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Research Article

Sharp Bounds for the General Sum-Connectivity Indices of Transformation Graphs

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Given a graph G, the general sum-connectivity index is defined as $\chi_\alpha(G) = \sum_{uv\in E(G)}(d_G(u) + d_G(v))^\alpha$, where $d_G(u)$ (or $d_G(v)$) denotes the degree of vertex u (or v) in the graph G and α is a real number. In this paper, we obtain the sharp bounds for general sum-connectivity indices of several graph transformations, including the semitotal-point graph, semitotal-line graph, total graph, and eight distinct transformation graphs G^{uvw}, where $u, v, w \in \{+, -\}$.

1. Introduction

In this paper, we consider simple, undirected, and connected graphs. Let G be the graph with vertex set $V(G)$ and edge set $E(G)$. The order and size of G are denoted by n and e, respectively. For a vertex $a \in V(G)$, $d_G(a)$ denotes the degree of a. Two vertices in G are adjacent if and only if they are end vertices of an edge, and each of the two vertices is called incident to the edge. Besides, two edges are adjacent to each other if and only if they share a common vertex. The minimum and maximum degrees of graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. We will use the notations P_n, C_n, and K_n for a path, cycle, and complete graph of order n [1], respectively.

The complement of G, denoted by $\overline{G}$, is the graph with $V(\overline{G}) = V(G)$ and two vertices in $\overline{G}$ are adjacent if and only if they are not adjacent in G. Thus, the size of $\overline{G}$ is $\binom{n}{2} - e$ and if $a \in V(\overline{G})$ then $d_{\overline{G}}(a) = n - 1 - d_G(a)$.

A topological index is a numeric quantity associated with a graph which characterizes the topology of graph. A topological index Top(G) of a graph G is equal to the topological index Top(H) of H, if and only if two graphs G and H are isomorphic. The idea of topological index appears from work done by Wiener in 1947, this index is called Wiener index. The first and second Zagreb indices have been introduced by Gutman and Trinajestić [2]. These indices are defined on the ground of vertex degrees as follows:

$$M_1(G) = \sum_{a\in V(G)} (d_G(a))^2,$$
$$M_2(G) = \sum_{ab\in E(G)} d_G(a)\, d_G(b). \quad (1)$$

The *Randić connectivity index* was defined in 1975 by Randić [3]. It has been extended to the general Randić connectivity index. The *general Randić connectivity index* (general product-connectivity index) was defined by Bollobás and Erdős [4] as follows:

$$R_{\alpha}(G)=\sum_{ab\in E(G)}\left(d_G(a)\,d_G(b)\right)^{\alpha}, \quad (2)$$

where α is a real number. Then $R_{-1/2}$ is the classical Randić connectivity index. The *sum-connectivity index* was proposed in [5]. This concept was extended to the *general sum-connectivity index* in [6], which is defined as

$$\chi_{\alpha}(G)=\sum_{ab\in E(G)}\left(d_G(a)+d_G(b)\right)^{\alpha}, \quad (3)$$

where α is a real number. Then $\chi_{-1/2}(G)$ is the classical sum-connectivity index. The *sum-connectivity index* and the *product-connectivity index* correlate well with the π-electron energy of benzenoid hydrocarbons [7].

The total graph $T(G)$ of the graph G is a graph whose vertex set is the union of $V(G)$ and $E(G)$ such that $ab \in E(T(G))$ if and only if a and b are either adjacent or incident in G [8]. Let u, v, and w be the variables having values + or −. The transformation graph G^{uvw} is a graph whose vertex set is the union of $V(G)$ and $E(G)$, and $ab \in E(G^{uvw})$ if and only if

(1) $a, b \in V(G)$; then $u = +$ or $u = -$ if a and b are adjacent or nonadjacent in G, respectively;

(2) $a, b \in E(G)$; then $v = +$ or $v = -$ if a and b are adjacent or nonadjacent in G, respectively;

(3) $a \in V(G)$ and $b \in E(G)$; then $w = +$ or $w = -$ if a and b are incident or nonincident in G, respectively.

There are eight different transformations of the given graph G. For instance, G^{+++} is the total graph $T(G)$ of G with number of vertices $n+e$ and number of edges $(1/2)M_1(G)+2e$, and G^{---} is the complement of total graph G^{+++}. For other transformations of graph, G^{++-}, G^{+-+}, and G^{+--} are the complements of G^{--+}, G^{-+-}, and G^{-++}, respectively.

