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Research Article

On the Complex Inversion Formula and Admissibility for a Class of Volterra Systems

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This paper studies Volterra integral evolution equations of convolution type from the point of view of complex inversion formula and the admissibility in the Salamon-Weiss sens. We first present results on the validity of the inverse formula of the Laplace transform for the resolvent families associated with scalar Volterra integral equations of convolution type in Banach spaces, which extends and improves the results in Hille and Philllips (1957) and Cioranescu and Lizama (2003, Lemma 5), respectively, including the stronger version for a class of scalar Volterra integrodifferential equations of convolution type on unconditional martingale differences UMD spaces, provided that the leading operator generates a C_0-semigroup. Next, a necessary and sufficient condition for L^p-admissibility ($p \in [1, \infty[$) of the system's control operator is given in terms of the UMD-property of its underlying control space for a wider class of Volterra integrodifferential equations when the leading operator is not necessarily a generator, which provides a generalization of a result known to hold for the standard Cauchy problem (Bounit et al., 2010, Proposition 3.2).

1. Introduction

The purpose of this paper is to analyze conditions for the inversion formula and the L^p-admissibility for control operators for the solution of the following integrodifferential equation:

$$\dot{x}(t) = Ax(t) + \int_0^t k(t-s)\,Ax(s)\,ds + Bu(t), \quad t \geq 0, \qquad x(0) = x_0 \in X, \tag{1}$$

which has a "big" intersection with the class of scalar Volterra integral equations. Here we assume that A is a closed linear densely defined operator in a Banach space X, and the kernel k belonging to $L^1_{\text{loc}}(\mathbb{R}^+)$ is real-valued and of at most exponential growth. B is a (possibly unbounded) linear operator on another Banach space U and the control function $u \in L^p_{\text{loc}}(\mathbb{R}^+; U)$.

It is further assumed that the uncontrolled system, that is,

$$\dot{x}(t) = Ax(t) + \int_0^t k(t-s)\,Ax(s)\,ds, \quad t \geq 0, \qquad x(0) = x_0, \tag{2}$$

is well-posed, which is equivalent to the existence of a unique family of bounded linear operators $(S(t))_{t\geq 0}$ on X called the resolvent or solution family for (2), where we write $(S(t))_{t\geq 0} \subset \mathscr{L}(X)$) (see Section 2) and that $(S(t))_{t\geq 0}$ is exponentially bounded.

Many authors have studied this class of Volterra integral equations by the classical approach and for different reasons, using the Laplace transform (see, e.g., Da Prato and Iannelli [1, 2], Grimmer and Pritchard [3], Lunardi [4], and Prüss [5, 6]). Maximal-L^p regularity results are due to Clément and Da Prato [7] and Prüss [6]. In addition to the classical approach, there is a semigroup approach which was used in, for example, Miller [8], Chen and Grimmer [9, 10], Desch and Grimmer [11], Desch and Schappacher [12], Di Blasio et al. [13], Nagel and Sinestrari [14], and Engel and Nagel [15]. It was the main objection against the semigroup approach for many years that it is not possible to obtain regularity of the solutions. This is not true, as it was proved recently in [16].

From the point of view of complex inversion formula, early Hille and Philllips have proved in [17, p. 349] the validity of the complex inversion formula of the Laplace transform for C_0-semigroups (i.e., (2) with $k(t) = 0$) on the domain $D(A)$. In 1995, Yao has proved [18], in Hilbert spaces, the validity of the complex inversion formula of the Laplace transform for C_0-semigroups on X. In 1999, Driouich and El-Mennaoui have proved [19] (see also [20, Proposition 3.12.2]) that this inversion result remains true on UMD spaces. These results have been extended to strongly continuous cosine families by Cioranescu and Keyantuo in [21] and to strongly continuous resolvent families by Cioranescu and Lizama in [22]. Recently, Haase has improved some results in [23] based on Fourier analysis and left as an open problem the corresponding result for convoluted semigroups, which has been solved affirmatively very recently in [24].

