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*Review Article*

# On Some Subclasses of Harmonic Functions Defined by Fractional Calculus

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The purpose of this paper is to study subclasses of normalized harmonic functions with positive real part using fractional derivative. Sharp estimates for coefficients and distortion theorems are given.



## 1. Introduction

A continuous function $f = u+iv$ is a complex-valued harmonic function in a complex domain $C$ if both $u$ and $v$ are real harmonic in $C$. In any simply connected domain $D \subseteq C$, we can write $f = h + \overline{g}$, where $h$ and $g$ are analytic in $D$. We call $h$ the analytic part and $g$ the coanalytic part of $f$. A necessary and sufficient condition for $f$ to be locally univalent and orientation-preserving in $D$ is that $|g'(z)| < |h'(z)|$ in $D$, see [1].

Denote by $H$ the class of functions $f = h + \overline{g}$ which are harmonic univalent and orientation-preserving in the open unit disk $U = \{z : |z| < 1\}$ so that $f = h + \overline{g}$ is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Therefore, for $f = h + \overline{g} \in H$, we can express $h$ and $g$ by the following power series expansion:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1.1}$$

Observe that $H$ reduces $S$, the class of normalized univalent analytic functions, if the coanalytic part of $f$ is zero.

For $f = h + \overline{g}$ given by (1.1) and $n > -1$, Murugusundaramoorthy [2] defined the Ruscheweyh derivative of the harmonic function $f = h + \overline{g}$ in $H$ by

$$D^n f(z) = D^n h(z) + \overline{D^n g(z)}, \tag{1.2}$$

where the Ruscheweh derivative of a power series $f(z) = z + \sum_{n=2}^{n} a_n z^n$ is given by

$$D^n f(z) = \frac{z}{(1-z)^{n+1}} * f. \tag{1.3}$$

The operator $*$ stands for the Hadamard product or convolution of two power series

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1.4}$$

defined by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n. \tag{1.5}$$

In [3], Owa introduced the following definition.

*Definition 1.1.* Let the function $f(z)$ be analytic in a simply connected domain of the $z$-plane containing the origin and let $0 \le \lambda < 1$. The fractional derivative of $f$ of order $\lambda$ is defined by

$$D_z^{\lambda} f(z) := \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^1 \frac{f(\zeta)}{(z-\zeta)^{\lambda}} d\zeta \quad (0 \le \lambda < 1), \tag{1.6}$$

where the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

In [4], Owa gave the relation between the fractional derivative and Ruscheweyh operator for the function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ as

$$\begin{aligned} D^{\lambda} f(z) &:= \frac{1}{\Gamma(1+\lambda)} D_z^{\lambda}\left[z^{\lambda-1} f(z)\right], \quad 0 < \lambda < 1, \\ D^0 f(z) &= \lim_{\lambda \to \infty} D^{\lambda} f(z), \\ D^1 f(z) &= \lim_{\lambda \to 1} D^{\lambda} f(z). \end{aligned} \tag{1.7}$$

Using (1.2) and the relation between the fractional derivative and Ruscheweyh operator, we define the fractional derivative of order $\lambda$, $0 \le \lambda < 1$, for the harmonic function $f = h + \overline{g}$ as

$$\begin{aligned} D_z^{\lambda}\left[z^{\lambda-1} f(z)\right] &= D_z^{\lambda}\left[z^{\lambda-1} h(z)\right] + \overline{D_z^{\lambda}\left[z^{\lambda-1} g(z)\right]}, \quad 0 < \lambda < 1, \\ D_z^0 f(z) &= \lim_{\lambda \to 0} D_z^{\lambda} f(z), \\ D_z^1 f(z) &= \lim_{\lambda \to 1} D_z^{\lambda} f(z). \end{aligned} \tag{1.8}$$

