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Research Article

Infinitely Many Homoclinic Orbits for 2nth-Order Nonlinear Functional Difference Equations Involving the p-Laplacian

Xiaofei He[1,2]

[1] *Department of Mathematics and Computer Science, Jishou University, Hunan, Jishou 416000, China*
[2] *Zhangjiajie College of Jishou University, Zhangjiajie 427000, China*

Correspondence should be addressed to Xiaofei He, hxfcsu@sina.com

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By establishing a new proper variational framework and using the critical point theory, we establish some new existence criteria to guarantee that the 2nth-order nonlinear difference equation containing both advance and retardation with p-Laplacian $\Delta^n(r(t-n)\varphi_p(\Delta^n u(t-1))) + q(t)\varphi_p(u(t)) = f(t,u(t+n),\dots,u(t),\dots,u(t-n))$, $n \in \mathbb{Z}(3)$, $t \in \mathbb{Z}$, has infinitely many homoclinic orbits, where $\varphi_p(s)$ is p-Laplacian operator; $\varphi_p(s) = |s|^{p-2}s(1 < p < \infty)$ r, q, f are nonperiodic in t. Our conditions on the potential are rather relaxed, and some existing results in the literature are improved.

1. Introduction

In this paper, we shall be concerned with the existence of homoclinic orbits of the nonlinear difference equation

$$\begin{aligned}&\Delta^n\left(r(t-n)\varphi_p(\Delta^n u(t-1))\right) + q(t)\varphi_p(u(t))\\ &\quad = f(t,u(t+n),\dots,u(t),\dots,u(t-n)), \quad n \in \mathbb{Z}(3),\ t \in \mathbb{Z},\end{aligned} \tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(t) = u(\mathrm{t}+1) - u(t)$, $\Delta^2 u(t) = \Delta(\Delta u(t))$. $\varphi_p(s)$ is p-Laplacian operator $\varphi_p(s) = |s|^{p-2}s(1 < p < \infty)$. As usual, we say that a solution $u(t)$ of (1.1) is homoclinic (to 0) if $u(t) \to 0$ as $t \to \pm\infty$. In addition, if $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution. It is well known that homoclinic orbits play an

important role in analyzing the chaos of dynamical systems; we refer the interested reader to [1–15].

We may think of (1.1) as being a discrete analogue of the $2n$th-order differential equation

$$\left[r(t)\varphi_p\left(x^{(n)}\right)\right]^{(n)} + q(t)\varphi_p(x) - f(t, x(t+n), \ldots, x(t), \ldots, x(t-n)) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

If $p = 2$, (1.1) reduces to the following equation:

$$\Delta^n(r(t-n)\Delta^n u(t-n)) + q(t)u(t) = f(t, u(t+n), \ldots, u(t), \ldots, u(t-n)), \quad n \in \mathbb{Z}(3),\ t \in \mathbb{Z}. \tag{1.3}$$

Recently, Chen and Tang [2] applied the critical point theory to prove the existence of homoclinic solutions for (1.3).

In some recent papers [1, 2, 8, 9, 16–18], the authors studied the existence of periodic and homoclinic solutions of second-order nonlinear difference equation by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions of second-order difference equations. Compared to one-order or second-order difference equations, the study of higher-order equations has received considerably less attention (see, e.g., [2, 3, 19–21] and references contained therein). But to the best knowledge of the authors, results on existence of homoclinic solutions of (1.1) have not been found in the literature.

Motivated by the recent papers [2, 5, 14, 22], the aim of this paper is to consider problem (1.1) in a more general sense. More exactly our results represent the extensions to equations with p-Laplacian. Throughout the paper, for a function G, we let $G_i'(x_1, x_2, \ldots, x_i, \ldots, x_n)$ denote the partial derivative of G on the i variable.

2. Main Results

Theorem 2.1. *Assume that q and F satisfy the following assumptions:*

(r) *For every $t \in \mathbb{Z}$, $r(t) > 0$;*

(q) *For every $t \in \mathbb{Z}$, $q(t) > 0$, and $\lim_{|t|\to+\infty} q(t) = +\infty$;*

(F1) *There exists a functional $F(t, x_n(t), \ldots, x_0(t))$ which is continuously differentiable in the variable from x_n to x_0 for every $t \in \mathbb{Z}$ and satisfy*

$$\begin{gathered}
\sum_{i=-n}^{0} F'_{2+n+i}(t+i, x_{n+i}, \ldots, x_i) = f(t, x_n, x_{n-1}, \ldots, x_0, x_{-1}, \ldots, x_{-n}), \\
\left|f(t, x_n, x_{n-1}, \ldots, x_0, x_{-1}, \ldots, x_{-n})\right| = o\left(\sum_{i=-n}^{n} |x_i|^{p-1}\right), \quad as \sum_{i=-n}^{n} |x_i|^{p-1} \longrightarrow 0, \\
|F(t, x_n, \ldots, x_0)| = o\left(\sum_{i=-n}^{n} |x_i|^{p}\right), \quad as \sum_{i=-n}^{n} |x_i|^{p} \longrightarrow 0
\end{gathered} \tag{2.1}$$

uniformly in $t \in \mathbb{Z} \setminus J$, where J is defined in (F2);

(F2) $F(t,x_n,\dots,x_0) = W(t,x_0) - H(t,x_n,\dots,x_0)$, *for every* $t \in \mathbb{Z}$, W, H *are continuously differentiable in* x_0 *and* $x_n,\dots,x_0$, *respectively. Moreover, there is a bounded set* $J \subset \mathbb{Z}$ *such that*

$$H(t,x_n,\dots,x_0) \geq 0; \tag{2.2}$$

(F3) *There is a constant* $\mu > p$ *such that*

$$0 < \mu W(t,x_0) \leq W_2'(t,x_0)x_0, \quad \forall (t,x_0) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\}); \tag{2.3}$$

(F4) $H(t,0,\dots,0) \equiv 0$, *and there is a constant* $\varrho \in (p,\mu)$ *such that*

$$\sum_{i=-n}^{0} H'_{2+n+i}(t,x_n,\dots,x_0)x_{-i} \leq \varrho H(t,x_n,\dots,x_0); \tag{2.4}$$

(F5) *There exists a constant* $b > 0$ *such that*

$$H(t,x_n,\dots,x_0) \leq b\gamma^{\varrho}, \quad \text{for } t \in \mathbb{Z},\ \gamma > 1, \tag{2.5}$$

where $\gamma = \left(\sum_{i=0}^{n}|x_i|^p\right)^{1/p}$.

