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Research Article

New Exact Solutions of the Brusselator Reaction Diffusion Model Using the Exp-Function Method

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Firstly, using a series of transformations, the Brusselator reaction diffusion model is reduced into a nonlinear reaction diffusion equation, and then through using Exp-function method, more new exact solutions are found which contain soliton solutions. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The results show the reliability and efficiency of the proposed method.

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1. Introduction

The Brusselator reaction model plays an important role both in biology and in chemistry. Since the model was put forward by Prigogine and Lefever in 1968, much attention had been paid to the model and many properties of it had been researched by many people via using different methods [1–5]. In this paper, we mainly consider the model improved by Lefever et al. in 1977 [1]. The model is described as follows:

$$\begin{aligned}K\psi_{xx} - \psi_t + \psi^2\varphi - B\psi &= 0, \\K\varphi_{xx} - \varphi_t - \psi^2\varphi + B\psi &= 0,\end{aligned}\tag{1.1}$$

where B is a constant, and K is the diffusion coefficient; the functions $\psi(x, t)$ and $\varphi(x, t)$ denote the concentrations. System (1.1) describes a biochemical model. Recently, many approaches have been suggested to solve the nonlinear equations, such as the variational iteration method [6–8], the homotopy perturbation method [9–11], the tanh-method [12], the extended tanh-method [13], the sinh-method [14], the homogeneous balance method [15, 16],

the F-expansion method [17], and the extended Fan's subequation method [18]. Recently, He and Wu [19] have proposed a straightforward method called Exp-function method to obtain the exact solutions of nonlinear evolution equations (NLEEs). It should be pointed out that the method is also valid for difference-different equations [20, 21]. The solution's procedure of this method is of utter simplicity, and this method has been successfully applied to many kinds of NLEEs [22–33]. The Exp-function method not only provides generalized solitary solutions but also provides periodic solutions. Taking advantage of the generalized solitary solutions, we can recover some known solutions obtained by the most existing methods such as decomposition method, tanh-function method, algebraic method, extended Jacobi elliptic function expansion method, F-expansion method, auxiliary equation method, and others [22–33].

2. Exp-Function Method and Exact Solutions

In this section we intend to find a solitary wave solution of (1.1). Therefore by using the following transformations:

$$\psi = v, \quad \varphi = -v, \quad x = \pm y, \quad t = t, \quad (2.1)$$

the system (1.1) is reduced to a nonlinear reaction diffusion equation with respect to $v(y, t)$:

$$Kv_{yy} - v_t - Bv - v^3 = 0. \quad (2.2)$$

After that we use the transformation

$$v = \phi(\xi), \quad \xi = \lambda(y - wt + \gamma), \quad (2.3)$$

where λ and w are constants to be determined later, and γ is arbitrary constant. Therefore (2.2) converts to

$$K\lambda^2 \frac{d^2\phi}{d\xi^2} + \lambda w \frac{d\phi}{d\xi} - B\phi - \phi^3 = 0. \quad (2.4)$$

By virtue of the Exp-function method [19], we assume that the solution of (2.4) is of the form

$$\phi(\xi) = \frac{a_c \exp(c\xi) + \cdots + a_{-d} \exp(-d\xi)}{b_f \exp(f\xi) + \cdots + b_{-g} \exp(-g\xi)}, \quad (2.5)$$

where c, d, f , and g are unknown positive integers and to be determined later; a_n and b_m are constants.

By balancing the highest order of linear term ϕ'' with the highest order of nonlinear term ϕ^3 , the values of c and f can be determined easily.

Since

$$\begin{aligned}\phi'' &= \frac{c_1 \exp[(c + 3f)\xi] + \dots}{c_2 \exp[4f\xi] + \dots}, \\ \phi^3 &= \frac{c_3 \exp[(3c + f)\xi] + \dots}{c_4 \exp[4f\xi] + \dots},\end{aligned}\tag{2.6}$$

setting

$$c + 3f = 3c + f\tag{2.7}$$

leads easily to $c = f$.