Since $D^\lambda f = D^\lambda h + \overline{D^\lambda g}$, it was proved in [1] that the harmonic function $D^\lambda f$ is starlike of order 1/2 if and only if the analytic function $D^\lambda h - D^\lambda g$ is starlike of order 1/2, and it was shown in [4, Theorem 3] that $D^\lambda h - D^\lambda g$ is starlike of order 1/2 if and only if $\operatorname{Re}\{D_z^{\lambda+1}[z^\lambda(h-g)]/D_z^\lambda[z^{\lambda-1}h-g)]\} > (1+\lambda)/2$ for $0<\lambda<1$. Since $\operatorname{Re}\{D_z^{\lambda+1}[z^\lambda(h-g)]/D_z^\lambda[z^{\lambda-1}(h-g)]\} = \operatorname{Re}(\Gamma(2+\lambda)D^{\lambda+1}(h-g)/\Gamma(1+\lambda)\underline{D^\lambda(h-g)})$, then $D^\lambda h - D^\lambda g$ is starlike of order $(1+\lambda)\Gamma(1+\lambda)/2\Gamma(2+\lambda)$, hence $D^\lambda f = D^\lambda h + \overline{D^\lambda g}$ is starlike of order $(1+\lambda)\Gamma(1+\lambda)/2\Gamma(2+\lambda)$. This means

$$\operatorname{Re}\frac{DD^\lambda f}{D^\lambda f} > \frac{(1+\lambda)\Gamma(1+\lambda)}{2\Gamma(2+\lambda)} \Longrightarrow \operatorname{Re}\frac{D_z^{\lambda+1}\left[z^\lambda f\right]}{D_z^\lambda\left[z^{\lambda-1} f\right]} > \frac{(1+\lambda)}{2}. \tag{1.9}$$

Recently, Owa and Srivastava [5] studied the linear $\Omega^\lambda$ defined by operator

$$\Omega^\lambda f(z) := \Gamma(2-\lambda) z^\lambda D_z^\lambda f(z) \quad (0 \le \lambda < 1), \tag{1.10}$$

where $f$ is normalized and analytic function on $U$.

It is easily seen that

$$\Omega^0 f = f, \qquad \Omega^1 f = zf'. \tag{1.11}$$

Analogously, we studied the linear operator $\Omega^\lambda$ defined on the harmonic function $f = h + \overline{g}$ by

$$\Omega^\lambda f(z) = \Omega^\lambda h(z) + \overline{\Omega^\lambda g(z)}, \tag{1.12}$$

where

$$\begin{aligned} \Omega^\lambda h(z) &:= \Gamma(2-\lambda) z^\lambda D_z^\lambda h(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} a_{n+1} z^{n+1}, \quad a_1 = 1, \\ \Omega^\lambda g(z) &:= \Gamma(2-\lambda) z^\lambda D_z^\lambda g(z) = \sum_{n=0}^{\infty} \frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+2-\lambda)} b_{n+1} z^{n+1}, \quad b_1 = 0. \end{aligned} \tag{1.13}$$

We will define subclasses of normalized harmonic functions obtained by the Hadamard product and using the fractional derivative.

## 2. Main results

Let $h$ and $g$ be analytic in $U$. Let $P_H$ stand for harmonic functions $f = h + \overline{g}$ so that $\operatorname{Re} f > 0$, $z \in U$ and $f(0) = 1$.

If the function $f_z + \overline{f_z} = h' + \overline{g'}$ belongs to $P_H$ for the analytic and normalized functions

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=2}^{\infty} b_n z^n, \tag{2.1}$$

then the class of functions $f = h + \overline{g}$ is denoted by $\widetilde{P}_H^0$ [6].

The function

$$t_\alpha(z) = z + \frac{1}{1+\alpha}z^2 + \cdots + \frac{1}{1+(n-1)\alpha}z^n + \cdots \tag{2.2}$$

is analytic on $U$ when $\alpha$ is a complex number different from $-1, -(1/2), -(1/3), \ldots$. For $\Omega^\lambda f \in \widetilde{P}_H^0$, we denote by $\widetilde{P}_H^{\lambda,0}(\alpha)$ the class of functions defined by

$$\Omega^\lambda F = \Omega^\lambda f * (t_\alpha + \overline{t_\alpha}). \tag{2.3}$$

Therefore,

$$\begin{aligned}
\Omega^\lambda F &= \Omega^\lambda H + \overline{\Omega^\lambda g} \\
&= z + \sum_{n=2}^{\infty} \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} a_n z^n + \sum_{n=2}^{\infty} \overline{\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} b_n z^n} \\
&= z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n}, \quad z \in U
\end{aligned} \tag{2.4}$$

is in $\widetilde{P}_H^{\lambda,0}(\alpha)$. Conversely, if $\Omega^\lambda F$ is in the form (2.4), with $a_n, b_n$ being the coefficients of $\Omega^\lambda f \in \widetilde{P}_H^0$, then $\Omega^\lambda F = \widetilde{P}_H^{\lambda,0}(\alpha)$.