The concepts of semitotal-point graph and semitotal-line graph are introduced by Sampathkumar and Chikkodimath [9]. The semitotal-point graph $T_1(G)$ is a graph whose vertex set is the union of $V(G)$ and $E(G)$, and $ab \in E(T_1(G))$ if and only if (i) a and b are adjacent vertices in G or (ii) one is a vertex of G and the other is an edge of G incident to it. Thus, semitotal-point graph has $n+e$ number of vertices and $3e$ number of edges.

The semitotal-line graph $T_2(G)$ is a graph whose vertex set is the union of $V(G)$ and $E(G)$, and $ab \in E(T_2(G))$ if and only if (i) a and b are adjacent edges in G and (ii) one is a vertex of G and the other is an edge of G incident to it. Thus, semitotal-line graph has $n+e$ number of vertices and $(1/2)M_1(G)+e$ number of edges.

Eventually, many properties of these transformation graphs can be determined. For example, the Zagreb indices of transformation graphs and total transformation graphs were calculated by Basavanagoud and Patil [10] and Hosamani and Gutman [11], respectively. Wu and Meng [12] investigated the basic properties (connectedness, graph equations and iteration, and diameter) of total transformation. Xu and Wu [13] determined the connectivity, the Hamiltonian, and the independence number of G^{-+-}. Yi and Wu [14] determined the connectivity, the Hamiltonian, and the independence number of G^{++-}.

In this paper, we obtain lower and upper bounds for the general sum-connectivity indices of the above-defined transformation graphs.

2. Main Results

In this section, we discuss the lower and upper bounds for the general sum-connectivity indices of transformation graphs defined in Section 1.

Theorem 1. *For $\alpha < 0$, we have $\gamma_1 \le \chi_{\alpha}(T_1(G)) \le \gamma_2$, where*

$$\begin{aligned}\gamma_1&=2^{\alpha}\chi_{\alpha}(G)+2^{\alpha+1}e(\Delta(G)+1)^{\alpha},\\ \gamma_2&=2^{\alpha}\chi_{\alpha}(G)+2^{\alpha+1}e(\delta(G)+1)^{\alpha};\end{aligned} \quad (4)$$

the equalities hold if and only if G is a regular graph.

Proof. Since $T_1(G)$ has $n+e$ vertices and $3e$ edges, it holds that

$$\begin{aligned}\chi_{\alpha}(T_1(G))&=\sum_{ab\in E(T_1(G))}\left(d_{T_1(G)}(a)+d_{T_1(G)}(b)\right)^{\alpha}\\ &=\sum_{\substack{ab\in E(T_1(G)),\\ a,b\in V(G)}}\left(d_{T_1(G)}(a)+d_{T_1(G)}(b)\right)^{\alpha}\\ &\quad+\sum_{\substack{ab\in E(T_1(G)),\\ a\in V(G),b\in E(G)}}\left(d_{T_1(G)}(a)+d_{T_1(G)}(b)\right)^{\alpha}.\end{aligned} \quad (5)$$

Note that if $a \in V(G)$ then $d_{T_1(G)}(a) = 2d_G(a)$ and if $a \in E(G)$ then $d_{T_1(G)}(a) = 2$. It is clear that $\delta(G) \le d_G(a)$ and $\Delta(G) \ge d_G(a)$. And these equalities hold if and only if G is a regular graph. Therefore,

$$\begin{aligned}\chi_{\alpha}(T_1(G))&=2^{\alpha}\sum_{\substack{ab\in E(T_1(G)),\\ a,b\in V(G)}}\left(d_G(a)+d_G(b)\right)^{\alpha}\\ &\quad+2^{\alpha}\sum_{\substack{ab\in E(T_1(G)),\\ a\in V(G),b\in E(G)}}\left(d_G(a)+1\right)^{\alpha}\\ &\ge 2^{\alpha}\chi_{\alpha}(G)+2^{\alpha+1}e(\Delta(G)+1)^{\alpha}.\end{aligned} \quad (6)$$

Similarly, we can compute

$$\chi_{\alpha}(T_1(G)) \le 2^{\alpha}\chi_{\alpha}(G)+2^{\alpha+1}e(\delta(G)+1)^{\alpha}. \quad (7)$$

The two equalities in (6) and (7) obviously hold if and only if G and H are regular, respectively. □

Example 2. By Theorem 1, the general sum-connectivity indices of some semitotal-point graphs are given below:

(1) $n(8^{\alpha}+2\times 6^{\alpha})-3\times 8^{\alpha} \le \chi_{\alpha}(T_1(P_n)) \le 2^{2\alpha}n(2^{\alpha}+2)+2\times 6^{\alpha}-3\times 8^{\alpha}-2^{2\alpha+1}$.

(2) $\chi_\alpha(T_1(C_n)) = 2^\alpha n(4^\alpha + 2 \times 3^\alpha)$.

