \lambda_n b_n |u_n|^{p-2}u_n + \gamma_n\left(\int_{\partial\Omega}|u_n|^{\widehat{\gamma}_n}d\sigma\right)\rho_n|u_n|^{\gamma_n-2}u_n \quad \text{in } \Omega, \\ |\nabla u_i|^{p-2}\frac{\partial u_i}{\partial \nu} &= \mu_i\widehat{b}_i|u_i|^{p-2}u_i + \rho_i\left|d_{i1}|u_i|^{\gamma_i} + d_{i2}|u_n|^{\gamma_i}\right|^{(q_i/\gamma_i)-2} \\ &\quad \times \left(d_{i1}|u_i|^{\gamma_i} + d_{i2}|u_n|^{\gamma_i}\right)d_{i1}|u_i|^{\gamma_i-2}u_i \quad \text{on } \partial\Omega \text{ as } i = 1, \dots, n-1, \\ |\nabla u_n|^{p-2}\frac{\partial u_n}{\partial \nu} &= \mu_n\widehat{b}_n|u_n|^{p-2}u_n + \sum_{j=1}^{n-1}\rho_j\left|d_{j1}\left|u_j\right|^{\gamma_j} + d_{j2}|u_n|^{\gamma_j}\right|^{(q_j/\gamma_j)-2} \\ &\quad \times \left(d_{j1}\left|u_j\right|^{\gamma_j} + d_{j2}|u_n|^{\gamma_j}\right)d_{j2}|u_n|^{\gamma_j-2}u_n \\ &\quad + \widehat{\gamma}_n\left(\int_{\Omega}\rho_n|u_n|^{\gamma_n}dx\right)|u_n|^{\widehat{\gamma}_n-2}u_n \quad \text{on } \partial\Omega. \end{aligned} \tag{3.76}$$

Let us introduce the conditions:

$$\rho_n^+ \not\equiv 0 \quad \left(\Longrightarrow V^+(D_n) \neq \emptyset \ \left(\text{Propositions A.1 and A.2}\right)\right), \tag{3.77}$$

$$\int_{\Omega}\rho_n dx < 0 \quad \left(\Longrightarrow D_n(0, \dots, 0, c_n) < 0 \ \forall c_n \in R^n \setminus \{0\}\right), \tag{3.78}$$

$$b_\ell \geq 0, \quad \widehat{b}_\ell \geq 0 \quad \text{as } \ell = 1, \dots, n-1, \tag{3.79}$$

$$\int_{\Omega}b_n dx > 0, \qquad \int_{\partial\Omega}\widehat{b}_n d\sigma > 0. \tag{3.80}$$

We have (Propositions 3.2 and 3.4)

$$
\begin{aligned}
&(3.77)\text{–}(3.79)\\
&\quad\Longrightarrow \big(\text{with } \lambda_\ell,\mu_\ell \le 0,\ \lambda_\ell+\mu_\ell<0 \textit{ as } \ell=1,\dots,n-1\ \exists\delta_1^*>0:\\
&\qquad (i_{14}) \text{ holds if } |\lambda_n|, \left|\mu_n\right| \le \delta_1^*\big),
\end{aligned}
\tag{3.81}
$$

$$
\begin{aligned}
&(3.78)\text{–}(3.80)\\
&\quad\Longrightarrow \big(\text{with } \lambda_\ell,\mu_\ell \le 0,\ \lambda_\ell+\mu_\ell<0 \text{ as } \ell=1,\dots,n-1\\
&\qquad \exists\delta_2^*>0:(i_{15}) \text{ holds if } \lambda_n,\mu_n\in\left[0,\delta_2^*\right] \text{ and } \lambda_n+\mu_n>0\big).
\end{aligned}
\tag{3.82}
$$

Proposition 3.21 (see ([1], Theorems 2.2 and 4.2; Remarks 2.3 and 4.4); Proposition A.4; [5, 6]). *Under assumption* (3.75)*, we have*