Several authors have been investigating the Cauchy problem from the point of view of admissibility of control operators (i.e., (2) with $k(t) = 0$) in the past and the present [25–32] et al. But the first studies on L^2-admissibility of control operator for Volterra integral scalar systems began with the paper of Jung [33]. The idea of treating L^2-admissibility for Volterra integral equations has been exploited in the past years by several authors, for example, [34–36]. In [33], the notion of finite-time L^2-admissibility for Volterra integral scalar system is linked with finite-time admissibility of the well-studied semigroups' (i.e., $k(t) = 0$) case for completely positive kernel. Likewise, in [34] infinite-time admissibility for a Volterra scalar system is linked with infinite-time admissibility for semigroups (i.e., $k(t) = 0$) for a large class of kernels and the result subsumes that of [33]. Other results are related to the case where the generator of the underlying semigroup has a Riesz basis of eigenvectors in [36]. In [35], the authors have given necessary and sufficient conditions for finite-time L^2-admissibility of linear Volterra integrodifferential systems (2) when the underlying semigroup is equivalent to a contraction semigroup, which generalizes an analogous result known to hold for the standard Cauchy problem and it subsumes the result in [33]. Recently, the authors in [37] have introduced the notion of Favard spaces with respect to resolvent families and have established a relationship between L^p-admissibilities to these Favard spaces. This extends the results obtained for the semigroups' case in [32]. Furthermore, it was proved in [37] that, for scalar Volterra integral systems with a creep kernel, finite- and infinite-time L^1-admissibility are equivalent to exponentially stable resolvent family, and if the state space X is reflexive then finite-time and uniformly finite-time L^1-admissibility are equivalent, extending well-known results for semigroups.

We proceed as follows. In Section 2, we review some well-known properties of resolvent families for scalar Volterra integral equations and their properties. Section 3 contains the definition of the UMD space and recalls some results on the complex inversion formula for wide classes of families of bounded linear operators on UMD spaces and prove the analogue of [17, Theorem 11.6.2] which is applied in Sections 4 and 5. Our hypotheses on the kernel differ from those considered by [22, 23] and can contain a class of completely positive functions (see [35, Example 4.5]). In Section 4, we are concerned with a class of scalar integrodifferential Volterra equations. First we embed this class in a larger Cauchy system, a technique originating in Engel and Nagel [15, VI.7], in order to prove some results concerning the validity of the complex inversion formula. In Section 5, we go back to the study of the admissibility of control operators for Volterra integrodifferential equations (1) in the same spirit of semigroups and we get a new criterion to judge L^p-admissibility ($p \in [1, \infty[$) of control operators in terms of UMD property of its underlying control space. If we set $k(t) = 0$ then we recover the result in [38] for the semigroups. Note that this paper involves in particular a nonscalar kernel of the form "$K(t) = k(t)A$" and so a natural question is whether the situation extends when $K(t)$ is a nonscalar kernel. In a forthcoming work, we will consider a class of nonscalar kernels.

2. Review on Resolvent Families

In this subsection, we collect some elementary facts about scalar Volterra integral equations and resolvent families. These topics have been covered in detail in [6]. We refer to these works for reference to the literature and further results.

Let $(X, \|\cdot\|_X)$ be a Banach space; let A be a linear closed densely defined operator in X; $a \in L^1_{\text{loc}}(\mathbb{R}^+)$ is a scalar kernel. We consider the linear Volterra integral equation

$$x(t) = x_0 + \int_0^t a(t-s) Ax(s)\, ds, \quad t \geq 0, \qquad x(0) = x_0 \in X. \tag{3}$$

We denote by $[D(A)]$ the domain of A equipped with the graph norm.

We define the convolution product of the scalar function a with a vector-valued function f by

$$(a * f)(t) := \int_0^t a(t-s) f(s)\, ds, \quad t \geq 0. \tag{4}$$

Definition 1. A function $x \in C(\mathbb{R}^+, X)$ is called

(i) strong solution of (3) if $x \in C(\mathbb{R}^+, [D(A)])$ and (3) is satisfied,

(ii) mild solution of (3) if $a * x \in C(\mathbb{R}^+, [D(A)])$ and

$$x = x_0 + A[a * x](t), \quad t \geq 0. \tag{5}$$

Obviously, every strong solution of (3) is a mild solution. Conditions under which mild solutions are strong solutions are studied in [6].