(F6) $F(t,-x_n,\dots,-x_0) = F(t,x_n,\dots,x_0)$, *for all* $(t,x_n,\dots,x_0) \in \mathbb{Z} \times \mathbb{R}^{n+1}$.

Then (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Theorem 2.2. *Assume that* r, q, *and* F *satisfy (r), (q), (F1), (F3)–(F5) and the following assumption:*

(F2′) $F(t,x_n,\dots,x_0) = W(t,x_0) - H(t,x_n,\dots,x_0)$, *for every* $t \in \mathbb{Z}$, W, H *are continuously differentiable in* x_0 *and* $x_n,\dots,x_0$, *respectively, and*

$$|F(t,x_n,\dots,x_0)| = o\left(\gamma^p\right) \quad \text{as } \gamma \longrightarrow 0, \tag{2.6}$$

where $\gamma = \left(\sum_{i=0}^{n}|x_i|^p\right)^{1/p}$ *uniformly in* $t \in \mathbb{Z}$.

Then (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Theorem 2.3. *Assume that* r, q, *and* F *satisfy (r), (q), (F1) and satisfy the following assumptions:*

(F7) *For any* $t \in \mathbb{Z}$,

$$F(t,x_n,\dots,x_0) \geq F(t,x_0) \geq 0; \tag{2.7}$$

(F8) *For any $r > 0$, there exist $a = a(r)$, $b = b(r) > 0$, and $\nu < p$ such that*

$$\left(p + \frac{1}{a + b\left(\sum_{i=0}^{n}|x_i|^p\right)^{\nu/p}}\right)F(t, x_n, \ldots, x_0) \leq \sum_{i=-n}^{0} F'_{2+n+i}(t, x_n, \ldots, x_0)x_{-i}, \quad \forall t \in \mathbb{Z}, \ \left(\sum_{i=0}^{n}|x_i|^p\right)^{1/p} \geq r; \tag{2.8}$$

(F9) *For any $t \in \mathbb{Z}$*

$$\lim_{s \to +\infty}\left[s^{-p}\min_{|x|=1}F(t, sx)\right] = +\infty. \tag{2.9}$$

Then there exists an unbounded sequence of homoclinic solutions for (1.1).

3. Preliminaries

To apply critical point theory to study the existence of homoclinic solutions of (1.1), we shall state some basic notations and lemmas, which will be used in the proofs of our main results.

Let

$$S = \{\{u(t)\}_{t\in\mathbb{Z}} : u(t) \in \mathbb{R}, \ t \in \mathbb{Z}\},$$
$$E = \left\{u \in S : \sum_{t\in\mathbb{Z}}\left[|r(t-1)\Delta^n u(t-1)|^p + q(t)|u(t)|^p\right] < +\infty\right\} \tag{3.1}$$

and for $u \in E$, let

$$\|u\| = \left\{\sum_{t\in\mathbb{Z}}\left[|r(t-1)\Delta^n u(t-1)|^p + q(t)|u(t)|^p\right]\right\}^{1/p}. \tag{3.2}$$

Then E is a uniform convex Banach space with this norm.

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{p}\|u\|^p - \sum_{t\in\mathbb{Z}}F(t, u(t+n), \ldots, u(t)). \tag{3.3}$$

If (q) and (F1) hold, then $I \in C^1(E, \mathbb{R})$ and one can easily check that

$$\langle I'(u), v\rangle = \sum_{t\in\mathbb{Z}}\Big[r(t-1)|\Delta^n u(t-1)|^{p-2}\Delta^n v(t-1) + q(t)|u(t)|^{p-2}v(t) - f(t), u(t+n), \ldots \mathrm{u}(t), \ldots u(t-n)v(t)\Big], \quad \forall u, v \in E. \tag{3.4}$$

By using

$$\Delta^n u(t-1) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} u(t+n-k-1), \tag{3.5}$$

we can compute the partial derivative as

$$\frac{\partial I(u)}{\partial u(t)} = \Delta^n \left(r(t-n)\varphi_p(\Delta^n u(t-1)) \right) + q(t)\varphi_p(u(t)) - f(t, u(t+n), \ldots, u(t), \ldots, u(t-n)). \tag{3.6}$$

So, the critical points of I in E are the solutions of (1.1) with $u(\pm\infty) = 0$.

Lemma 3.1 (see [12]). *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS)-condition with I even. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$;

(ii) *There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;*

(iii) *For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \leq 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in E of radius r centered at 0.*

Then I possesses an unbounded sequence of critical values.

Lemma 3.2. *For $u \in E$,*

$$\beta \|u\|_\infty^p \leq \beta \|u\|_{l^p}^p \leq \|u\|^p, \tag{3.7}$$

where $\beta = \inf_{t\in\mathbb{Z}} q(t)$.

Lemma 3.3. *Assume that (F3) holds. Then for every $(t, x) \in \mathbb{Z} \times \mathbb{R}$, $s^{-\mu} W(t, sx)$ is nondecreasing on $(0, +\infty)$.*

The proof of Lemmas 3.2 and 3.3 is routine and so we omit it.

4. Proofs of Theorems

Proof of Theorem 2.1. It is clear that $I(0) = 0$. Our proof is devided into three steps.

Step 1 (PS Condition).

Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_k)| \leq c, \quad \left\| I'(u_k) \right\|_{E^*} \leq \varrho c \quad \text{for } k \in \mathbb{N}. \tag{4.1}$$

From (2.2), (2.3), (2.4), (4.1), (F3), and (F4), we obtain

$$\begin{aligned} pc + pc\|u_k\| &\geq pI(u_k) - \frac{p}{\varrho}\langle I'(u_k), u_k\rangle \\ &= \frac{\varrho - p}{\varrho}\|u_k\|^p - p\sum_{t\in\mathbb{Z}}\left[W(t,u_k(t)) - \frac{1}{\varrho}W_2'\ (t,u_k(t))u_k(t)\right] \\ &\quad + p\sum_{t\in\mathbb{Z}}H(t,u_k(t+n),\ldots,u_k(t)) \\ &\quad - \frac{p}{\varrho}\sum_{t\in\mathbb{Z}}\sum_{i=-n}^{0}H'_{2+n+i}(t+i,u_k(t+n+i),\ldots,u_k(t+i))u_k(t) \\ &= \frac{\varrho - p}{\varrho}\|u_k\|^p - p\sum_{n\in\mathbb{Z}}\left[W(t,u_k(t)) - \frac{1}{\varrho}W_2'(t,u_k(t))u_k(t)\right] \\ &\quad + p\sum_{t\in\mathbb{Z}}H(t,u_k(t+n),\ldots,u_k(t)) \\ &\quad - \frac{p}{\varrho}\sum_{t\in\mathbb{Z}i=-n}^{0}H'_{2+n+i}(t,u_k(t+n),\ldots,u_k(t))u_k(t-i) \\ &\geq \frac{\varrho - p}{\varrho}\|u_k\|^p, \quad k\in\mathbb{N}. \end{aligned} \tag{4.2}$$

By (4.2), there exists a constant $A > 0$ such that

$$\|u_k\| \leq A \quad \text{for } k\in\mathbb{N}. \tag{4.3}$$

It can be assumed that $u_k \rightharpoonup u_0$ in E. For any given number $\varepsilon > 0$, by (F1), we can choose $\zeta > 0$ such that

$$\begin{aligned} &\left|f(t,u(t+n),\ldots,u(t),\ldots,u(t-n))\right| \\ &\qquad \leq \varepsilon\xi^{p-1} \quad \text{for } t\in\mathbb{Z}\setminus J,\ (u(t+n),\ldots,u(t),\ldots,u(t-n))\in\mathbb{R}^{n+1}, \end{aligned} \tag{4.4}$$

where $\xi = \left(\sum_{i=-n}^{n}|u(t+i)|^p\right)^{1/p} \leq \zeta$.

Since $q(t) \to \infty$, we can also choose an integer $\Pi > \max\{|k| : k\in J\}$ such that

$$q(t) \geq \frac{(2n+1)A^p}{\zeta^p}, \quad |t| \geq \Pi. \tag{4.5}$$

By (4.2) and (4.4), we have

$$|u_k(t)|^p = \frac{1}{q(t)}q(t)(u_k(t))^p \leq \frac{\zeta^p}{(2n+1)A^p}\|u_k\|^p \leq \frac{\zeta^p}{2n+1} \quad \text{for } |t|\geq\Pi,\ k\in\mathbb{N}. \tag{4.6}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(t)$ converges to $u_0(t)$ pointwise for all $t \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_k(t) = u_0(t), \quad \forall t \in \mathbb{Z}. \tag{4.7}$$

Hence, we have by (4.6) and (4.7)

$$|u_0(t)|^p \le \frac{\zeta^p}{2n+1} \quad \text{for } |t| \ge \Pi. \tag{4.8}$$

It follows from (4.7) and the continuity of $f(t, u(t+1), \ldots, u(t), \ldots, u(t-n))$ on $u(t+1)$, $\ldots, u(t), \ldots, u(t-n)$ that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{t=-\Pi}^{\Pi} \left| f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) - f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)) \right| < \varepsilon \quad \text{for } k \ge k_0. \tag{4.9}$$

On the other hand, it follows from (F1), (2.4), (4.2), (4.4), (4.6), and (4.8) that

$$\begin{aligned}
&\sum_{|t|>\Pi} \left| f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) - f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)) \right| \\
&\qquad \times |u_k(t) - u_0(t)| \\
&\le \sum_{|t|>\Pi} \Bigg(\left| f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) \right| \\
&\qquad\qquad + \left| f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)) \right| \Bigg) \times (|u_k(t)| + |u_0(t)|) \\
&\le \varepsilon \sum_{|t|>\Pi} \left[\left(\sum_{i=-n}^{n} |u_k(t+i)|^{p-1} \right) + \left(\sum_{i=-n}^{n} |u_0(t+i)|^{p-1} \right) \right] (|u_k(t)| + |u_0(t)|) \\
&\le (2n+1)\varepsilon \sum_{t\in\mathbb{Z}} \left(|u_k(t)|^{p-1} + |u_0(t)|^{p-1} \right) (|u_k(t)| + |u_0(t)|) \\
&\le (4n+2)\varepsilon \sum_{t\in\mathbb{Z}} \left(|u_k(t)|^p + |u_0(t)|^p \right) \\
&\le \frac{(4n+2)\varepsilon}{\beta} \left(A^p + \|u_0\|^p \right).
\end{aligned} \tag{4.10}$$

Since ε is arbitrary, combining (4.9) with (4.10), we get

$$\sum_{t\in\mathbb{Z}} \left| f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) - f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)) \right| |u_k(t) - u_0(t)| \longrightarrow 0 \quad \text{as } k \longrightarrow \infty. \tag{4.11}$$