(i) *When* (3.77)–(3.79) *hold, with λ_ℓ,μ_ℓ as in* (3.81)*, system* (3.76) *has at least two weak solutions u^0 and $-u^0$ ($u^0=\tau^0 v^0$, $\tau^0=$ const. >0, $v^0\in S_{\lambda\mu}\cap V^+(D_n)$), and it results in $u_i^0\ge 0$ $(i=1,\dots,n)$, $u_n^0\not\equiv 0$. If $\gamma_n<\widehat{p}$, then*

$$
\begin{aligned}
&u_i^0\in L^\infty(\Omega)\cap C_{\ell oc}^{1,\alpha_i^0}(\Omega)\quad (i=1,\dots,n),\\
&u_i^0\not\equiv 0\Longrightarrow u_i^0>0\quad (i=1,\dots,n-1);
\end{aligned}
\tag{3.83}
$$

(ii) *When* (3.78)–(3.80) *hold, with λ_ℓ,μ_ℓ as in* (3.82)*, system* (3.76) *has at least two weak solutions $\overline{u}$ and $-\overline{u}$ ($\overline{u}=\overline{\tau}\overline{v}$, $\overline{\tau}=$ const. >0, $\overline{v}\in V_{\lambda\mu}^-\cap S(D_n)$), and it results in $\overline{u}_i\ge 0$ $(i=1,\dots,n)$, $\overline{u}_n\not\equiv 0$. If $\gamma_n<\widehat{p}$, then*

$$
\begin{aligned}
&\overline{u}_i\in L^\infty(\Omega)\cap C_{\ell oc}^{1,\overline{\alpha}_i}(\Omega)\quad (i=1,\dots,n),\\
&\overline{u}_i\not\equiv 0\Rightarrow \overline{u}_i>0\quad (i=1,\dots,n-1).
\end{aligned}
\tag{3.84}
$$

Consequently, when (3.77)–(3.80) *hold, with λ_ℓ,μ_ℓ as in* (3.82)*, and* $\min\{\delta_1^*,\delta_2^*\}$ *instead of δ_2^* system* (3.76) *has at least four different weak solutions. Obviously, $u_n^0>0$ and $\overline{u}_n>0$ if $p\le\gamma_n<\widehat{p}$.*

The following proposition gives a sufficient condition to

$$u_i^0>0\quad \text{as } i=1,\dots,n-1, \tag{3.85}$$

$$\overline{u}_i>0\quad \text{as } i=1,\dots,n-1. \tag{3.86}$$

Proposition 3.22. *Let $\gamma_n<\widehat{p}$. If $\gamma_j<p$ and $d_{j1}\cdot d_{j2}<0$ as $j=1,\dots,n-1$, then* (3.85) *and* (3.86) *hold.*

Proof. Since

$$(v_1,\dots,v_n)\in V^+(D_n)\Longrightarrow (\exists\Gamma\subseteq\partial\Omega:\sigma(\Gamma)>0,\ |v_n|>0\ \text{ on }\ \Gamma), \tag{3.87}$$

using Propositions A.1 and A.2, we can verify that

$$\left(i_{16}^{h}\right) \text{ holds as } h = 1,\dots,n-1 \quad \text{with } \mathfrak{F} = S_{\lambda\mu} \cap V^{+}(D_n), \tag{3.88}$$

from which (Remark 1.1) we get (3.85).

Let us prove (3.86). Reasoning by contradiction, let us set, for example, $\overline{v}_1 \equiv 0$. If $v(s) = ((1-s)^{1/p}\overline{v}_n, \overline{v}_2, \dots, \overline{v}_n)$, we have

$$D_n(v(s)) = -1 \quad \forall s \in [0,1], \qquad \exists s_0 \in [0,1[\, : H_{\lambda\mu}(v(s)) < 0 \quad \forall s \in [s_0, 1],$$
$$\lim_{s\to 1^-} \frac{d}{ds} D_1(v(s)) = -\infty. \tag{3.89}$$

Then as in Proposition 3.16, we get a contradiction. □

Remark 3.23. Making in (3.75) the change:

$$1 < \gamma_j < q_j \quad \text{as } j = 1,\dots,n-1, \qquad 1 < \gamma_n, \qquad 1 < \widehat{\gamma}_n, \qquad q_1 < \cdots < q_n = \gamma_n + \widehat{\gamma}_n < p, \tag{3.90}$$

system (3.76) has at least the two weak solutions $\overline{u}$ and $-\overline{u}$ ([1], Theorem 4.2; Remark 4.4). The components of $\overline{u}$, all bounded, are locally Hölderian with their first derivatives. If $d_{j1} \cdot d_{j2} < 0$ as $j = 1,\dots,n-1$, then (3.86) holds.