(3) $\chi_\alpha(T_1(K_n)) = 2^\alpha n(n-1)[2^{\alpha-1}(n-1)^\alpha + n^\alpha]$.

Theorem 3. *If $\alpha < 0$ then $\gamma_1 \le \chi_\alpha(T_2(G)) \le \gamma_2$, where*

$$\gamma_1 = 2^{2\alpha-1} M_1(G)\Delta^\alpha(G) + e\Delta^\alpha(G)\left[2\times 3^\alpha - 4^\alpha\right],$$
$$\gamma_2 = 2^{2\alpha-1} M_1(G)\delta^\alpha(G) + e\delta^\alpha(G)\left[2\times 3^\alpha - 4^\alpha\right]; \quad (8)$$

the equalities hold if and only if G is a regular graph.

Proof. Since $|V(T_2(G))| = n + e$ and $|E(T_2(G))| = (1/2)M_1(G) + e$, we have

$$\begin{aligned}\chi_\alpha(T_2(G)) &= \sum_{ab\in E(T_2(G))} \left(d_{T_2(G)}(a) + d_{T_2(G)}(b)\right)^\alpha \\ &= \sum_{\substack{ab\in E(T_2(G)),\\ a,b\in E(G)}} \left(d_{T_2(G)}(a) + d_{T_2(G)}(b)\right)^\alpha \\ &\quad + \sum_{\substack{ab\in E(T_2(G)),\\ a\in V(G), b\in E(G)}} \left(d_{T_2(G)}(a) + d_{T_2(G)}(b)\right)^\alpha.\end{aligned} \quad (9)$$

Note that if $a \in E(G)$ then $d_{T_2(G)}(a) = d_G(w_i) + d_G(w_j)$ and if $a \in V(G)$ then $d_{T_2(G)}(a) = d_G(a)$. Therefore, we have

$$\begin{aligned}\chi_\alpha(T_2(G)) &= \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G),\\ w_i\neq w_k}} \left[\left(d_G(w_i) + d_G(w_j)\right)\right. \\ &\quad \left. + \left(d_G(w_j) + d_G(w_k)\right)\right]^\alpha + \sum_{\substack{ab\in E(T_2(G)),\\ a\in V(G),\\ b=ax\in E(G),\\ x\in V(G)}} \left[d_G(a)\right. \\ &\quad \left. + \left(d_G(a) + d_G(x)\right)\right]^\alpha = \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G),\\ w_i\neq w_k}} \left[d_G(w_i)\right. \\ &\quad \left. + 2d_G(w_j) + d_G(w_k)\right]^\alpha + \sum_{\substack{a\in V(G),\\ b=ax\in E(G),\\ x\in V(G)}} \left[d_G(a)\right. \\ &\quad \left. + \left(d_G(a) + d_G(x)\right)\right]^\alpha.\end{aligned} \quad (10)$$

Since $d_G(a) \ge \delta(G)$ and $d_G(a) \le \Delta(G)$, each equality holds if and only if G is a regular graph.

After simplification we get

$$\begin{aligned}\chi_\alpha(T_2(G)) &\ge [4\Delta(G)]^\alpha \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G),\\ w_i\neq w_k}} 1 \\ &\quad + [3\Delta(G)]^\alpha \sum_{\substack{ab\in E(T_2(G)),\\ a\in V(G),\\ b=ax\in E(G),\\ x\in V(G)}} 1 \\ &= [4\Delta(G)]^\alpha \cdot \left[E\left(T_2(G)\right) - 2e\right] \\ &\quad + [3\Delta(G)]^\alpha \cdot (2e) \\ &= 2^{2\alpha-1} M_1(G)\Delta^\alpha(G) \\ &\quad + e\Delta^\alpha(G)\left[2\times 3^\alpha - 4^\alpha\right].\end{aligned} \quad (11)$$

Similarly, we can calculate

$$\begin{aligned}\chi_\alpha(T_2(G)) &\le 2^{2\alpha-1} M_1(G)\delta^\alpha(G) \\ &\quad + e\delta^\alpha(G)\left[2\times 3^\alpha - 4^\alpha\right].\end{aligned} \quad (12)$$

Obviously the equalities in (11) and (12) hold if and only if G is a regular graph. □

Example 4. By Theorem 3, the general sum-connectivity indices of some semitotal-line graphs are given below:

(1) $2^\alpha n(4^\alpha + 2\times 3^\alpha) - 2^{\alpha+1}(4^\alpha - 3^\alpha) \le \chi_\alpha(T_2(P_n)) \le n(4^\alpha + 2\times 3^\alpha) - 2(4^\alpha + 2\times 3^\alpha)$.