Using the Hölder's inequality

$$ac + bd \leq (a^p + b^p)^{1/p}(c^q + d^q)^{1/q}, \tag{4.12}$$

where a, b, c, d are nonnegative numbers and $1/p + 1/q = 1$, $p > 1$, it follows from (3.3) and (3.4) that

$$\begin{aligned}
&\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \\
&\quad= \sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_k(t-1)|^{p-2}(\Delta^n u_k(t-1), \Delta^n u_k(t-1) - \Delta^n u_0(t-1)) \\
&\qquad+ \sum_{t\in\mathbb{Z}} q(t)|u_k(t)|^{p-2}(u_k(t), u_k(t) - u_0(t)) \\
&\qquad- \sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_0(t-1)|^{p-2}(\Delta^n u_0(t-1), \Delta^n u_k(t-1) - \Delta^n u_0(t-1)) \\
&\qquad- \sum_{t\in\mathbb{Z}} q(t)|u_0(t)|^{p-2}(u_0(n), u_k(t) - u_0(t)) \\
&\qquad- \sum_{t\in\mathbb{Z}} \big(f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) \\
&\qquad\qquad - f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)), u_k(n) - u_0(n)\big) \\
&\quad= \|u_k\|^p + \|u_0\|^p - \sum_{n\in\mathbb{Z}} |r(t-1)\Delta^n u_k(t-1)|^{p-2}(\Delta^n u_k(t-1), \Delta^n u_0(t-1)) \\
&\qquad- \sum_{t\in\mathbb{Z}} q(t)|u_k(t)|^{p-2}(u_k(t), u_0(t)) \\
&\qquad- \sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_0(t-1)|^{p-2}(\Delta^n u_0(t-1), \Delta^n u_k(t-1)) \\
&\qquad- \sum_{t\in\mathbb{Z}} q(t)|u_0(t)|^{p-2}(u_0(t), u_k(t)) \\
&\qquad- \sum_{t\in\mathbb{Z}} \big(f(t, u_k(t+n), \ldots, u_k(t), \ldots, u_k(t-n)) \\
&\qquad\qquad - f(t, u_0(t+n), \ldots, u_0(t), \ldots, u_0(t-n)), u_k(n) - u_0(n)\big) \\
&\quad\geq \|u_k\|^p + \|u_0\|^p - \left(\sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_0(t-1)|^p\right)^{1/p} \left(\sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_k(t-1)|^p\right)^{1/q} \\
&\qquad- \left(\sum_{t\in\mathbb{Z}} q(t)|u_0(t)|^p\right)^{1/p} \left(\sum_{t\in\mathbb{Z}} q(t)|u_k(t)|^p\right)^{1/q} \\
&\qquad- \left(\sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_k(t-1)|^p\right)^{1/p} \left(\sum_{t\in\mathbb{Z}} |r(t-1)\Delta^n u_0(t-1)|^p\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
&- \left(\sum_{t\in\mathbb{Z}} q(t)|u_k(t)|^p\right)^{1/p} \left(\sum_{t\in\mathbb{Z}} q(t)|u_0(t)|^p\right)^{1/q} \\
&- \sum_{t\in\mathbb{Z}} \big(f(t,u_k(t+n),\ldots,u_k(t),\ldots,u_k(t-n)) \\
&\qquad -f(t,u_0(t+n),\ldots,u_0(t),\ldots,u_0(t-n)),u_k(t)-u_0(t)\big) \\
&\geq \|u_k\|^p + \|u_0\|^p - \left(\sum_{t\in\mathbb{Z}} \left[|r(t-1)\Delta^n u_0(t-1)|^p + q(t)|u_0(t)|^p\right]\right)^{1/p} \\
&\cdot \left(\sum_{t\in\mathbb{Z}} \left[|r(t-1)\Delta^n u_k(t-1)|^p + q(t)|u_k(t)|^p\right]\right)^{1/q} \\
&- \left(\sum_{t\in\mathbb{Z}} \left[|r(t-1)\Delta^n u_k(n-1)|^p + q(t)|u_k(t)|^p\right]\right)^{1/p} \\
&\cdot \left(\sum_{t\in\mathbb{Z}} \left[|r(t-1)\Delta^n u_0(n-1)|^p + q(t)|u_0(t)|^p\right]\right)^{1/q} \\
&- \sum_{t\in\mathbb{Z}} \big(f(t,u_k(t+n),\ldots,u_k(t),\ldots,u_k(t-n)) \\
&\qquad -f(t,u_0(t+n),\ldots,u_0(t),\ldots,u_0(t-n)),u_k(t)-u_0(t)\big) \\
&= \|u_k\|^p + \|u_0\|^p - \|u_0\|\|u_k\|^{p-1} - \|u_k\|\|u_0\|^{p-1} \\
&- \sum_{t\in\mathbb{Z}} \big(f(t,u_k(t+n),\ldots,u_k(t),\ldots,u_k(t-n)) \\
&\qquad -f(t,u_0(t+n),\ldots,u_0(t),\ldots,u_0(t-n)),u_k(t)-u_0(t)\big) \\
&= \left(\|u_k\|^{p-1} - \|u_0\|^{p-1}\right)(\|u_k\| - \|u_0\|) \\
&- \sum_{t\in\mathbb{Z}} \big(f(t,u_k(t+n),\ldots,u_k(t),\ldots,u_k(t-n)) \\
&\qquad -f(t,u_0(t+n),\ldots,u_0(t),\ldots,u_0(t-n)),u_k(t)-u_0(t)\big).
\end{aligned}
\tag{4.13}
$$

Since $I'(u_k) \to 0$ as $k \to +\infty$ and $u_k \rightharpoonup u_0$ in E, it follows from (4.11) and (4.13) that

$$
\langle I'(u_k) - I'(u_0), u_k - u_0\rangle \longrightarrow 0 \quad \text{as } k \to \infty, \tag{4.14}
$$

which yields that $\|u_k\| \to \|u\|$ as $k \to +\infty$. By the uniform convexity of E and the fact that $u_k \rightharpoonup u_0$ in E, it follows from the Kadec-Klee property that $u_k \to u_0$ in E. Hence, I satisfies (PS)-condition.

Step 2 (Condition (ii) of Lemma 3.1).