Definition 2. Equation (3) is called well-posed if, for each $v \in D(A)$, there is a unique strong solution $x(t, v)$ on $\mathbb{R}^+$ of

$$x(t, v) = v + (a * Ax)(t), \quad t \geq 0, \tag{6}$$

and, for a sequence $(x_n) \subset D(A)$, $x_n \to 0$ implies $x(t, x_n) \to 0$ in X, uniformly on compact intervals.

Definition 3. Let $a \in L^1_{loc}(\mathbb{R}^+)$. A strongly continuous family $(S(t))_{t\geq 0} \subset \mathscr{L}(X)$ is called resolvent family for (3), if the following three conditions are satisfied:

(S1) $S(0) = I$;

(S2) $S(t)$ commutes with A, which means that $S(t)(D(A)) \subset D(A)$, for all $t \geq 0$, and $AS(t)x = S(t)Ax$ for all $x \in D(A)$ and $t \geq 0$;

(S3) for each $x \in D(A)$ and all $t \geq 0$ the resolvent equations hold:

$$S(t)x = x + \int_0^t a(t-s) S(s) Axds. \quad (7)$$

Note that the resolvent for (3) is uniquely determined. The proofs of these results and further information on resolvent can be found in the monograph by Prüss [6]. We also notice that the choice of the kernel a classifies different families of strongly continuous solution operators in $\mathscr{L}(X)$. For instance, when $a(t) = 1$, then $S(t)$ corresponds to a C_0-semigroup and when $a(t) = t$, then $S(t)$ corresponds to cosine operator function. In particular, when $a(t) = t^{\alpha-1}/\Gamma(\alpha)$ with $0 < \alpha \leq 2$, they are the α-times resolvent families studied by [39] and correspond to the solution families for fractional evolution equations, that is, evolution equations where the integer derivative with respect to time is replaced by a derivative of fractional order.

The existence of a resolvent family allows one to find the solution for (3). Several properties of resolvent families have been discussed in [6, 40].

The following well-known result [6, Proposition 1.1] establishes the relation between well-posedness and existence of a resolvent family.

Theorem 4. *Equation* (3) *is well-posed if and only if* (3) *admits a resolvent family* $(S(t))_{t\geq 0}$. *If this is the case one has in addition Range* $(a * S)(t) \subset D(A)$, *for all* $t \geq 0$, *and*

$$S(t)x = x + A\int_0^t a(t-s) S(s) xds, \quad (8)$$

for each $x \in X$, $t \geq 0$.

From this, we obtain that if $(S(t))_{t\geq 0}$ is a resolvent family of (3), we have $A(a * S)(\cdot)$ which is strongly continuous and the so-called mild solution $x(t) = S(t)x_0$ solves (3).

A resolvent family $(S(t))_{t\geq 0}$ is called exponentially bounded, if there exist $M > 0$ and $\omega \in \mathbb{R}$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$, and the pair (M, ω) is called type of $(S(t))_{t\geq 0}$. The growth bound of $(S(t))_{t\geq 0}$ is $\omega_0(S) := \inf\{\omega \in \mathbb{R}, \|S(t)\| \leq Me^{\omega t}, t \geq 0, M > 0\}$. The resolvent family is called exponentially stable if $\omega_0(S) < 0$.

Note that, contrary to the case of C_0-semigroup, the resolvent for (3) needs not to be exponentially bounded; a counterexample can be found in [6, 41]. However, there are checkable conditions guaranteeing that (3) possesses an exponentially bounded resolvent operator.

We will use the Laplace transform at times. Suppose $g : \mathbb{R}^+ \to X$ is measurable and there exist $M > 0$ and $\omega \in \mathbb{R}$, such that $\|g(t)\| \leq Me^{\omega t}$ for almost $t \geq 0$. Then, the Laplace transform

$$\widehat{g}(\lambda) = \int_0^\infty e^{-\lambda t} g(t)\, dt \quad (9)$$

exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$.

The following well-known generation result, stated in [6], is quite important in this paper. It establishes the relation between resolvent family and Laplace transform.