Note that $\widetilde{P}_H^{0,0}(\alpha) \equiv \widetilde{P}_H^0(\alpha)$ [7] and $\widetilde{P}_H^{0,0}(0) \equiv \widetilde{P}_H^0$.

**Theorem 2.1.** *If $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f \in \widetilde{P}_H^0$ so that*

$$\alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}(\Omega^\lambda F)_{\overline{z}}(z)\right] + (1-\alpha)\Omega^\lambda F(z) = \Omega^\lambda f(z). \tag{2.5}$$

*Conversely, for any function $f$ such that $\Omega^\lambda f \in \widetilde{P}_H^0$, there exists $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$ satisfying* (2.5).

*Proof.* Let $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$. If $\Omega^\lambda f \in \widetilde{P}_H^0$, then since

$$\alpha z t'_\alpha(z) + (1-\alpha)t_\alpha(z) = t_0(z) \tag{2.6}$$

as $\Omega^\lambda F = \Omega^\lambda f * (t_\alpha + \overline{t_\alpha})$, we obtain that

$$\Omega^\lambda f(z) = \alpha\left[\Omega^\lambda f(z) * \left(z t'_\alpha + \overline{z t'_\alpha}\right)\right] + (1-\alpha)\left[\Omega^\lambda f(z) * \left(t_\alpha + \overline{t_\alpha}\right)\right]. \tag{2.7}$$

Therefore,

$$\Omega^\lambda f(z) = \alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}\left(\Omega^\lambda F\right)_{\overline{z}}(z)\right] + (1-\alpha)\Omega^\lambda F(z). \tag{2.8}$$

Conversely, for $\Omega^\lambda f \in \widetilde{P}_H^0$, from (2.1), (2.2), and (2.5),

$$\begin{aligned} &z + \sum_{n=2}^{\infty} \frac{\Gamma(n+2)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} a_n z^n + \sum_{n=2}^{\infty} \overline{\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)} b_n z^n} \\ &\quad = z + \sum_{n=2}^{\infty} [1+(n-1)\alpha] A_n z^n + \sum_{n=2}^{\infty} \overline{[1+(n-1)\alpha] B_n z^n}, \end{aligned} \tag{2.9}$$

where

$$A_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} a_n, \qquad B_n = \frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)[1+(n-1)\alpha]} b_n. \tag{2.10}$$

Therefore,

$$\Omega^\lambda F = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=2}^{\infty} \overline{B_n z^n} = \Omega^\lambda f * \left[t_\alpha(z) + \overline{t_\alpha}(z)\right]. \tag{2.11}$$

□

**Theorem 2.2.** *A function $\Omega^\lambda F$ of the form* (2.4) *belongs to $\widetilde{P}_H^{\lambda,0}(\alpha)$, if and only if*

$$\operatorname{Re}\left\{z(\Omega^\lambda H(z))'' + \overline{\alpha}(\Omega^\lambda G(z))'' + (\Omega^\lambda H(z))' + (\Omega^\lambda G(z))'\right\} > 0, \quad z \in U. \tag{2.12}$$

*Proof.* If $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then from Theorem 2.1,

$$\alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0 \tag{2.13}$$

and $(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'} \in P_H$. Hence

$$\begin{aligned} 0 &< \operatorname{Re}\left\{(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'}\right\} \\ &\times \operatorname{Re}\left\{\alpha z(\Omega^\lambda H)'' + \alpha(\Omega^\lambda H)' + (1-\alpha)(\Omega^\lambda H)' + \overline{\alpha} z(\Omega^\lambda G)'' + \overline{\alpha}(\Omega^\lambda G)' + (1-\overline{\alpha})(\Omega^\lambda G)'\right\} \\ &\times \operatorname{Re}\left\{z(\alpha(\Omega^\lambda H)'' + \overline{\alpha}(\Omega^\lambda G)'') + (\Omega^\lambda H)' + (\Omega^\lambda G)'\right\}. \end{aligned} \tag{2.14}$$

Conversely, if the function $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G}$ of the form (2.4) satisfies (2.10), then by Theorem 2.1 $(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'} \in P_H$ and the following function holds:

$$\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} = \alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] \in \widetilde{P}_H^0. \tag{2.15}$$