By (F1), there exists $\eta \in (0,1)$ such that

$$|F(t,u(t+n),\dots,u(t))| \le \frac{2^{-p}\beta}{4p(n+1)}\sum_{i=0}^{n}|u(t+i)|^p \quad \text{for } t \in \mathbb{Z}\setminus J,\ \left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p} \le \eta. \tag{4.15}$$

Set

$$M = \sup\{W(t,u) \mid\ t \in J,\ u \in \mathbb{R},\ |u| = 1\}, \tag{4.16}$$

and $\delta = \min\{(\beta/(4pM+1))^{(\mu-p)}, \eta\}$. If $\|u\| = \beta^{1/p}\delta := \rho$, then by Lemma 3.2, $|u(t)| \le \delta \le \eta < 1$ for $t \in \mathbb{Z}$. By (4.16) and Lemma 3.2, we have

$$\begin{aligned}
\sum_{t\in J}W(t,u(t)) &\le \sum_{t\in J,u(t)\neq 0}W\left(t,\frac{u(t)}{|u(t)|}\right)|u(t)|^{\mu}\\
&\le M\sum_{t\in J}|u(t)|^{\mu}\\
&\le M\delta^{\mu-p}\sum_{t\in J}|u(t)|^{p}\\
&\le \frac{M\delta^{\mu-p}}{\beta}\sum_{t\in J}q(t)|u(t)|^{p}\\
&\le \frac{1}{4p}\sum_{t\in J}q(t)|u(t)|^{p}.
\end{aligned} \tag{4.17}$$

Set $\alpha = \beta\delta^p/2p$. Hence, from (3.3), (4.15), (4.17), (q), (F1), and (F2), we have

$$\begin{aligned}
I(u) &= \frac{1}{p}\|u\|^p - \sum_{t\in\mathbb{Z}}F(t,u(t+n),\dots,u(t))\\
&= \frac{1}{p}\|u\|^p - \sum_{t\in\mathbb{Z}\setminus J}F(t,u(t+n),\dots,u(t)) - \sum_{t\in J}F(t,u(t+n),\dots,u(t))\\
&\ge \frac{1}{p}\|u\|^p - \frac{2^{-p}\beta}{4p(n+1)}\sum_{t\in\mathbb{Z}\setminus J}\left(\sum_{i=0}^{n}|u(t+i)|^p\right) - \sum_{t\in J}W(t,u(t))\\
&\quad + \sum_{t\in J}H(t,u(\mathrm{t}+n),\dots,u(t))\\
&\ge \frac{1}{p}\|u\|^p - \frac{\beta}{4p}\sum_{t\in\mathbb{Z}}|u(t)|^p - \frac{1}{4p}\sum_{t\in J}q(t)|u(t)|^p\\
&\ge \frac{1}{p}\|u\|^p - \frac{1}{4p}\sum_{t\in\mathbb{Z}}q(t)|u(t)|^p - \frac{1}{4p}\sum_{t\in J}q(t)|u(t)|^p
\end{aligned}$$

$$\begin{aligned} &\geq \frac{1}{p}\|u\|^p - \frac{1}{4p}\|u\|^p - \frac{1}{4p}\|u\|^p \\ &= \frac{1}{2p}\|u\|^p \\ &= \alpha. \end{aligned} \tag{4.18}$$

equation (4.18)shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, that is, I satisfies assumption (ii) of Lemma 3.1.

Step 3 (Condition (iii) of Lemma 3.1).

Let E' be a finite dimensional subspace of E. Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ is a base of E'.

For any $u \in E'$, there exist $\lambda_i \in \mathbb{R}$, $i = 1, 2, \ldots, m$ such that

$$u(t) = \sum_{i=1}^{m} \lambda_i u_i(t) \quad \text{for } t \in \mathbb{Z}. \tag{4.19}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i| \|u_i\|. \tag{4.20}$$

It is easy to verify that $\|\cdot\|_*$ defined by (4.20) is a norm of E'. Since all the norms of a finite dimensional normed space are equivalent, so there exists a constant $d > 0$ such that

$$\|u\|_* \leq d\|u\|_\infty \quad \text{for } u \in E'. \tag{4.21}$$

Assume that

$$\|u_i\| = d, \quad i = 1, 2, \ldots, m. \tag{4.22}$$

Since $u_i \in E$, by Lemma 3.2, we can choose an integer $\Pi_1 > 0$ such that

$$|u_i(t)| < \frac{\eta}{m}, \quad i = 1, 2, \ldots, m, \ |t| > \Pi_1, \tag{4.23}$$

where η is given in (4.15). Set

$$\Theta = \left\{ \sum_{i=1}^{m} \lambda_i u_i(t) : \lambda_i \in \mathbb{R}, \ i = 1, 2, \ldots, m; \ \sum_{i=1}^{m} |\lambda_i| = 1 \right\} = \left\{ u \in E' : \|u\|_* = d \right\}. \tag{4.24}$$

Hence, for $u \in \Theta$, let $t_0 = t_0(u) \in \mathbb{Z}$ such that

$$|u(t_0)| = \|u\|_\infty. \tag{4.25}$$

Then, by (4.19)–(4.25), we have

$$\begin{aligned} d = d\sum_{i=1}^{m}|\lambda_i| &= \sum_{i=1}^{m}|\lambda_i|\|u_i\| = \|u\|_* \\ &\le d\|u\|_\infty = d|u(t_0)| \\ &\le d\sum_{i=1}^{m}|\lambda_i||u_i(t_0)|, \quad u \in \Theta. \end{aligned} \tag{4.26}$$

This shows that $|u(t_0)| \ge 1$ and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(t_0)| \ge 1/m$, which, together with (4.23), implies that $|t_0| \le \Pi_1$.