(2) $\chi_\alpha(T_2(C_n)) = 2^\alpha n(4^\alpha + 2\times 3^\alpha)$.

(3) $\chi_\alpha(T_2(K_n)) = n^\alpha(n-1)^{\alpha+1}[2^{2\alpha-1}n + 2^{2\alpha} + 3^\alpha]$.

Theorem 5. *Let $\alpha < 0$. Then $\gamma_1 \le \chi_\alpha(T(G)) \le \gamma_2$, where*

$$\gamma_1 = 2^\alpha \chi_\alpha(G) + 2^{2\alpha-1} M_1(G)\Delta^\alpha(G) + 4^\alpha e\Delta^\alpha(G),$$
$$\gamma_2 = 2^\alpha \chi_\alpha(G) + 2^{2\alpha-1} M_1(G)\delta^\alpha(G) + 4^\alpha e\delta^\alpha(G); \quad (13)$$

the equalities hold if and only if G is a regular graph.

Proof. Since $|V(T(G))| = n+e$ and $|E(T(G))| = (1/2)M_1(G) + 2e$, we have

$$\begin{aligned}\chi_\alpha(T(G)) &= \sum_{ab\in E(T(G))} \left(d_{T(G)}(a) + d_{T(G)}(b)\right)^\alpha \\ &= \sum_{\substack{ab\in E(T(G)),\\ a,b\in V(G)}} \left(d_{T(G)}(a) + d_{T(G)}(b)\right)^\alpha \\ &\quad + \sum_{\substack{ab\in E(T(G)),\\ a,b\in E(G)}} \left(d_{T(G)}(a) + d_{T(G)}(b)\right)^\alpha \\ &\quad + \sum_{\substack{ab\in E(T(G)),\\ a\in V(G), b\in E(G)}} \left(d_{T(G)}(a) + d_{T(G)}(b)\right)^\alpha.\end{aligned} \quad (14)$$

Note that $d_{T(G)}(a) = 2d_G(a)$ for $a \in V(G)$ and $d_{T(G)}(a) = d_G(w_i) + d_G(w_j)$ for $a \in E(G)$. So

$$\begin{aligned}\chi_\alpha(T(G)) &= 2^\alpha \sum_{\substack{ab\in E(G),\\ a,b\in V(G)}} (d_G(a)+d_G(b))^\alpha \\ &+ \sum_{\substack{a=w_iw_j\in E(G),\\ b=w_jw_k\in E(G),\\ w_i\neq w_k}} \left[\left(d_G(w_i)+d_G(w_j)\right) + \left(d_G(w_j)+d_G(w_k)\right)\right]^\alpha + \sum_{\substack{b=ax\in E(G),\\ a\in V(G),x\in V(G)}} [2d_G(a) + (d_G(a)+d_G(x))]^\alpha = 2^\alpha \cdot \chi_\alpha(G) \\ &+ \sum_{\substack{a=w_iw_j\in E(G),\\ b=w_jw_k\in E(G),\\ w_i\neq w_k}} \left[d_G(w_i)+2d_G(w_j)+d_G(w_k)\right]^\alpha \\ &+ \sum_{\substack{b=ax\in E(G),\\ a\in V(G),x\in V(G)}} [2d_G(a)+(d_G(a)+d_G(x))]^\alpha .\end{aligned} \tag{15}$$

Note that $d_G(a) \le \Delta(G)$ and $d_G(a) \ge \delta(G)$. The equalities hold if and only if G is a regular graph.

After simplification, we get

$$\begin{aligned}\chi_\alpha(T(G)) &\ge 2^\alpha \chi_\alpha(G) + [4\Delta(G)]^\alpha \left[\frac{1}{2}M_1(G) - e\right] + [4\Delta(G)]^\alpha [2e] \\ &= 2^\alpha \chi_\alpha(G) + 2^{2\alpha-1}\Delta^\alpha(G) M_1(G) + 2^{2\alpha} e \Delta^\alpha(G).\end{aligned} \tag{16}$$

Similarly, we can compute

$$\chi_\alpha(T(G)) \le 2^\alpha \chi_\alpha(G) + 2^{2\alpha-1}\delta^\alpha(G) M_1(G) + 2^{2\alpha} e \delta^\alpha(G). \tag{17}$$

Since $\chi_\alpha(G) \ge 2^\alpha e \Delta^\alpha(G)$, we can also write the results above as

$$4^\alpha \delta^\alpha(G)\left[\frac{1}{2}M_1(G)+2e\right] \le \chi_\alpha(T(G)) \le 4^\alpha \Delta^\alpha(G)\left[\frac{1}{2}M_1(G)+2e\right]. \tag{18}$$

Thus, if G is a regular graph, then we obtain the equality in (16), (17), and (18). □

Example 6. By Theorem 5, the general sum-connectivity indices of some total graphs are given below:

(1) $2^{3\alpha+2}n + 2\times 6^\alpha - 7\times 8^\alpha \le \chi_\alpha(T(P_n)) \le 4^\alpha n(2^\alpha+1) + 2^\alpha(2\times 3^\alpha - 3\times 4^\alpha + -2^{\alpha+2})$.