Then by Theorem 2.1, $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$. □

**Theorem 2.3.** $\widetilde{P}_H^{\lambda,0}(\alpha)$ *is convex and compact.*

*Proof.* Let $\Omega^\lambda F_1 = \Omega^\lambda H_1 + \overline{\Omega^\lambda G_1}$, $\Omega^\lambda F_2 = \Omega^\lambda H_2 + \overline{\Omega^\lambda G_2} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and let $\mu \in [0,1]$. Then

$$\begin{aligned}
&\operatorname{Re}\{z[\alpha(\mu(\Omega^\lambda H_1(z))'' + (1-\mu)(\Omega^\lambda H_2(z))'') + \overline{\alpha}(\Omega^\lambda G_1(z))'' + (1-\mu)(\Omega^\lambda G_2(z))'')] \\
&\quad + \mu[(\Omega^\lambda H_1(z))' + (\Omega^\lambda G_1(z))'] + (1-\mu)[(\Omega^\lambda H_2(z))' + (\Omega^\lambda G_2(z))']\} \\
&\quad = \mu \operatorname{Re}\{z[\alpha(\Omega^\lambda H_1(z))'' + \overline{\alpha}(\Omega^\lambda G_1(z))''] + (\Omega^\lambda H_1(z))' + (\Omega^\lambda G_1(z))'\} \\
&\qquad + (1-\mu)\operatorname{Re}\{z[\alpha(\Omega^\lambda H_2(z))'' + \overline{\alpha}(\Omega^\lambda G_2(z))''] + (\Omega^\lambda H_2(z))' + (\Omega^\lambda G_2(z))'\} \\
&\quad > 0.
\end{aligned} \tag{2.16}$$

Hence from Theorem 2.2, $\mu\Omega^\lambda F_1 + (1-\mu)\Omega^\lambda F_2 \in \widetilde{P}_H^{\lambda,0}(\alpha)$. Therefore, $\widetilde{P}_H^{\lambda,0}(\alpha)$ is convex.

Now, let $\Omega^\lambda F_n = \Omega^\lambda H_n + \overline{\Omega^\lambda G_n} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and let $\Omega^\lambda F_n \to \Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G}$. By Theorem 2.2,

$$\alpha\left[z(\Omega^\lambda H_n)' + \overline{z(\Omega^\lambda G_n)'}\right] + (1-\alpha)\left[\Omega^\lambda H_n + \overline{\Omega^\lambda G_n}\right] \in \widetilde{P}_H^0. \tag{2.17}$$

Since $\widetilde{P}_H^0$ is compact, see [6],

$$\alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] \in \widetilde{P}_H^0. \tag{2.18}$$

Hence by Theorem 2.1, $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$, therefore $\widetilde{P}_H^{\lambda,0}(\alpha)$ is compact. □

**Theorem 2.4.** *If* $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ *and* $|z| = r < 1$, *then*

$$\begin{aligned}
-r + 2\operatorname{In}(1+r) &\le \operatorname{Re}\{\alpha[z(\Omega^\lambda H_n)' + \overline{z(\Omega^\lambda G_n)'}] + (1-\alpha)[\Omega^\lambda H_n + \overline{\Omega^\lambda G_n}]\} \\
&\le -r - 2\operatorname{In}(1-r).
\end{aligned} \tag{2.19}$$

*Equality is obtained for the function* (2.3) *where*

$$\Omega^\lambda f = 2z + \ln(1-z) - 3\overline{z} - 3\ln(1-\overline{z}), \quad z \in U. \tag{2.20}$$

*Proof.* From Theorem 2.1, if $\Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0$ so that

$$\alpha\left[z(\Omega^\lambda H)' + \overline{z(\Omega^\lambda G)'}\right] + (1-\alpha)\left[\Omega^\lambda H + \overline{\Omega^\lambda G}\right] = \Omega^\lambda f. \tag{2.21}$$

Since by [5, Proposition 2.2]

$$-r + 2\ln(1+r) \le \operatorname{Re}(\Omega^\lambda f) \le -r - 2\ln(1-r), \tag{2.22}$$

this completes the proof. □

**Theorem 2.5.** *If $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and* Re $\alpha > 0$*, then there exists $\Omega^\lambda f \in \widetilde{P}_H^0$ so that*

$$\Omega^\lambda F = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda f)(z\zeta)d\zeta, \quad z \in U. \tag{2.23}$$