Set

$$\tau = \min\{W(t, x) : |t| \le \Pi_1, |x| = 1\}. \tag{4.27}$$

Since $W(t, x) > 0$ for all $t \in \mathbb{Z}$ and $x \in \mathbb{R} \setminus \{0\}$, and $W(t, x)$ is continuous in x, so $\tau > 0$. It follows from (4.27) and Lemma 3.3 that

$$\begin{aligned} \sum_{n=-\Pi_1}^{\Pi_1} W(t, u(t)) &\ge W(t_0, u(t_0)) \\ &\ge W\left(t_0, \frac{u(t_0)}{|u(t_0)|}\right)|u(t_0)|^\mu \\ &\ge \left[\min_{|x|=1} W(t_0, x)\right]|u(t_0)|^\mu \\ &\ge \tau \quad \text{for } u \in \Theta. \end{aligned} \tag{4.28}$$

For any $u \in E$, it follows from (F5) that

$$\begin{aligned} &\sum_{t=-\Pi_1}^{\Pi_1} H(t, u(t+n), \ldots, u(t)) \\ &\quad = \sum_{\left\{t \in \mathbb{Z}(-\Pi_1, \Pi_1), \left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p} > 1\right\}} H(t, u(t+n), \ldots, u(t)) \\ &\qquad + \sum_{\left\{t \in \mathbb{Z}(-\Pi_1, \Pi_1), \left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\right\} \le 1} H(t, u(t+n), \ldots, u(t)) \end{aligned}$$

$$\begin{aligned}
&\le b \sum_{\left\{t\in\mathbb{Z}(-\Pi_1,\Pi_1),\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}>1\right\}} \left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{\varrho/p} \\
&\quad + \sum_{t=-\Pi_1}^{\Pi_1} \max_{\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\le 1} H(t,u(t+n),\dots,u(t)) \\
&\le (n+1)^{\varrho/p+1} b \sum_{t\in\mathbb{Z}} |u(t)|^{\varrho} + \sum_{t=-\Pi_1}^{\Pi_1} \max_{\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\le 1} H(t,u(t+n),\dots,u(t)) \\
&\le (n+1)^{\varrho/p+1} \beta^{-\varrho/p} b \|u\|^{\varrho} + \sum_{t=-\Pi_1}^{\Pi_1} \max_{\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\le 1} H(t,u(t+n),\dots,u(t)) \\
&= M_1\|u\|^{\varrho} + M_2,
\end{aligned} \tag{4.29}$$

where

$$M_1 = (n+1)^{\varrho/p+1}\beta^{-\varrho/p}\, b, \qquad M_2 = \sum_{t=-\Pi_1}^{\Pi_1} \max_{\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\le 1} H(t,u(t+n),\dots,u(t)). \tag{4.30}$$

From (3.3), (3.7), (4.28), and (4.29), we have for $u \in \Theta$ and $\sigma > 1$

$$\begin{aligned}
I(\sigma u) &= \frac{\sigma^p}{p}\|u\|^p - \sum_{t\in\mathbb{Z}} F(t,\sigma u(t+n),\dots,\sigma u(t)) \\
&= \frac{\sigma^p}{p}\|u\|^p - \sum_{|t|>\Pi_2} F(t,\sigma u(t+n),\dots,\sigma u(t)) - \sum_{|t|\le\Pi_2} F(t,\sigma u(t+n),\dots,\sigma u(t)) \\
&\le \frac{\sigma^p}{p}\|u\|^p + \sum_{|t|>\Pi_2}\sum_{i=0}^{n} \frac{\beta\sigma^p}{4p(n+1)}|u(t+i)|^p - \sum_{|t|\le\Pi_2} F(t,\sigma u(t+n),\dots,\sigma u(t)) \\
&\le \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^p}{4p}\|u\|^p - \sum_{|t|\le\Pi_2} F(t,\sigma u(t+n),\dots,\sigma u(t)) \\
&= \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^p}{4p}\|u\|^p - \sum_{|t|\le\Pi_2} W(t,\sigma u(t)) + \sum_{|t|\le\Pi_2} H(t,\sigma u(t+n),\dots,\sigma u(t)) \\
&\le \frac{\sigma^p}{p}\|u\|^p + \frac{\sigma^p}{4p}\|u\|^p + \sigma^{\varrho}\left(M_1\|u\|^{\varrho} + M_2\right) - \tau\sigma^{\mu} \\
&= \frac{(c\sigma)^p}{p} + \frac{c^p\sigma^p}{4p} + M_1(c\sigma)^{\varrho} + M_2\sigma^{\varrho} - \tau\sigma^{\mu}.
\end{aligned} \tag{4.31}$$

Since $\mu > \varrho > p$, we deduce that there is $\sigma_0 = \sigma_0(d, M_1, M_2, \tau) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta,\ \sigma \geq \sigma_0. \tag{4.32}$$

That is

$$I(u) < 0 \quad \text{for } u \in E',\ \|u\| \geq d\sigma_0. \tag{4.33}$$

This shows that (iii) of Lemma 3.1 holds. By Lemma 3.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leq B \quad \text{for } k \in \mathbb{N}. \tag{4.34}$$

By a similar fashion for the proof of (4.15), for the given η in (4.15), there exists $\Pi_3 > 0$ such that

$$|u_k(t)| \leq \eta \quad \text{for } |t| \geq \Pi_3,\ k \in \mathbb{N}. \tag{4.35}$$

Thus, from (F2), (3.7), (4.34), and (4.35), we have

$$\begin{aligned}
\frac{1}{p}\|u_k\|^p &= d_k + \sum_{t\in\mathbb{Z}} F(t, u_k(t+n), \ldots, u_k(t)) \\
&= d_k + \sum_{|t|>\Pi_3} F(t, u_k(t+n), \ldots, u_k(t)) + \sum_{t=-\Pi_3}^{\Pi_3} F(t, u_k(t+n), \ldots, u_k(t)) \\
&\geq d_k - \frac{\beta}{4p(n+1)} \sum_{|t|>\Pi_3} \sum_{i=0}^{n} |u_k(t+i)|^p - \sum_{n=-\Pi_3}^{\Pi_3} H(t, u_k(t+n), \ldots, u_k(t)) \\
&\geq d_k - \frac{1}{4p}\|u_k\|^p - \sum_{n=-\Pi_3}^{\Pi_3} \max_{|u_k|\leq B/\sqrt{\beta}} |H(t, u_k(t+n), \ldots, u_k(t))|.
\end{aligned} \tag{4.36}$$

It follows that

$$d_k \leq \frac{5}{4p}\|u_k\|^p + \sum_{t=-\Pi_3}^{\Pi_3} \max_{|u_k|\leq B/\sqrt{\beta}} |H(t, u_k(t+n), \ldots, u_k(t))| < +\infty. \tag{4.37}$$

This contradicts the fact that $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, and so $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded.