Application 3.24. Let for each $v = (v_1,\dots,v_n) \in W$:

$$A(v) = p^{-1}\sum_{\ell=1}^{n}\int_\Omega \left(|\nabla v_\ell|^\gamma + \int_\Omega |v_\ell|^\gamma dx\right)^{p/\gamma} dx,$$
$$D_j(v) = q_j^{-1}\int_{\partial\Omega}\left(\sum_{\ell=1}^{n}|v_\ell|^\gamma\right)^{q_j/\gamma} d\sigma \quad \text{as } j = 1,\dots,m-1, \quad D_m(v) = q_m^{-1}\int_\Omega\left(\sum_{\ell=1}^{n}|v_\ell|^\gamma\right)^{q_m/\gamma} dx, \tag{3.91}$$

where

$$1 < \gamma < q_1 < \cdots < q_m < p. \tag{3.92}$$

Let us consider the system:

$$
\begin{aligned}
&-\operatorname{div}\left[\left(|\nabla u_i|^\gamma+\int_\Omega |u_i|^\gamma dx\right)^{(p/\gamma)-1}|\nabla u_i|^{\gamma-2}\nabla u_i\right]\\
&\quad=\lambda_i b_i|u_i|^{p-2}u_i-\left(\int_\Omega\left(|\nabla u_i|^\gamma+\int_\Omega|u_i|^\gamma dx\right)^{(p/\gamma)-1}dx\right)|u_i|^{\gamma-2}u_i\\
&\qquad+\left(\sum_{\ell=1}^n|u_\ell|^\gamma\right)^{(q_m/\gamma)-1}|u_i|^{\gamma-2}u_i \quad\text{in } \Omega,\\
&\left(|\nabla u_i|^\gamma+\int_\Omega|u_i|^\gamma dx\right)^{(p/\gamma)-1}|\nabla u_i|^{\gamma-2}\frac{\partial u_i}{\partial \nu}\\
&\quad=\mu_i\widehat{b}_i|u_i|^{p-2}u_i+\sum_{j=1}^{m-1}\left(\sum_{\ell=1}^n|u_\ell|^\gamma\right)^{(q_j/\gamma)-1}|u_i|^{\gamma-2}u_i \quad \text{on } \partial\Omega \text{ as } i=1,\dots n.
\end{aligned}
\tag{3.93}
$$

We advance the condition:

$$
b_\ell\geq 0,\quad \widehat{b}_\ell\geq 0\quad \text{as } \ell=1,\dots n,
\tag{3.94}
$$

and we note that (Proposition 3.1)

$$
(3.94)\Longrightarrow \big((i_{13})\text{ holds if } \lambda_\ell,\mu_\ell\leq 0,\ \lambda_\ell+\mu_\ell<0\quad \text{as } \ell=1,\dots,n\big).
\tag{3.95}
$$

Proposition 3.25. *Under conditions* (3.92) *and* (3.94)*, with* λ_ℓ,μ_ℓ *as in* (3.95)*, system* (3.93) *has at least two weak solutions* u^0 *and* $-u^0$ ($u^0=\tau^0 v^0$, $\tau^0=$ *const.* >0, $v^0\in S^+(D_1,\dots,D_m)$)*, and it results in*

$$
u_i^0\in L^\infty(\Omega),\ u_i^0\geq 0,\qquad u_i^0\not\equiv 0\quad \text{as } i=1,\dots,n.
\tag{3.96}
$$

Proof. We recall that ([1], Section 2), set $\psi(t,v)=pt^{p-1}H_{\lambda\mu}(v)-\sum_{j=1}^m q_j t^{q_j-1}D_j(v)$, we have

$$
\begin{gathered}
\forall v\in V^+(D_1,\dots,D_m)\qquad \exists\mid t(v)>0:\psi(t(v),v)=0,\\
\text{the functional } t(v) \text{ is } C^1 \text{ in } V^+(D_1,\dots,D_m).
\end{gathered}
\tag{3.97}
$$