Proposition 5. *Let* $a \in L^1_{loc}(\mathbb{R}^+)$ *be* ω*-exponentially bounded. Then,* (3) *admits a resolvent family* $(S(t))_{t\geq 0}$ *of type* (M, ω) *if and only if the following conditions hold:*

(i) $\widehat{a}(\lambda) \neq 0$ *and* $1/\widehat{a}(\lambda) \in \rho(A)$ *(the resolvent set of A), for all* $\lambda > \omega$;

(ii) $H(\lambda) := (1/\lambda\widehat{a}(\lambda))((1/\widehat{a}(\lambda))I - A)^{-1}$ *called the resolvent associated with* $(S(t))_{t\geq 0}$ *satisfies*

$$\left\|H^{(n)}(\lambda)\right\| \leq Mn!(\lambda-\omega)^{-(n+1)} \quad \forall\lambda > \omega,\ n \in \mathbb{N}. \quad (10)$$

Under these assumptions the Laplace transform of $S(\cdot)$ is well-defined and it is given by $\widehat{S}(\lambda) = H(\lambda)$ for all $\operatorname{Re}(\lambda) > \omega$. Note that, for $a(t) = 1$, Proposition 5 becomes the well-known Hille-Yosida theorem.

3. The Complex Inversion Formula and UMD Spaces

In this section, we review some results on the complex inversion formula of the Laplace transform, in the strong sense, for wide classes of families of bounded linear operators on UMD spaces. There are several equivalent definitions of a UMD space, one of which involves the so-called unconditional Martingale differences, but we will use a different characterization, due to [42–44] involving the vector-valued Hilbert transform (see [45]) for more about UMD spaces.

Let $p \in]1, \infty[$, and define the operator $\mathscr{H}_\varepsilon$ on $L^p(\mathbb{R}; X)$ by

$$\mathscr{H}_\varepsilon f(t) := \frac{1}{\pi}\int_{|t-s|\geq\varepsilon} \frac{f(s)}{t-s} ds \quad \forall t \in \mathbb{R}. \quad (11)$$

A Banach space X is called a UMD space (or said to have the UMD-property) if for some (and hence all) $p \in]1, \infty[$ (see [44, 46]) $\mathscr{H}f := \lim_{\varepsilon\searrow 0}\mathscr{H}_\varepsilon f$ exists in $L^p(\mathbb{R}; X)$ and defines a bounded operator $\mathscr{H}$ on $L^p(\mathbb{R}; X)$. The operator $\mathscr{H}$ is called the Hilbert transform on $L^p(\mathbb{R}; X)$. Every UMD space is reflexive and its dual is also a UMD space. Typical examples of UMD spaces are $L^p(\Omega)$-spaces, Sobolev spaces $W^s_p(\Omega)$, and Besov spaces $B^s_{p,q}$ for $p, q \in]1, \infty[$ and their closed subspaces.

The natural question that comes in mind is the following: let $S(t)_{t\geq 0}$ be the (exponentially bounded) resolvent family for (3) on a Banach space X, under what conditions does the complex inversion formula

$$\lim_{N\to\infty}\left[S_N(t)x := \frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}H(\lambda)x\,d\lambda\right] = S(t)x \quad (\alpha > \omega_0(S),\ t \ge 0) \tag{12}$$

holds true?

For the C_0-semigroups' case (i.e., $a(t) = 1$) the classical result [17, p. 349] is that one always has the strong convergence (12) if $x \in D(A)$. Recently, the authors in [38, Proposition 4.8] have used the notion of the admissibility and proved in [38, Proposition 4.8] that this inversion takes place on larger spaces than $D(A)$. In Hilbert setting, the inversion (12) has been generalized to all $x \in X$ in [18] using Plancherel's Theorem. In the paper [19] Driouich and El-Mennaoui have extended the result in [18] in the case where X has the UMD-property and it has been proved that the UMD property is essential by exhibiting an example for which the inverse Laplace transform does not always converge. This was subsequently generalized from semigroups to solution families for scalar type Volterra integral equations (3) by Cioranescu and Lizama in [22] under some regularity assumptions on the kernel $a(t)$. In particular, it has been proved in [22, Proposition 2] that for $a \in C^1(\mathbb{R}^+)$ the inversion for solution families for Volterra equations (3) holds on $D(A)$. On UMD spaces, Haase in [23] has presented new and much shorter proofs of these results (under less strong assumptions on $a(t)$), eventually, even generalizing them. His approach uses some elementary Fourier analysis. Recently, this was generalized from resolvent families to convoluted semigroups in [24]. This inversion problem will be studied for a class of integrodifferential Volterra equations in the next section. Note that the class of integrodifferential Volterra equations has a "big" intersection with the class of Volterra integral one (partial results had been obtained earlier). Let us signalize that early, and in Hilbert setting, the first result to our knowledge on the inversion formula on $D(A)$ for scalar Volterra integrodifferential equations (3) under the conditions that $k \in C^1(\mathbb{R}^+)$ and both k and k' are exponentially bounded is implicitly contained in [8].