□

Proof of Theorem 2.2. In the proof of Theorem 2.1, the condition that $H(t, u(t+n), \ldots, u(t)) \geq 0$ for $(t, u(t+n), \ldots, u(t)) \in J \times \mathbb{R}^{n+1}$, $\gamma = \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{1/p} \leq 1$ in (F1) is only used in the proofs

of assumption (ii) of Lemma 3.1. Therefore, we only prove that assumption (ii) of Lemma 3.1 still holds using (F2′) instead of (F2). By (F2′), it follows that

$$|F(t,u(t+n),\dots,u(t))| \le \frac{2^{-p}\beta}{2p(n+1)}\left(\sum_{i=0}^{n}|u(t+i)|^p\right) \quad \text{for } t\in\mathbb{Z},\ \left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p} \le \eta. \tag{4.38}$$

If $\|u\| = \beta^{1/p}\eta := \rho$, then by Lemma 3.2 , $|u(t)| \le \eta$ for $t \in \mathbb{Z}$. Set $\alpha = \beta\eta^p/2p$. Hence, from (3.3), (4.38), and Lemma 3.2, we have

$$\begin{aligned} I(u) &= \frac{1}{p}\|u\|^p - \sum_{t\in\mathbb{Z}}F(t,u(t+n),\dots,u(t)) \\ &\ge \frac{1}{p}\|u\|^p - \frac{2^{-p}\beta}{2p(n+1)}\left(\sum_{i=0}^{n}|u(t+i)|^p\right) \\ &\ge \frac{1}{p}\|u\|^p - \frac{1}{2p}\|u\|^p \\ &= \frac{1}{2p}\|u\|^p \\ &= \alpha. \end{aligned} \tag{4.39}$$

equation (4.39) shows that $\|u\| = \rho$ implies that $I(u) \ge \alpha$, that is, assumption (ii) of Lemma 3.1 holds. The proof of Theorem 2.2 is completed. □

Proof of Theorem 2.3. We first show that I satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (C) sequence of I, that is, $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $(1+\|u_k\|)\|I'(u_k)\| \to 0$ as $k \to +\infty$. Then it follows from (3.3) and (3.4) that

$$\begin{aligned} C_1 &\ge pI(u_k) - \langle I'(u_k), u_k\rangle \\ &= \sum_{t\in\mathbb{Z}}\left[\sum_{i=-n}^{0}F'_{2+n+i}(t,u_k(t+n),\dots,u_k(t))u_k(t-i) - pF(t,u_k(t+n),\dots,u_k(t))\right]. \end{aligned} \tag{4.40}$$

It follows from (F8) that there exists $\eta \in (0,1)$ such that (4.15) holds. By (F7) and (F8), we have

$$\sum_{i=-n}^{0}F'_{2+n+i}(t,u(t+n),\dots,u(t))u(t-i) > pF(t,u_k(t+n),\dots,u(t)) \ge 0, \tag{4.41}$$

and for $t \in \mathbb{Z}$, $\sum_{i=0}^{n}|u(t+i)|^p \geq \eta^p$, we have

$$F(t,u_k(t+n),\dots,u_k(t)) \leq \left[a+b\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{\nu/p}\right] \times \left[\sum_{i=-n}^{0}F'_{2+n+i}(t,u_k(t+n),\dots,u_k(t))u_k(t-i)-pF(t,u_k(t+n),\dots,u_k(t))\right]. \tag{4.42}$$

It follows from (3.2), (3.3), (4.38), (4.40), (4.41), and (4.42) that

$$\begin{aligned}
\frac{1}{p}\|u_k\|^p &= I(u_k)+\sum_{t\in\mathbb{Z}}F(t,u_k(t+n),\dots,u(t)) \\
&= I(u_k)+\sum_{t\in\mathbb{Z}\left(\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\leq\eta\right)}F(t,u_k(t+n),\dots,u(t)) \\
&\quad+\sum_{t\in\mathbb{Z}\left(\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}>\eta\right)}F(t,u_k(t+n),\dots,u(t)) \\
&\leq I(u_k)+\frac{2^{-p}\beta}{2p(n+1)}\sum_{t\in\mathbb{Z}\left(\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}\leq\eta\right)}|u_k(t+i)|^p \\
&\quad+\sum_{t\in\mathbb{Z}\left(\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{1/p}>\eta\right)}\left[a+b\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{\nu/p}\right] \\
&\quad\times\left[\sum_{i=-n}^{0}F'_{2+n+i}(t,u_k(t+n),\dots,u_k(t))u_k(t-i)-pF(t,u_k(t+n),\dots,u_k(t))\right] \\
&\leq C_2+\frac{1}{2p}\|u_k\|^2+\sum_{t\in\mathbb{Z}}\left[a+b\left(\sum_{i=0}^{n}|u(t+i)|^p\right)^{\nu/p}\right] \\
&\quad\times\left[\sum_{i=-n}^{0}F'_{2+n+i}(t,u_k(t+n),\dots,u_k(t))u_k(t-i)-pF(t,u_k(t+n),\dots,u_k(t))\right] \\
&\leq C_2+\frac{1}{2p}\|u_k\|^p+\left(a+b\|u_k\|_\infty^\nu\right)\sum_{t\in\mathbb{Z}}\left[(\nabla W(n,u_k(n)),u_k(n))-pW(n,u_k(n))\right] \\
&\leq C_2+\frac{1}{2p}\|u_k\|^p+C_1\left(a+b(n+1)\|u_k\|_\infty^\nu\right) \\
&\leq C_2+\frac{1}{2p}\|u_k\|^p+C_1\left\{a+\beta^{-\nu/p}b(n+1)\|u_k\|^\nu\right\},\quad k\in\mathbb{N}.
\end{aligned} \tag{4.43}$$

Since $\nu < p$, it follows from (4.43) that $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded. Similar to the proof of Theorem 2.1, we can prove that $\{u_k\}$ has a convergent subsequence in E. Hence, I satisfies condition (C).