We introduce the functional $\widetilde{\widetilde{E}}(v)=(t(v))^pH_{\lambda\mu}(v)-\sum_{j=1}^m(t(v))^{q_j}D_j(v)$ which is C^1 in $V^+(D_1,\dots,D_m)$. We still remember that ([1], Theorem 2.3; Remark 2.5)

$$
\begin{gathered}
\exists v^0\in S^+(D_1,\dots,D_m),\quad \text{with } v_i^0\geq 0 \text{ as } i=1,\dots,n, \text{ such that}\\
\widetilde{\widetilde{E}}\left(v^0\right)=\inf\left\{\widetilde{\widetilde{E}}(v):v\in S^+(D_1,\dots,D_m)\right\},\\
u^0=t\left(v^0\right)v^0 \text{ is a weak solution to system (3.93).}
\end{gathered}
\tag{3.98}
$$

The property $u_i^0 \in L^\infty(\Omega)$ is due to Proposition A.4. Let us verify that $u_i^0 \not\equiv 0$ as $i = 1,\ldots,n$. Reasoning by contradiction, let us set, for example, $v_1^0 \equiv 0$ and $v_2^0 \not\equiv 0$. As $v(s) = ((1-s)^{1/\gamma} v_2^0,\ s^{1/\gamma} v_2^0,\ v_3^0,\ldots,v_n^0)$, we have

$$\sum_{j=1}^{m} D_j(v(s)) = 1 \quad \forall s \in [0,1], \qquad \left[\frac{d}{ds} H_{\lambda\mu}(v(s))\right]_{s=1} > 0. \tag{3.99}$$

Then, since $(d/ds)\tilde{\tilde{E}}(v(s)) = (t(v(s)))^p (d/ds) H_{\lambda\mu}(v(s))$, there exists $s_0 \in [0,1[$ such that $(d/ds)\tilde{\tilde{E}}(v(s)) > 0$ for all $s \in [s_0,1]$, from which the contradiction:

$$\tilde{\tilde{E}}\left(v^0\right) \le \tilde{\tilde{E}}(v(s)) < \tilde{\tilde{E}}\left(v^0\right), \quad \forall s \in [s_0,1[. \tag{3.100}$$

□

Application 3.26. Let for each $v = (v_1,\ldots,v_n) \in W$:

$$\begin{gathered} A(v) = p^{-1} \sum_{\ell=1}^{n} \int_\Omega |\nabla v_\ell|^p dx, \qquad D_1(v) = q_1^{-1} \int_\Omega \rho \left| \sum_{\ell=1}^{n} d_\ell v_\ell \right|^{q_1} dx, \\ D_2(v) = \left(\int_\Omega \left[\sum_{\ell=1}^{n} \tilde{d}_\ell |v_\ell|^\gamma \right] dx \right) \left(\int_{\partial\Omega} \hat{\rho} \left[\sum_{\ell=1}^{n} \hat{d}_\ell |v_\ell|^{\hat{\gamma}} \right] d\sigma \right), \end{gathered} \tag{3.101}$$

where

$$\begin{gathered} 1 < \gamma < \tilde{p}, \quad 1 < \hat{\gamma} < \hat{p}, \quad 1 < q_1 < \min\{\tilde{p}, q_2 = \gamma + \hat{\gamma}\}, \\ p < q_2, \quad \rho, d_\ell \in L^\infty(\Omega) \setminus \{0\}, \quad \rho \le 0, \ \rho d_\ell \not\equiv 0 \\ \text{as some } \ell, \tilde{d}_\ell \in L^\infty(\Omega) \setminus \{0\}, \quad \tilde{d}_\ell \ge 0, \quad \hat{\rho} \in L^\infty(\Omega) \setminus \{0\}, \quad \hat{d}_\ell = \text{const.} > 0. \end{gathered} \tag{3.102}$$