(2) $\chi_\alpha(T(C_n)) = 4\times 8^\alpha n$.

(3) $\chi_\alpha(T(K_n)) = 2^{2\alpha-1}n(n-1)^{\alpha+1}(n+1)$.

Theorem 7. *Let $\alpha < 0$. Then $\gamma_1 \le \chi_\alpha(G^{---}) \le \gamma_2$, where*

$$\begin{aligned}\gamma_1 &= 2^\alpha\left[\frac{1}{2}M_1(G)+2e\right](e+n-1-2\Delta^\alpha(G))^\alpha, \\ \gamma_2 &= 2^\alpha\left[\frac{1}{2}M_1(G)+2e\right](e+n-1-2\delta^\alpha(G))^\alpha;\end{aligned} \tag{19}$$

the equalities hold if and only if G is a regular graph.

Proof. For a given graph G, since $G^{---} \cong \overline{G^{+++}}$ and $G^{+++} = T(G)$, then $|V(G^{---})| = n+e$, $|E(G^{---})| = \binom{e+n}{2} - (1/2)M_1(G) - 2e$, and $2\Delta(\overline{G}) = e+n-1-2\Delta(G)$. Using these values, we can compute the required results. □

Theorem 8. *Let $\alpha < 0$. Then $\gamma_1 \le \chi_\alpha(G^{++-}) \le \gamma_2$, where*

$$\begin{aligned}\gamma_1 &= 2^\alpha e^{\alpha+1} + 2^\alpha\left[\frac{1}{2}M_1(G)-e\right][n-4+2\Delta(G)]^\alpha \\ &+ \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right] \cdot [e+n-4+2\Delta(G)]^\alpha, \\ \gamma_2 &= 2^\alpha e^{\alpha+1} + 2^\alpha\left[\frac{1}{2}M_1(G)-e\right][n-4+2\delta(G)]^\alpha \\ &+ \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right] \cdot [e+n-4+2\delta(G)]^\alpha;\end{aligned} \tag{20}$$

the equalities hold if and only if G is a regular graph.

Proof. Since $|V(G^{++-})| = n+e$ and $|E(G^{++-})| = \binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} + (1/2)M_1(G) - 2m$,

$$\begin{aligned}\chi_\alpha(G^{++-}) &= \sum_{ab\in E(G^{++-})} (d_{G^{++-}}(a)+d_{G^{++-}}(b))^\alpha \\ &= \sum_{\substack{ab\in E(G^{++-}),\\ a,b\in V(G)}} (d_{G^{++-}}(a)+d_{G^{++-}}(b))^\alpha \\ &+ \sum_{\substack{ab\in E(G^{++-}),\\ a,b\in E(G)}} (d_{G^{++-}}(a)+d_{G^{++-}}(b))^\alpha \\ &+ \sum_{\substack{ab\in E(G^{++-}),\\ a\in V(G),b\in E(G)}} (d_{G^{++-}}(a)+d_{G^{++-}}(b))^\alpha .\end{aligned} \tag{21}$$

Note that if $a \in V(G)$ then $d_{G^{++-}}(a) = e$ and if $a \in E(G)$ then $d_{G^{++-}}(a) = d_G(w_i) + d_G(w_j) + n - 4$

$$\begin{aligned}\chi_\alpha(G^{++-}) &= \sum_{\substack{ab\in E(G),\\ a,b\in V(G)}} [2e]^\alpha + \sum_{\substack{a=w_iw_j\in E(G),\\ b=w_jw_k\in E(G),\\ w_i\neq w_k}} \left[\left(d_G(w_i) + d_G(w_j) + n - 4\right)\right. \\ &\quad \left. + \left(d_G(w_j) + d_G(w_k) + n - 4\right)\right]^\alpha + \sum_{\substack{b=xy\in E(G),\\ a\in V(G),\\ a\notin\{x,y\}}} [e \\ &\quad + \left(d_G(x) + d_G(y) + n - 4\right)]^\alpha = 2^\alpha e^{\alpha+1} \\ &\quad + \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G)}} \left[2n - 8 + d_G(w_i) + 2d_G(w_j)\right. \\ &\quad \left. + d_G(w_k)\right]^\alpha + \sum_{\substack{b=xy\in E(G),\\ a\in V(G),\\ a\notin\{x,y\}}} [e + n - 4 + d_G(x) \\ &\quad + d_G(y)]^\alpha .\end{aligned} \tag{22}$$