We will have the occasion to use the following observation which is the exact generalization of the corresponding well-known case $k(t) = 0$ (i.e., the semigroups) proved in [17, Theorem 11.6.1]. We sketch the proof for the reader's convenience.

Proposition 6. *Let $a \in L^1_{loc}(R^+)$ be exponentially bounded and let $(S(t))_{t\ge0}$ be the exponentially bounded resolvent family for* (3). *Then, for each $x \in D(A)$ and $\alpha > \max(\omega_0(a), \omega_0(S))$, we have*

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}H(\lambda)x\,d\lambda = \begin{cases} S(t)x & \forall t > 0, \\ \dfrac{x}{2} & \text{for } t = 0 \end{cases} \tag{13}$$

in X, uniformly in t from compact subsets of $[0,\infty[$.

Proof. Let $x \in D(A)$ and $\alpha > \max(w_0(a), w_0(S))$ then by the resolvent equation (S3), we have $S(t)x - x = (a * S)(t)Ax$, and by virtue of Proposition 5 we obtain

$$\begin{aligned}\widehat{(a * S)}(\lambda)Ax &= \widehat{S}(\lambda)x - \widehat{1}(\lambda)x \\ &= H(\lambda)x - \frac{1}{\lambda}x\end{aligned} \tag{14}$$

for $\operatorname{Re}\lambda > \alpha$.

Thanks to [23, Proposition 2.1], for all $t \ge 0$ we have

$$\begin{aligned}(a * S)(t)Ax &= \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}\widehat{(a * S)}(\lambda)Ax\,d\lambda, \\ &= \lim_{N\to\infty}\frac{1}{2i\pi}\left(\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}H(\lambda)x\,d\lambda \right. \\ &\quad \left. - \int_{\alpha-iN}^{\alpha+iN}\frac{e^{\lambda t}}{\lambda}x\,d\lambda\right),\end{aligned} \tag{15}$$

where the limit exists uniformly with respect to t in $[0,\infty[$.

On the other hand, we have

$$\begin{aligned}(a * S)(t)Ax = {} & \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}H(\lambda)x\,d\lambda \\ & - \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN}\frac{e^{\lambda t}}{\lambda}x\,d\lambda.\end{aligned} \tag{16}$$

It is well-known that the second limit is x if $t > 0$ but $x/2$ if $t = 0$ (see, e.g., [17]).

Indeed, we known for all $x \in D(A)$ and $\alpha > w_0(A)$ that

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}R(\lambda, A)x\,d\lambda = \begin{cases} T(t)x & \forall t > 0, \\ \dfrac{x}{2} & \text{for } t = 0, \end{cases} \tag{17}$$

in X, uniformly in t from compact subsets of $[0,\infty[$ where $R(\lambda, A) := (\lambda I - A)^{-1}$ is the resolvent of the semigroup $(T(t))_{t\ge0}$.

In particular, for $T(t) = I$, the generator is $A = 0$ and $R(\lambda, 0) = 1/\lambda$, thus we obtain

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN}\frac{e^{\lambda t}}{\lambda}x\,d\lambda = \begin{cases} x & \forall t > 0, \\ \dfrac{x}{2} & \text{for } t = 0, \end{cases} \tag{18}$$

in X, uniformly in t from compact subsets of $[0,\infty[$.