Let as $\ell = 1,\ldots,n$ $F_\ell = f_\ell + \hat{f}_\ell$, where $f_\ell \in L^{p'}(\Omega)$ $(p' = p/(p-1))$ and $\hat{f}_\ell \in (W^{1-(1/p),p}(\partial\Omega))^*$ (dual space of $W^{1-(1/p),p}(\partial\Omega)$). Let $\langle\langle F, v\rangle\rangle = \sum_{\ell=1}^{n} \langle F_\ell, v_\ell\rangle$ for all $v = (v_1,\ldots,v_n) \in W$. Let us consider the system:

$$\begin{aligned} &-\operatorname{div}\left(|\nabla u_i|^{p-2} \nabla u_i\right) = \lambda_i b_i |u_i|^{p-2} u_i + \rho \left| \sum_{\ell=1}^{n} d_\ell u_\ell \right|^{q_1-2} \left(\sum_{\ell=1}^{n} d_\ell u_\ell \right) d_i \\ &\quad + \gamma \left(\int_{\partial\Omega} \hat{\rho} \left[\sum_{\ell=1}^{n} \hat{d}_\ell |u_\ell|^{\hat{\gamma}} \right] d\sigma \right) \tilde{d}_i |u_i|^{\gamma-2} u_i + f_i \quad \text{in } \Omega, \\ &|\nabla u_i|^{p-2} \frac{\partial u_i}{\partial \nu} \\ &= \mu_i \hat{b}_i |u_i|^{p-2} u_i + \hat{\gamma} \left(\int_\Omega \left[\sum_{\ell=1}^{n} \tilde{d}_\ell |u_\ell|^\gamma \right] dx \right) \hat{\rho} \hat{d}_i |u_i|^{\hat{\gamma}-2} u_i + \hat{f}_i \quad \text{on } \partial\Omega \text{ as } i = 1,\ldots,n. \end{aligned} \tag{3.103}$$

Let us introduce the conditions:

$$(\hat{\rho})^+ \not\equiv 0 \quad (\Longrightarrow V^+(D_2) \neq \emptyset \text{ (Proposition A.1 and A.2)}),$$
$$\int_{\partial\Omega} \hat{\rho}\, d\sigma < 0 \quad (\Longrightarrow D_2(c) < 0\ \forall c \in R^n \setminus \{0\}), \tag{3.104}$$

and let us note that (Proposition 3.3)

$$(3.104) \Longrightarrow (\exists \delta^* > 0 : (i_{14}) \text{ holds if } |\lambda_\ell|, |\mu_\ell| \leq \delta^* \text{ as } \ell = 1, \ldots, n). \tag{3.105}$$

Proposition 3.27. *Under assumptions* (3.102) *and* (3.104)*, if* $F \not\equiv 0$ *and* $\|F\|_*$ *is sufficiently small, then with* λ_ℓ, μ_ℓ *as in* (3.105)*, system* (3.103) *has at least one weak solution* $\tilde{u}$ ($\tilde{u} = \tilde{\tau}\tilde{v}$, $\tilde{\tau} = const. > 0, \tilde{v} \in S_{\lambda\mu} \cap V^+(D_2)$)*. When* $\gamma < p \leq q_1$*, it results in*

$$\tilde{u}_h \not\equiv 0 \text{ even if } F_h \equiv 0. \tag{3.106}$$

Proof. The existence of $\tilde{u}$ is due to ([1], Theorem 3.2). About (3.106), it is sufficiently (Remark 1.1) to verify that

$$\left(i_{16}^h\right) \text{ holds as } h = 1, \ldots, n \quad \text{with } \mathfrak{F} = S_{\lambda\mu} \cap V^+(D_2). \tag{3.107}$$

Let $v = (v_1, \ldots, v_n) \in V^+(D_2) \cap S_{\lambda\mu}$ with, for example, $v_1 \equiv 0$. Let $\psi = \sum_{\ell \neq 1} d_\ell v_\ell$. Let $\mathbb{K} \subseteq \Omega$ be a compact set having positive measure such that

$$\tilde{d}_1 > 0 \text{ in } \mathbb{K} \text{ if } \rho d_1 \psi \equiv 0, \quad \text{either } \rho d_1 \psi > 0 \text{ in } \mathbb{K} \text{ or } \rho d_1 \psi < 0 \text{ in } \mathbb{K} \text{ if } \rho d_1 \psi \not\equiv 0. \tag{3.108}$$