Note that $d_G(a) \le \Delta(G)$ and $d_G(a) \ge \delta(G)$. The equalities hold if and only if G is a regular graph. After simplification, we get

$$\begin{aligned}\chi_\alpha(G^{++-}) &\ge 2^\alpha e^{\alpha+1} + 2^\alpha \left[\frac{1}{2}M_1(G) - e\right] \\ &\quad \cdot [n - 4 + 2\Delta(G)]^\alpha \\ &\quad + \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right] \\ &\quad \cdot (e + n - 4 + 2\Delta(G))^\alpha .\end{aligned} \tag{23}$$

Similarly, we can compute

$$\begin{aligned}\chi_\alpha(G^{++-}) &\le 2^\alpha e^{\alpha+1} + 2^\alpha \left[\frac{1}{2}M_1(G) - e\right] \\ &\quad \cdot (n - 4 + 2\delta(G))^\alpha \\ &\quad + \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right] \\ &\quad \cdot (e + n - 4 + 2\delta(G))^\alpha .\end{aligned} \tag{24}$$

The equalities in (23) and (24) obviously hold if and only if G is a regular graphs. □

Theorem 9. *Let $\alpha < 0$. Then $\gamma_1 \le \chi_\alpha(G^{-+-}) \le \gamma_2$, where*

$$\begin{aligned}\gamma_1 &= 2^\alpha [e + n - 1 - 2\Delta(G)]^\alpha \left[\binom{n}{2} - e\right] \\ &\quad + 2^\alpha [n - 4 + 2\Delta(G)]^\alpha \left[\frac{1}{2}M_1(G) - e\right] \\ &\quad + (e + 2n - 5)^\alpha \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right], \\ \gamma_2 &= 2^\alpha [e + n - 1 - 2\delta(G)]^\alpha \left[\binom{n}{2} - e\right] \\ &\quad + 2^\alpha [n - 4 + 2\delta(G)]^\alpha \left[\frac{1}{2}M_1(G) - e\right] \\ &\quad + (e + 2n - 5)^\alpha \left[\binom{e+n}{2} - \binom{n}{2} - \binom{e}{2} - 2e\right];\end{aligned} \tag{25}$$

the equalities hold if and only if G is a regular graph.

Proof. Since $|V(G^{-+-})| = n + e$ and $|E(G^{-+-})| = \binom{e+n}{2} - \binom{e}{2} + (1/2)M_1(G) - 4e$,

$$\begin{aligned}\chi_\alpha(G^{-+-}) &= \sum_{ab\in E(G^{-+-})} \left(d_{G^{-+-}}(a) + d_{G^{-+-}}(b)\right)^\alpha \\ &= \sum_{\substack{ab\in E(G^{-+-}),\\ a,b\in V(G)}} \left(d_{G^{-+-}}(a) + d_{G^{-+-}}(b)\right)^\alpha \\ &\quad + \sum_{\substack{ab\in E(G^{-+-}),\\ a,b\in E(G)}} \left(d_{G^{-+-}}(a) + d_{G^{-+-}}(b)\right)^\alpha \\ &\quad + \sum_{\substack{ab\in E(G^{-+-}),\\ a\in V(G), b\in E(G)}} \left(d_{G^{-+-}}(a) + d_{G^{-+-}}(b)\right)^\alpha .\end{aligned} \tag{26}$$

Note that $d_{G^{-+-}}(a) = e + n - 1 - 2d_G(a)$ for $a \in V(G)$ and $d_{G^{-+-}}(a) = d_G(w_i) + d_G(w_j) + n - 4$ for $a \in E(G)$. Then

$$\begin{aligned}\chi_\alpha(G^{-+-}) &= \sum_{\substack{ab\in E(G^{-+-}),\\ a,b\in V(G)}} \left[(e + n - 1 - 2d_G(a))\right. \\ &\quad \left. + (e + n - 1 - 2d_G(b))\right]^\alpha \\ &\quad + \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G),\\ w_i\neq w_k}} \left[\left(d_G(w_i) + d_G(w_j) + n - 4\right)\right. \\ &\quad \left. + \left(d_G(w_j) + d_G(w_k) + n - 4\right)\right]^\alpha\end{aligned}$$

$$+ \sum_{\substack{ab\in E(G^{-+-}),\\ a\in V(G), b=xy\in E(G)}} [(e+n-1-2d_G(a)) + (d_G(x)+d_G(y)+n-4)]^\alpha = 2^\alpha \sum_{\substack{ab\notin E(G),\\ a,b\in V(G)}} [e+n-1-d_G(a)-d_G(b)]^\alpha + \sum_{\substack{w_iw_j\in E(G),\\ w_jw_k\in E(G),\\ w_i\neq w_k}} [2n-8+d_G(w_i)+2d_G(w_j)+d_G(w_k)]^\alpha + \sum_{\substack{b=xy\in E(G),\\ a\notin\{x,y\},\\ a\in V(G)}} [e+2n-5-2d_G(a)+d_G(x)+d_G(y)]^\alpha. \quad (27)$$