*Proof.* Since

$$t_\alpha(z) = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-1}\frac{z}{1-z\zeta}d\zeta, \quad |\zeta| \leq 1, \ \text{Re}\ \alpha > 0, \tag{2.24}$$

and for $\Omega^\lambda f = \Omega^\lambda h + \overline{\Omega^\lambda g} \in \widetilde{P}_H^0$,

$$(\Omega^\lambda h)(z) * \frac{z}{1-z\zeta} = \frac{(\Omega^\lambda h)(z\zeta)}{\zeta}, \qquad (\Omega^\lambda g)(z) * \frac{z}{1-z\zeta} = \frac{(\Omega^\lambda g)(z\zeta)}{\zeta}, \tag{2.25}$$

we have

$$\begin{aligned}(\Omega^\lambda H)(z) &= (\Omega^\lambda h)(z) * t_\alpha = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda h)(z\zeta)d\zeta,\\ (\Omega^\lambda G)(z) &= (\Omega^\lambda g)(z) * t_\alpha = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}(\Omega^\lambda g)(z\zeta)d\zeta.\end{aligned} \tag{2.26}$$

Hence $\Omega^\lambda F$ is type (2.23). □

**Theorem 2.6.** *If* Re $\alpha > 0$*, then $\widetilde{P}_0^{\lambda,0}(\alpha) \subset \widetilde{P}_H^0$.*

*Proof.* Let $\Omega^\lambda F \in \widetilde{P}_H^{\lambda,0}(\alpha)$ and Re $\alpha > 0$. Then there exists $\Omega^\lambda f \in \widetilde{P}_H^0$ so that

$$\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} = \Omega^\lambda f * \left(t_\alpha + \overline{t_\alpha}\right) = \left(\Omega^\lambda h * t_\alpha\right) + \left(\Omega^\lambda g * \overline{t_\alpha}\right). \tag{2.27}$$

Hence $0 < \text{Re}\{(\Omega^\lambda h)' + \overline{(\Omega^\lambda g)'}\} = \text{Re}\{(\Omega^\lambda h)' + (\Omega^\lambda g)'\}$ and since Re $\alpha > 0$, $\text{Re}\{(\Omega^\lambda H)' + (\Omega^\lambda G)'\} > 0$ and $\Omega^\lambda H(0) = 0$, $(\Omega^\lambda H)'(0) = 1$, $\Omega^\lambda G(0) = 0$, $(\Omega^\lambda G)'(0) = 0$. And hence $\Omega^\lambda F \in \widetilde{P}_H^0$. □

**Theorem 2.7.** *Let $\Omega^\lambda F = \Omega^\lambda H + \overline{\Omega^\lambda G} \in \widetilde{P}_H^{\lambda,0}(\alpha)$. Then*

(i)

$$\left||A_n| - |B_n|\right| \leq \frac{2\Gamma(n+1)\Gamma(2-\lambda)}{n\Gamma(n+1-\lambda)|(1+(n-1)\alpha|}, \quad n \geq 1, \tag{2.28}$$

(ii) *if* $\Omega^\lambda F$ *is sense-preserving, then*

$$\left|A_n\right| \leq \frac{2n-1}{n}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n=1,2,\ldots,$$
$$|B_n| \leq \frac{2n-3}{n}\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}, \quad n=2,3,\ldots. \tag{2.29}$$

*Proof.* By (2.10),

$$\left\|A_n\right|-|B_n\|=\frac{\Gamma(n+1)\Gamma(2-\lambda)}{\Gamma(n+1-\lambda)|1+(n-1)\alpha|}\left\|a_n\right|-|b_n\|. \tag{2.30}$$

Also, by [6, Theorem 2.3], we have

$$\left\|a_n\right|-|b_n\| \leq \frac{2}{n}. \tag{2.31}$$

The required results are obtained.

On the other hand, from (2.10), it is known [6, Corollary 2.5] that

$$\left|a_n\right| \leq \frac{2n-1}{n}, \qquad \left|b_n\right| \leq \frac{2n-3}{n}. \tag{2.32}$$

Then we get the coefficient inequalities for $\widetilde{P}_0^{\lambda,0}(\alpha)$. □

*Remark 2.8.* Taking $\lambda=0$ in Theorems 2.1–2.7, we get the similar results in [7].