Similarly, to determine d and g , we balance the lowest-order linear term of Exp-function in (2.4):

$$\begin{aligned}\phi'' &= \frac{\dots + d_1 \exp[-(d + 3g)\xi]}{\dots + d_2 \exp[-3g\xi]}, \\ \phi^3 &= \frac{\dots + d_3 \exp[-(3d + g)\xi]}{\dots + d_4 \exp[-3g\xi]}.\end{aligned}\tag{2.8}$$

This requires

$$-(d + 3g) = -(3d + g),\tag{2.9}$$

which leads to $g = d$.

We can freely choose the values of c and d , but we will illustrate that the final solution does not strongly depend on the choice of the values of c and d [19, 28]. Choosing $f = c = 1$ and $d = g = 1$ for simplicity causes the trial function (2.5) to become

$$\phi(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}.\tag{2.10}$$

By substituting (2.10) into (2.4), we get

$$\frac{1}{Q} (k_3 \exp(3\xi) + k_2 \exp(2\xi) + k_1 \exp(\xi) + k_0 + k_{-1} \exp(-\xi) + k_{-2} \exp(-2\xi) + k_{-3} \exp(-3\xi)) = 0,\tag{2.11}$$

where $Q = (b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi))^3$; k_i ($i = -3, -2, \dots, 2, 3$) are constants. If the coefficients of $\exp(n\xi)$ are set to zero, we have

$$\begin{aligned} k_3 &= 0, & k_2 &= 0, & k_1 &= 0, \\ k_0 &= 0, \\ k_{-3} &= 0, & k_{-2} &= 0, & k_{-1} &= 0. \end{aligned} \quad (2.12)$$

By solving this system of algebraic equations, we obtain the following sets of solutions.

Case A.

$$\begin{aligned} a_1 &= 0, & a_0 &= a_0, & a_{-1} &= 0, & \lambda^2 &= \frac{B}{K}, \\ b_1 &= b_1, & b_0 &= b_0, & b_{-1} &= -\frac{a_0^2}{8Bb_1}, & w &= 0. \end{aligned} \quad (2.13)$$

Case B.

$$\begin{aligned} a_1 &= 0, & a_0 &= a_0, & a_{-1} &= \frac{Bb_{-1}}{\sqrt{-B}}, & \lambda^2 &= \frac{-2B}{K}, \\ b_1 &= 0, & b_0 &= \frac{a_0}{\sqrt{-B}}, & b_{-1} &= b_{-1}, & w &= 0. \end{aligned} \quad (2.14)$$

Case C.

$$\begin{aligned} a_1 &= a_1, & a_0 &= a_0, & a_{-1} &= \frac{Bb_0^2 + a_0^2}{4a_1}, & \lambda^2 &= \frac{-2B}{K}, \\ b_1 &= \frac{a_1}{\sqrt{-B}}, & b_0 &= b_0, & b_{-1} &= \frac{1}{4} \frac{Ba_0b_0^2 + a_0^3 + b_0^3(B^2/\sqrt{-B}) + b_0(Ba_0^2/\sqrt{-B})}{a_1B(a_0/\sqrt{-B} - b_0)}, & w &= 0. \end{aligned} \quad (2.15)$$

Case D.

$$\begin{aligned} a_0 &= a_0, & a_{-1} &= 0, & a_1 &= 0, & \lambda^2 &= \frac{-B}{2K}, \\ b_0 &= \frac{a_0}{\sqrt{-B}}, & b_{-1} &= b_{-1}, & b_1 &= 0, & w &= \frac{3B}{\sqrt{-2B/K}}. \end{aligned} \quad (2.16)$$

Case E.