Note that $d_G(a) \le \Delta(G)$ and $d_G(a) \ge \delta(G)$. The equalities hold if and only if G is a regular graph.

After simplification, we get

$$\chi_\alpha(G^{-+-}) \ge 2^\alpha [e+n-1-2\Delta(G)]^\alpha \left[\binom{n}{2}-e\right] + 2^\alpha [n-4+2\Delta(G)]^\alpha \left[\frac{1}{2}M_1(G)-e\right] + (e+2n-5)^\alpha \left[\binom{e+n}{2}-\binom{n}{2}-\binom{e}{2}-2e\right]. \quad (28)$$

Similarly, we calculate

$$\chi_\alpha(G^{-+-}) \le 2^\alpha [e+n-1-2\delta(G)]^\alpha \cdot [n-4+2\delta(G)]^\alpha \left[\binom{n}{2}-e\right] + 2^\alpha \left[\frac{1}{2}M_1(G)-e\right] + \left[\binom{e+n}{2}-\binom{n}{2}-(e+2n-5)^\alpha\binom{e}{2}-2e\right]. \quad (29)$$

If G is a regular graph then we obtain the equalities in (28) and (29). □

In fully analogous manner, we also arrive at the following.

Theorem 10. *If $\alpha < 0$ then*

(1) $\gamma_1 \le \chi_\alpha(G^{+--}) \le \gamma_2$, *where*

$$\gamma_1 = 2^\alpha e^{\alpha+1} + 2^\alpha \left[\binom{e}{2}-\frac{1}{2}M_1(G)+e\right] \cdot [n+e-1-2\Delta(G)]^\alpha + \left[\binom{e+n}{2}-\binom{n}{2}-\binom{e}{2}-2e\right] \cdot [2e+n-1-2\Delta(G)]^\alpha,$$

$$\gamma_2 = 2^\alpha e^{\alpha+1} + 2^\alpha \left[\binom{e}{2}-\frac{1}{2}M_1(G)+e\right] \cdot [n+e-1-2\delta(G)]^\alpha + \left[\binom{e+n}{2}-\binom{n}{2}-\binom{e}{2}-2e\right] \cdot [2e+n-1-2\delta(G)]^\alpha; \quad (30)$$

(2) $\gamma_1 \le \chi_\alpha(G^{--+}) \le \gamma_2$, *where*

$$\gamma_1 = 2^\alpha \left[\binom{n}{2}-e\right](n-1)^\alpha + 2^\alpha \left[\binom{e}{2}-\frac{1}{2}M_1(G)+e\right][e-3-2\Delta(G)]^\alpha + 2e[e+n+2-2\Delta(G)]^\alpha,$$

$$\gamma_2 = 2^\alpha \left[\binom{n}{2}-e\right](n-1)^\alpha + 2^\alpha \left[\binom{e}{2}-\frac{1}{2}M_1(G)+e\right][e-3-2\delta(G)]^\alpha + 2e[e+n+2-2\delta(G)]^\alpha; \quad (31)$$

(3) $\gamma_1 \le \chi_\alpha(G^{-++}) \le \gamma_2$, *where*

$$\gamma_1 = 2^\alpha \left[\binom{n}{2}-e\right](n-1)^\alpha + 4^\alpha\Delta^\alpha(G)\left[\frac{1}{2}M_1(G)-e\right] + 2e[n-1+2\Delta(G)]^\alpha,$$

$$\gamma_2 = 2^\alpha \left[\binom{n}{2}-e\right](n-1)^\alpha + 4^\alpha\delta^\alpha(G)\left[\frac{1}{2}M_1(G)-e\right] + 2e[n-1+2\delta(G)]^\alpha; \quad (32)$$

(4) $\gamma_1 \le \chi_\alpha(G^{+-+}) \le \gamma_2$, *where*

$$\gamma_1 = 2^{2\alpha} e\Delta^\alpha(G) + \left[\binom{e}{2}-\frac{1}{2}M_1(G)+e\right][3-2\Delta(G)]^\alpha + 2e(e+3)^\alpha,$$

$$\begin{aligned}\gamma_2 &= 2^{2\alpha}e\delta^{\alpha}(G) \\ &+ \left[\binom{e}{2} - \frac{1}{2}M_1(G) + e\right][3 - 2\delta(G)]^{\alpha} \\ &+ 2e(e+3)^{\alpha}.\end{aligned} \quad (33)$$

In all the above cases, the equalities hold if and only if G is a regular graph, respectively.