**Theorem 2.9.** *Let* $\Omega^\lambda F=\Omega^\lambda H=\overline{\Omega^\lambda G}\in\widetilde{P}_H^{\lambda,0}(\alpha)$ *and sense-preserving in* $U$*, then for* $|z|=r<1$*,*

$$\left|\alpha z(\Omega^\lambda H)'+(1-\alpha)\Omega^\lambda H\right| \leq \frac{2r}{1-r}+\ln(1-r),$$
$$\left|\overline{\alpha} z(\Omega^\lambda G)'+(1-\overline{\alpha})\Omega^\lambda G\right| \leq \frac{3r-r^2}{1-r}+3\ln(1-r). \tag{2.33}$$

*Proof.* From Theorems 2.1 and 2.2, if $\Omega^\lambda F=\Omega^\lambda H+\overline{\Omega^\lambda G}\in\widetilde{P}_H^{\lambda,0}(\alpha)$, then there exists $\Omega^\lambda f=\Omega^\lambda h+\overline{\Omega^\lambda g}\in\widetilde{P}_H^0$ such that

$$\alpha z(\Omega^\lambda H)'+(1-\alpha)\Omega^\lambda H=\Omega^\lambda h,$$
$$\overline{\alpha} z(\Omega^\lambda G)'+(1-\overline{\alpha})\Omega^\lambda G=\Omega^\lambda g. \tag{2.34}$$

By [6, Theorem 3.5], we obtain the results. □

*Remark 2.10.* Taking $\lambda = 0$ and $\alpha = 0$ in Theorem 2.9, we get [6, Theorem 2.4].

## 3. Positive order

We say that the harmonic function $f = h + \overline{g}$ of the form (2.1) is in the class $P_H(\beta)$, $0 \le \beta < 1$ for $|z| = r$ if $\operatorname{Re} f > \beta$ and $f(0) = 1$.

If the function $f_z + \overline{f_{\overline{z}}} = h' + \overline{g'}$ belongs to $P_H(\beta)$ for the analytic and normalized functions $h$ and $g$ of the form (2.1), then the class of functions $f = h + \overline{g}$ is denoted by $\widehat{P}_H^{0}(\beta)$.

Denote by $\widehat{P}_H^{\lambda,0}(\beta,\alpha)$ the class of functions defined By (2.3) where $\Omega^\lambda f \in \widehat{P}_H^{0}(\beta)$.

Many of our results can be rewritten for functions in the class $\widehat{P}_H^{\lambda,0}(\beta,\alpha)$. For instance, see the following theorems.

**Theorem 3.1.** *If $\Omega^\lambda F \in \widehat{P}_H^{\lambda,0}(\beta,\alpha)$, then there exists $\Omega^\lambda f \in \widehat{P}_H^{0}(\beta)$ so that*

$$\alpha\left[z\left(\Omega^\lambda F\right)_z(z) + \overline{z}\left(\Omega^\lambda F\right)_{\overline{z}}(z)\right] + (1-\alpha)\Omega^\lambda F(z) = \Omega^\lambda f(z). \tag{3.1}$$

*Conversely, for any function $f$ such that $\Omega^\lambda f \in \widehat{P}_H^{0}(\beta)$, there exists $\Omega^\lambda F \in \widehat{P}_H^{\lambda,0}(\beta,\alpha)$ satisfying* (3.1).

**Theorem 3.2.** *A function $\Omega^\lambda F$ belongs to $\widehat{P}_H^{\lambda,0}(\beta,\alpha)$ if and only if*

$$\operatorname{Re}\left\{z\left(\Omega^\lambda H(z)\right)'' + \overline{\alpha}\left(\Omega^\lambda G(z)\right)'' + \left(\Omega^\lambda H(z)\right)' + \left(\Omega^\lambda G(z)\right)'\right\} > \beta, \quad z \in U. \tag{3.2}$$

**Theorem 3.3.** *If $\Omega^\lambda F \in \widehat{P}_H^{\lambda,0}(\beta,\alpha)$ and $\operatorname{Re}\,\alpha > 0$, then there exists $\Omega^\lambda f \in \widehat{P}_H^{0}(\beta)$ so that*

$$\Omega^\lambda F = \frac{1}{\alpha}\int_0^1 \zeta^{1/\alpha-2}\left(\Omega^\lambda f\right)(z\zeta)\,d\zeta, \quad z \in U. \tag{3.3}$$

**Theorem 3.4.** *If $\operatorname{Re}\,\alpha > 0$, then $\widehat{P}_H^{\lambda,0}(\beta,\alpha) \subset \widehat{P}_H^{0}(\beta)$.*

## References


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