$$\begin{aligned} a_0 &= a_0, & a_{-1} &= 0, & a_1 &= a_1, & \lambda^2 &= \frac{-B}{2K}, \\ b_0 &= \frac{-a_0^2 + Ba_1b_{-1}/\sqrt{-B}}{Ba_0/\sqrt{-B}}, & b_{-1} &= b_{-1}, & b_1 &= \frac{a_1}{\sqrt{-B}}, & w &= \frac{3B}{\sqrt{-2B/K}}. \end{aligned} \quad (2.17)$$

Substituting (2.13)–(2.17) into (2.10) gives the generalized solitary solution

$$\phi(\xi) = \frac{a_0}{b_1 \exp(\xi) - (a_0^2/8Bb_1) \exp(-\xi)}, \quad (2.18)$$

where $\xi = \sqrt{B/K}(\pm x + \gamma)$ and the solutions

$$\phi(\xi) = \frac{a_0 + (Bb_{-1}/\sqrt{-B}) \exp(-\xi)}{a_0/\sqrt{-B} + b_{-1} \exp(-\xi)}, \quad (2.19)$$

where $\xi = \sqrt{-2B/K}(\pm x + \gamma)$, and

$$\phi(\xi) = \frac{a_1 \exp(\xi) + a_0 + (Bb_0^2 + a_0^2/4a_1) \exp(-\xi)}{(a_1/\sqrt{-B}) \exp(\xi) + b_0 + \mathfrak{S} \exp(-\xi)}, \quad (2.20)$$

where \mathfrak{S} denotes $1/4((Ba_0b_0^2 + a_0^3 + b_0^3B^2/\sqrt{-B} + b_0Ba_0^2/\sqrt{-B})/(a_1B(a_0/\sqrt{-B} - b_0)))$, and $\xi = \sqrt{-2B/K}(\pm x + \gamma)$, and

$$\phi(\xi) = \frac{a_0}{a_0/\sqrt{-B} + b_{-1} \exp(-\xi)}, \quad (2.21)$$

where $\xi = \sqrt{-B/2K}(\pm x - (3B/\sqrt{-2B/K})t + \gamma)$, and

$$\phi(\xi) = \frac{a_0 + a_1 \exp(\xi)}{(a_1/\sqrt{-B}) \exp(\xi) + (-a_0^2 + Ba_1b_{-1}/\sqrt{-B}) / (Ba_0/\sqrt{-B}) + b_{-1} \exp(-\xi)}, \quad (2.22)$$

where $\xi = \sqrt{-B/2K}(\pm x - (3B/\sqrt{-2B/K})t + \gamma)$, respectively. The choice of $a_0 = 2\sqrt{-2B}$ and $b_1 = 1$ in our solution (2.18) gives the same bell solitary wave solution presented in [34] which was obtained on using the sine-cosine method

$$v(x, t) = \sqrt{-2B} \operatorname{sech} \sqrt{\frac{B}{K}} (\pm x + \gamma). \quad (2.23)$$

Also if we set

$$\begin{aligned} b_1 &= \pm 1, & a_0 &= 2\sqrt{2B}, \\ b_1 &= -1, & a_0 &= 4\sqrt{-B}, \end{aligned} \quad (2.24)$$

in (2.18), we obtain the new solutions

$$\begin{aligned} v(x, t) &= \pm \sqrt{2B} \operatorname{csch} \sqrt{\frac{B}{K}} (\pm x + \gamma), \\ v(x, t) &= \frac{4\sqrt{B}}{\cosh \sqrt{B/K} (\pm x + \gamma) - 3 \sinh \sqrt{B/K} (\pm x + \gamma)}. \end{aligned} \quad (2.25)$$

Also, setting $a_0 = \sqrt{-B}$ and $b_{-1} = \pm 1$ causes (2.19) to lead to the new kink solitary wave solution

$$\begin{aligned} v(x, t) &= \sqrt{-B} \operatorname{coth} \sqrt{\frac{-B}{2K}} (\pm x + \gamma), \\ v(x, t) &= \sqrt{-B} \tanh \sqrt{\frac{-B}{2K}} (\pm x + \gamma). \end{aligned} \quad (2.26)$$

This solution is similar to the solution obtained in [34].