3. Conclusion

In this paper, we obtain the sharp lower and upper bounds for general sum-connectivity indices of the semitotal-point graph, the semitotal-line graph, the total graph, and the eight distinct transformation graphs G^{uvw}, where $u, v, w \in \{+, -\}$ in terms of the order, minimum degree, and maximum degree of a graph. Moreover, the extremal graphs achieving these bounds have been described.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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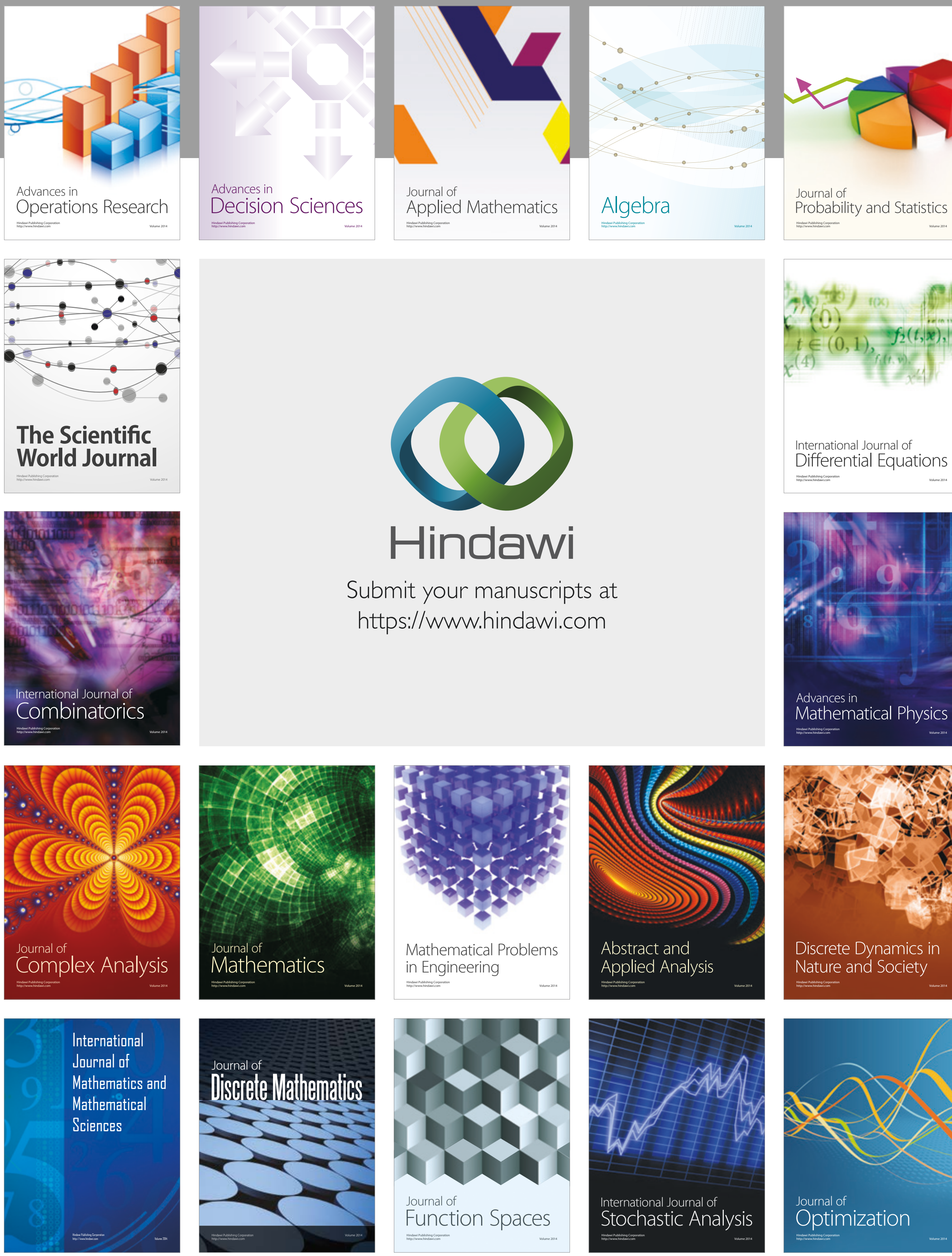
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