Proposition A.1 lets us choose $\varphi \in C_0^\infty(R^N)$ satisfying the following conditions:

$$\delta = p^{-1}\left[\int_\Omega |\nabla\varphi|^p dx - \lambda_1 \int_\Omega b_1 |\varphi|^p dx\right] > 0, \qquad \int_\Omega \tilde{d}_1 \varphi^\gamma dx > 0 \text{ if } \rho d_1 \psi \equiv 0,$$
$$\int_\Omega \rho d_1 |\psi|^{q_1 - 2} \psi\varphi\, dx > 0 \text{ if } \rho d_1 \psi \not\equiv 0. \tag{3.109}$$

Then with $v(s) = ((1-s)^{1/p} \delta^{-1/p} \varphi,\ s^{1/p} v_2, \ldots, s^{1/p} v_n)$, we have

$$H_{\lambda\mu}(v(s)) = 1 \quad \forall s \in [0,1], \qquad D_2(v(s)) > 0 \quad \forall s \in [s_0, 1]\ (0 \leq s_0 < 1),$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_1(v(s)) \in R, \quad \lim_{s \to 1^-} \frac{d}{ds} D_2(v(s)) = -\infty \quad \text{if } \rho d_1 \psi \equiv 0,$$
$$\lim_{s \to 1^-} \frac{d}{ds} D_1(v(s)) = -\infty, \quad \lim_{s \to 1^-} \frac{d}{ds} D_2(v(s)) \in R \quad \text{if } \rho d_1 \psi \not\equiv 0. \tag{3.110}$$

Now we replace conditions (3.104) with the following:

$$\hat{\rho} \geq 0, \qquad b_\ell \geq 0, \quad \hat{b}_\ell \geq 0 \quad \text{as } \ell = 1, \ldots, n. \tag{3.111}$$

□

Proposition 3.28. *Under assumptions* (3.102) *and* (3.111), *if* $F \not\equiv 0$ *and* $\|F\|_*$ *is sufficiently small, then with* $\lambda_\ell, \mu_\ell \leq 0$ *and* $\lambda_\ell + \mu_\ell < 0$ *as* $\ell = 1, \ldots, n$ *system* (3.103) *has at least two different weak solution* u^1 *and* u^2 $(u^i = \tau^i v^i,\ \tau^i = const. > 0, v^1 \in S_{\lambda\mu} \cap V^+(F), v^2 \in S_{\lambda\mu} \cap V^+(D_2))$. *When* $\gamma < p \leq q_1$, *it results in*

$$u_h^2 \not\equiv 0 \text{ even if } F_h \equiv 0. \tag{3.112}$$

Proof. The existence of u^1 and u^2 is due to ([1], Theorems 3.1, 3.2, and 3.3; Remark 3.1). Relation (3.112) is proved as in Proposition 3.27. □

Appendix

In this appendix, we present some results used previously. The first one is trivial. The second one is easy to prove. It is possible to show the third one and the fourth one with the technique developed by Drabek in ([7, Lemma 3.2]). The symbols σ, $\widehat{p}$, and $\widetilde{p}$ are the same introduced in Section 3.

Proposition A.1. *Let* Ω *be an open set of* R^N. *Let* $\mathbb{K} \subseteq \Omega$ *be a compact set with* $|\mathbb{K}|_N > 0$. *If* Ω' *is an open set such that* $\mathbb{K} \subseteq \Omega' \subseteq \Omega$, *then there exists a family of functions* $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(\Omega)$ *such that*

$$\begin{gathered} 0 \leq \varphi_\varepsilon \leq 1,\ \text{supp}\ \varphi_\varepsilon \subseteq \Omega', \quad \varphi_\varepsilon \longrightarrow \chi \text{ strongly in } L^s(\Omega), \\ \int_\Omega |\nabla \varphi_\varepsilon|^s dx \longrightarrow +\infty \quad \text{as } \varepsilon \longrightarrow 0^+\ \forall s \in [1, +\infty[, \end{gathered} \tag{A.1}$$

where χ *is the characteristic function of* $\mathbb{K}$.