By setting

$$b_0 = 0, \quad a_1 = \sqrt{-B}, \quad a_0 = \pm 2i\sqrt{-B}, \quad (2.27)$$

in (2.20), we will have the solitary wave solution

$$v(x, t) = \sqrt{-B} \left(\tanh \sqrt{\frac{-2B}{K}} (\pm x + \gamma) \pm i \operatorname{sech} \sqrt{\frac{-2B}{K}} (\pm x + \gamma) \right), \quad (2.28)$$

where $i = \sqrt{-1}$. Now if we set $a_0 = \sqrt{-B}$ and $b_{-1} = 1$ in (2.21), we will have another kink solitary wave solution

$$v(x, t) = \frac{\sqrt{-B}}{2} + \frac{\sqrt{-B}}{2} \tanh \sqrt{\frac{-B}{8K}} \left(\pm x - \frac{3B}{\sqrt{-2B/K}} t + \gamma \right), \quad (2.29)$$

which is similar to the solution obtained in [34].

If we also set $b_{-1} = 1$ and $a_0 = -\sqrt{-B}$ in (2.21), we obtain the new kink solitary wave solution

$$v(x, t) = \frac{\sqrt{-B}}{2} + \frac{\sqrt{-B}}{2} \coth \sqrt{\frac{-B}{8K}} \left(\pm x - \frac{3B}{\sqrt{-2B/K}} t + \gamma \right). \quad (2.30)$$

Also, setting $b_{-1} = a_0$ causes (2.21) to lead to the new soliton solution

$$v(x, t) = \frac{1}{-\left(\sqrt{-B}/B\right) + \cosh \sqrt{-B/2K} \left(\pm x - \left(3B/\sqrt{-2B/K}\right) t + \gamma \right) - \xi}, \quad (2.31)$$

where ξ denotes $\sinh \sqrt{-B/2K} \left(\pm x - \left(3B/\sqrt{-2B/K}\right) t + \gamma \right)$.

By choosing $a_1 = \sqrt{-B}$, $a_{-1} = 1$, and $a_0 = i\sqrt{-B}$ in (2.22), we can find solitary wave solution

$$v(x, t) = \frac{\sqrt{-B}}{2} \left(1 + \tanh \sqrt{\frac{-B}{2K}} \left(\pm x - \frac{3B}{\sqrt{-2B/K}} t + \gamma \right) + i \operatorname{sech} \sqrt{\frac{-B}{2K}} \left(\pm x - \frac{3B}{\sqrt{-2B/K}} t + \gamma \right) \right), \quad (2.32)$$

where $i = \sqrt{-1}$.

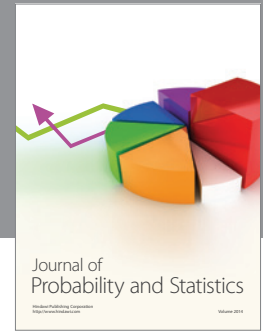
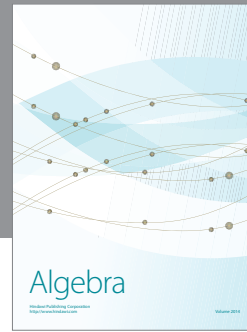
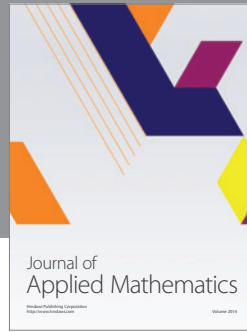
3. Conclusion

An investigation on the Brusselator reaction diffusion model was established by using the Exp-function method. During this procedure some new exact solitary wave solutions, mostly solitons and kinks solutions, were obtained as well as some special cases. In particular, Yan's solution [34] can be considered as a special case of our result, and our result can turn into kink, soliton, and bell solutions with a suitable choice of the parameters. The study reveals the power of the method.

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