Then according to the resolvent equation (S1) and (S3) in Definition 3, for all $x \in D(A)$ and $\alpha > \max(w_0(a), w_0(S))$ we obtain

$$\begin{aligned}S(t)x - x = {} & \lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN} e^{\lambda t}H(\lambda)x\,d\lambda \\ & - \begin{cases} x & \forall t > 0, \\ \dfrac{x}{2} & \text{for } t = 0. \end{cases}\end{aligned} \tag{19}$$

Hence we deduce that for all $x \in D(A)$ and $\alpha > \max(w_0(a), w_0(S))$, we have

$$\lim_{N\to\infty} \frac{1}{2i\pi} \int_{\alpha-iN}^{\alpha+iN} e^{\lambda t} H(\lambda)\, x\, d\lambda - \begin{cases} S(t)\, x & \forall t > 0, \\ \dfrac{x}{2} & \text{for } t = 0, \end{cases} \quad (20)$$

in X, uniformly in t from compact subsets of $[0, \infty[$. □

4. Integrodifferential Equation with Bounded Variation Kernels

The purpose here is to prove some complex inversion of Laplace transform for the resolvent families of the integrodifferential Volterra equations (2). Although our hypotheses on kernel and the approach differ from the one considered by [22, 23], there is a big overlap in the fundamental results. For this class, it is known that the integrodifferential equation (2) can be converted to an abstract Cauchy problem on a product space (see, e.g., [15, VI.7]). This technique has been widely used (see, e.g., [9, 10, 12, 16, 35]). In the following, we restate some notations and related results for the sake of convenience.

This idea in the following will be applied to the case of the complex inversion formula for (2). Although it is a special case of the situation considered in the above section, it is worthwhile to deal with the partial inverse formula case first, which generalizes the result from semigroups to solution families for scalar Volterra equations (2) and constitutes an extension of a result in [8]. Next, we prove the strong convergence of the complex inversion formula for this class on UMD spaces.

It is easy to see that (2) is equivalent to

$$x(t) = x_0 + \int_0^t (1 + 1 * k)(t - s)\, Ax(s)\, ds, \quad t \geq 0. \quad (21)$$

In what follows, we assume that $k \in W^{1,p}(\mathbb{R}^+)$ and A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X. Recall that with $k \in BV_{\text{loc}}(\mathbb{R}^+)$ (the space of functions locally of bounded variation) only, the operator A has to be a generator of C_0-semigroups to obtain the well-posedness of (2), but this condition is not much restrictive, since generation of A is a necessary and sufficient condition for the well-posedness of (21) (which is equivalent to (2)) (see [6, Corollary 1.4]).

From now on, we denote by $(S(t)_{t\geq 0})$ the resolvent family associated with (21). As in [15, VI.7, Part C], we first introduce the product space $\mathcal{X} := X \times L^p(\mathbb{R}^+; X) 1 \leq p < \infty$, which is a Banach space with the norm

$$\left\| \begin{pmatrix} x \\ f \end{pmatrix} \right\|^2 = \|x\|_X^2 + \|f\|_{L^p(\mathbb{R}^+, X)}^2, \quad x \in X,\ f \in L^p(\mathbb{R}^+; X). \quad (22)$$

Next, we define on $\mathcal{X}$ the operator matrices

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S_l(t) \end{pmatrix}, \quad t \geq 0, \quad (23)$$

where $(S_l(t))_{t\geq 0}$ (the left shift semigroup) and $R(t)$ are defined on $L^p(\mathbb{R}^+; X)$:

$$\begin{gathered} (S_l(t) f)(\tau) := f(t + \tau), \quad \tau \geq 0 \\ R(t) f := \int_0^t T(t - s) f(s)\, ds, \quad f \in L^p(\mathbb{R}^+; X). \end{gathered} \quad (24)$$

Then $(\mathcal{T}(t))_{t\geq 0}$ forms a C_0-semigroup on $\mathcal{X}$ with the generator given by

$$\mathcal{A} := \begin{pmatrix} A & \delta_0 \\ 0 & \dfrac{d}{ds} \end{pmatrix}, \qquad D(\mathcal{A}) := D(A) \times W^{1,p}(\mathbb{R}^+; X), \quad (25)$$

where d/ds denotes the generator of the semigroup $(S_l(t))_{t\geq 0}$ with domain $D(d/ds) = W^{1,p}(\mathbb{R}^+; X)$, the vector-valued Sobolev space, and δ_0 the Dirac distribution, that is, $\delta_0(f) = f(0)$ for each $f \in W^{1,p}(\mathbb{R}^+; X)$. Finally, define $M \in \mathcal{L}(D(A); W^{1,p}(\mathbb{R}^+; X))$ as follows:

$$\begin{gathered} Mx := k(\cdot) Ax, \quad x \in D(A), \\ \mathcal{M} \begin{pmatrix} x \\ f \end{pmatrix} := \begin{pmatrix} 0 & 0 \\ M & 0 \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} = \begin{pmatrix} 0 \\ Mx \end{pmatrix}, \\ \begin{pmatrix} x \\ f \end{pmatrix} \in D(\mathcal{M}) := D(\mathcal{A}), \end{gathered} \quad (26)$$

and denote

$$\mathcal{A}^V := \mathcal{A} + \mathcal{M}, \qquad D\left(\mathcal{A}^V\right) := D(\mathcal{A}). \quad (27)$$

In [15, VI.7], it is shown that $\mathcal{A}^V$ generates a C_0-semigroup $(\mathcal{T}^V(t))$ on $\mathcal{X}$. In the sequel, we denote

$$\mathcal{T}^V(t) = \begin{pmatrix} T_{11}(t) & T_{12}(t) \\ T_{21}(t) & T_{22}(t) \end{pmatrix}. \quad (28)$$

We now rewrite the Volterra system (2) into an equivalent Cauchy system as follows:

$$\begin{gathered} \dot{\mathcal{Z}}(t) = \mathcal{A}^V \mathcal{Z}(t), \quad t \geq 0, \\ \mathcal{Z}(0) = \mathcal{Z}_0 \in \mathcal{X}. \end{gathered} \quad (29)$$

Note that it was early observed in [6] (see also [35]) that the solutions of (21) are given by $x(t) = \pi_1 \mathcal{T}^V(t) x_0$, where $\pi_1 : \mathcal{X} \to X$ is the projection mapping (x, f) to x (see [15, VI.7]) for the L^p case and [12] for $Y = C_{ub}$, the space of bounded and uniformly continuous functions. That is, $T_{11}(t) = S(t)$ for all $t \geq 0$.

Observe that the resolvent $(S(t))_{t\geq 0}$ of (21) is exponentially bounded; this extends the result in [35, when $p = 2$] which has been obtained by a direct method. Indeed, the semigroup $(\mathcal{T}^V(t))_{t\geq 0}$ is always exponentially bounded (see, e.g., [15, Proposition 5.5, p.39]); thus there exist constants $M > 0$ and $\alpha \in \mathbb{R}$ such that $\|\mathcal{T}^V(t)\| \leq Me^{\alpha t}$. Making use of the fact that $T_{11}(t) = S(t)$, we deduce that $(S(t))_{t\geq 0}$ is also exponentially bounded and that $\omega_0(S) \leq \omega_0(\mathcal{A}^V)$. Thus it is permissible to consider its Laplace transform.

By virtue of Proposition 5, the Laplace transform of $S(\cdot)x_0$ is well defined and it is given by

$$\int_0^\infty e^{-\lambda t} S(t)\, x_0 dt = H(\lambda)\, x_0 := \left(\lambda I - \left(1 + \widehat{k}(\lambda)\right) A\right)^{-1} x_0, \tag{30}$$

for all $\operatorname{Re}\lambda > \max(\omega_0(k), \omega_0(S))$ and $x_0 \in X$.

The following lemma is quite useful.

Lemma 7. *(i) For* $\operatorname{Re}\lambda > \omega^* := \max(0, \omega_0(S))$ *and* $x \in X$, *we have*

$$R\left(\lambda, \mathscr{A}^V\right)\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} H(\lambda)\, x \\ R\left(\lambda, \dfrac{d}{ds}\right) MH(\lambda)\, x \end{pmatrix}. \tag{31}$$

(ii) For $\lambda \in \mathbb{C}_0 \cap \rho(\mathscr{A}^V) \cap \rho(\mathscr{A})$, *we have*

$$R\left(\lambda, \mathscr{A}^V\right) = \begin{pmatrix} H(\lambda) & H(\lambda)\,\delta_0 R\left(\lambda, \dfrac{d}{ds}\right) \\ R\left(\lambda, \dfrac{d}{ds}\right) MH(\lambda) & R\left(\lambda, \dfrac{d}{ds}\right) MH(\lambda)\,\delta_0 R\left(\lambda, \dfrac{d}{ds}\right) + R\left(\lambda, \dfrac{d}{ds}\right) \end{pmatrix}, \tag{32}$$

where $R(\lambda, d/ds)$ *is the resolvent of* $(S_l(t))_{t\geq 0}$.