Proposition A.2. *Let* $\Omega \subseteq R^N$ *be an open, bounded, connected and* $C^{0,1}$ *set. Let* $\mathcal{U}$ *be an open neighborhood of* $\partial\Omega$. *If* Γ *is a subset of* $\partial\Omega$ *with* $\sigma(\Gamma) > 0$, *then there exist a compact set* $\widehat{\Gamma} \subseteq \Gamma$ *with* $\sigma(\widehat{\Gamma}) > 0$ *and a family of functions* $(\varphi_\varepsilon)_{0<\varepsilon<\varepsilon_0} \subseteq C_0^\infty(R^N)$ *such that*

$$\begin{gathered} 0 \leq \varphi_\varepsilon \leq 1, \quad \text{supp}\ \varphi_\varepsilon \subseteq \mathcal{U}, \quad \varphi_\varepsilon \longrightarrow \widehat{\chi} \text{ strongly in } L^s(\partial\Omega), \\ \int_{R^N} \varphi_\varepsilon^s\, dx \longrightarrow 0 \quad \text{as } \varepsilon \longrightarrow 0^+\ \forall s \in [1, +\infty[, \end{gathered} \tag{A.2}$$

where $\widehat{\chi}$ *is the characteristic function of* $\widehat{\Gamma}$.

Let $\Omega \subseteq R^N$ be an open, bounded, connected and $C^{0,1}$ set. Let as $i = 1, \ldots, n$ $A_i(x, \xi, \eta^1, \ldots, \eta^n)$ be a Carathèodory function into R^N defined for $x \in \Omega$, for $\xi \in R^n$ and for $(\eta^1, \ldots, \eta^n) \in (R^N)^n$ such that

$$A_i\left(x, \xi, \eta^1, \ldots, \eta^n\right) \cdot \eta^i \geq c_0 \left|\eta^i\right|^p, \tag{A.3}$$

where $1 < p < +\infty$, $c_0 = \text{const.} > 0$.

Proposition A.3. *Let* $(u_1,\dots,u_n) \in (W_0^{1,p}(\Omega))^n$ *with* $u_i \geq 0$*. If there exist* $r \in]p,\widetilde{p}[$ *and* $g \in L^{r/(r-p)}(\Omega)$ *with* $g \geq 0$ *such that*

$$\sum_{i=1}^{n}\int_{\Omega} A_i(x,u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n)\cdot\nabla v_i dx \leq \int_{\Omega} g\left(\sum_{i=1}^{n} u_i\right)^{p-1}\left(\sum_{i=1}^{n} v_i\right)dx \quad \forall (v_1,\dots,v_n) \in \left(W_0^{1,p}(\Omega)\cap L^{\infty}(\Omega)\right)^n \text{ with } v_i \geq 0, \tag{A.4}$$

then $u_i \in L^{\infty}(\Omega)$ *as* $i = 1,\dots n$.

Proposition A.4. *Let* $(u_1,\dots,u_n) \in (W^{1,p}(\Omega))^n$ *with* $u_i \geq 0$*. If there exist* $r \in]p,\widehat{p}[$*,* $g \in L^{r/(r-p)}(\Omega)$ *with* $g \geq 0$*,* $\widehat{g} \in L^{r/(r-p)}(\partial\Omega)$ *with* $\widehat{g} \geq 0$ *such that*

$$\sum_{i=1}^{n}\int_{\Omega} A_i(x,u_1,\dots,u_n,\nabla u_1,\dots,\nabla u_n)\cdot\nabla v_i dx \leq \int_{\Omega} g\left(1+\sum_{i=1}^{n} u_i\right)^{p-1}\left(\sum_{i=1}^{n} v_i\right)dx + \int_{\partial\Omega} \widehat{g}\left(1+\sum_{i=1}^{n} u_i\right)^{p-1}\left(\sum_{i=1}^{n} v_i\right)d\sigma \quad \forall (v_1,\dots,v_n) \in \left(W^{1,p}(\Omega)\cap L^{\infty}(\Omega)\right)^n \quad \text{with } v_i \geq 0, \tag{A.5}$$

then $u_i \in L^{\infty}(\Omega)$ *as* $i = 1,\dots,n$.

Remark A.5. If $\widehat{g} \equiv 0$, we can suppose $r \in]p,\widetilde{p}[$.

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