It is obvious that I is even and $I(0) = 0$, and so assumption (i) of Lemma 3.1 holds. The proof of assumption (ii) of Lemma 3.1 is the same as in the proof of Theorem 2.2.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 3.1. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $d > 0$ such that (4.21) holds. Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ are the bases of E' such that (4.22) holds. Let η, Π_1, and Θ be the same as in the proof of Theorem 2.1. Then (4.23), (4.25), and (4.26) hold. For the Π_1 given in the proof of Theorem 2.1. By (F9), there exists $\sigma_0 = \sigma_0(d, \Pi_1) > 1$ such that

$$s^{-p} \min_{|x|=1} W(n, sx) \geq (2d)^p \quad \text{for } s \geq \frac{\sigma_0}{2},\ n \in \mathbb{Z}(-\Pi_1, \Pi_1). \tag{4.44}$$

It follows from (F7), (F9), (3.3), and (4.44) that

$$\begin{aligned} I(\sigma u) &= \frac{\sigma^p}{p}\|u\|^p - \sum_{t\in\mathbb{Z}} F(t, \sigma u(t+n), \ldots, \sigma u(t)) \\ &\leq \frac{\sigma^p}{p}\|u\|^p - F(t_0, \sigma u(t_0)) \\ &\leq \frac{\sigma^p}{p}\|u\|^p - \min_{|x|=1} F(t_0, \sigma |u(t_0)| x) \\ &\leq \frac{(d\sigma)^p}{p} - (d\sigma)^p |u(t_0)|^p \\ &\leq \frac{(d\sigma)^p}{p} - (d\sigma)^p \\ &= -\frac{(d\sigma)^p}{q}, \quad u \in \Theta,\ \sigma \geq \sigma_0. \end{aligned} \tag{4.45}$$

We deduce that there is $\sigma_0 = \sigma_0(d, \Pi_1) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta,\ \sigma \geq \sigma_0. \tag{4.46}$$

That is

$$I(u) < 0 \quad \text{for } u \in E',\ \|u\| \geq d\sigma_0. \tag{4.47}$$

This shows that condition (iii) of Lemma 3.1 holds. By Lemma 3.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. Since

$$\frac{1}{p}\|u_k\|^p = d_k + \sum_{n\in\mathbb{Z}} F(t, u_k(t+n), \ldots, u_k(t)) \geq d_k \tag{4.48}$$

and $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, it follows that $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded. The proof is complete. □

5. Examples

In this section, we give some examples to illustrate our results.

Example 5.1. In (1.1), let $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F(t,u(t+n),\dots,u(t)) = \left[|u(t)|^{4+|t|} - \left(\sum_{i=0}^{n} u^2(t+i)\right)^{(5+3|t|)/2(4+|t|)}\right]. \tag{5.1}$$

Let $p = 2$, $\mu = 4$, $\varrho = 3$, $J = \{-3,-2,-1,0,1,2,3\}$ and

$$W(t,u(t)) = |u(t)|^{4+|t|}, \qquad H(t,u(t+n),\dots,u(t)) = \left(\sum_{i=0}^{n} u^2(t+i)\right)^{(5+3|t|)/2(4+|t|)}. \tag{5.2}$$

Then it is easy to verify that all conditions of Theorem 2.1 are satisfied. By Theorem 2.1, (1.1) has a nontrivial homoclinic solution.

Example 5.2. In (1.1), let $r(t) > 0$, $q(t) \to +\infty$ as $|t| \to +\infty$ and

$$F(t,u(t+n),\dots,u(t)) = \left(\sum_{k=1}^{m_1} a_k |u(t)|^{\mu_k} - \sum_{j=1}^{m_2} b_j \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{\varrho_j/p}\right), \tag{5.3}$$

where $\mu_1 > \mu_2 > \cdots > \mu_{m_1} > \varrho_1 > \varrho_2 > \cdots > \varrho_{m_2} > p$, $a_k, b_j > 0$, $k = 1,2,\dots,m_1$; $j = 1,2,\dots,m_2$. Let $\mu = \mu_{m_1}$, $\varrho = \varrho_1$, and

$$W(t,u(t)) = \sum_{k=1}^{m_1} a_k |u(t)|^{\mu_k}, \qquad H(t,u(t+n),\dots,u(t)) = \sum_{j=1}^{m_2} b_j \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{\varrho_j/p}. \tag{5.4}$$

Then it is easy to verify that all conditions of Theorem 2.2 are satisfied. By Theorem 2.2, (1.1) has a nontrivial homoclinic solution.

Example 5.3. In (1.1), let

$$F(t,u(t+n),\dots,u(t)) = q(t)\sum_{i=0}^{n} |u(t+i)|^p \ln\left[1 + \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{1/p}\right], \tag{5.5}$$

where $q : \mathbb{Z} \to (0,\infty)$ such that $q(t) \to +\infty$ as $|t| \to +\infty$. Since

$$\begin{aligned}
&\sum_{i=-n}^{0} F'_{2+n+i}(t,u(t+n),\dots,u(t))u(t-i) \\
&\quad = q(t)\left[p\sum_{i=0}^{n} |u(t+i)|^p \ln\left[1 + \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{1/p}\right] + \frac{\left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{p+1/p}}{1 + \left(\sum_{i=0}^{n} |u(t+i)|^p\right)^{1/p}}\right]
\end{aligned}$$

$$\geq \left(p + \frac{1}{1 + \left(\sum_{i=0}^{n} |u(t+i)|^p \right)^{1/p}} \right) F(t, u(t+n), \ldots, u(t))$$
$$\geq 0, \quad \forall (t, u(t+n), \ldots, u(t)) \in \mathbb{Z} \times \mathbb{R}^{n+1}. \tag{5.6}$$

This shows that (F8) holds with $a = b = \nu = 1$. It is easy to verify that assumptions (q), (F1), and (F7) of Theorem 2.3 are satisfied. By Theorem 2.3, (1.1) has an unbounded sequence of homoclinic solutions.

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