Proof. (i) Invoking (28) (see [6, page 339] for more information on $(\mathscr{T}^V(t))_{t\geq 0}$) we have

$$T_{21}(t) = A\int_0^t k(t-\tau+\cdot)\, S(\tau)\, d\tau \quad \forall t > 0. \tag{33}$$

Thus, using (28) for all $\operatorname{Re}\lambda > \omega^* := \max(0, \omega_0(S))$, we have

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} \mathscr{T}^V(t)\begin{pmatrix} x \\ 0 \end{pmatrix} dt \\
&= \begin{pmatrix} \displaystyle\int_0^\infty e^{-\lambda t} S(t)\, x dt \\ \displaystyle\int_0^\infty e^{-\lambda t}\left(A\int_0^t k(t-\tau+\cdot)\, S(\tau)\, x d\tau\right) dt \end{pmatrix} \\
&= \begin{pmatrix} H(\lambda)\, x \\ \displaystyle\int_0^\infty e^{-\lambda t}\left(A\int_0^t k(t-\tau+\cdot)\, S(\tau)\, x d\tau\right) dt \end{pmatrix} \\
&= \begin{pmatrix} H(\lambda)\, x \\ \displaystyle\int_0^\infty e^{-\lambda t} A\int_0^t (S_l(t-\tau)\, k)(\cdot)\, S(\tau)\, x d\tau\, dt \end{pmatrix} \\
&= \begin{pmatrix} H(\lambda)\, x \\ \displaystyle e^{\lambda\cdot}\int_\cdot^\infty e^{-\lambda t}\left((S_l(\cdot)\, M)(\cdot) * S\right)(t)\, x dt \end{pmatrix} \\
&= \begin{pmatrix} H(\lambda)\, x \\ R\left(\lambda, \dfrac{d}{ds}\right) MH(\lambda)\, x \end{pmatrix},
\end{aligned} \tag{34}$$

which implies (31).

(ii) For each $\lambda \in \rho(A)$ with $\operatorname{Re}(\lambda) > 0$, from [15, Lemmas VI.7.23-24], we know that

$$\begin{aligned}
R(\lambda, \mathscr{A}) &= \begin{pmatrix} R(\lambda, A) & R(\lambda, A)\,\delta_0 R\left(\lambda, \dfrac{d}{ds}\right) \\ 0 & R\left(\lambda, \dfrac{d}{ds}\right) \end{pmatrix}, \\
I - R(\lambda, \mathscr{A})\,\mathscr{M} &= \begin{pmatrix} I - \widehat{k}(\lambda)\, R(\lambda, A)\, A & 0 \\ -R\left(\lambda, \dfrac{d}{ds}\right) M & I \end{pmatrix}.
\end{aligned} \tag{35}$$

Using

$$\begin{aligned}
\lambda I - \mathscr{A}^V &= \lambda I - \mathscr{A} - \mathscr{M}, \\
&= (\lambda I - \mathscr{A})(I - R(\lambda, \mathscr{A})\,\mathscr{M}),
\end{aligned} \tag{36}$$

we get, since $\lambda \in \rho(\mathscr{A}) \cap \rho(\mathscr{A}^V)$, that $[I - R(\lambda, \mathscr{A})\mathscr{M}]$ is invertible in $\mathscr{L}(X)$ and by, for example, [31, Lemma A.4.2], we have

$$\begin{aligned}
&[I - R(\lambda, \mathscr{A})\,\mathscr{M}]^{-1} \\
&= \begin{bmatrix} \left(I - \widehat{k}(\lambda)\, R(\lambda, A)\, A\right)^{-1} & 0 \\ R\left(\lambda, \dfrac{d}{ds}\right) M\left(I - \widehat{k}(\lambda)\, R(\lambda, A)\, A\right)^{-1} & I \end{bmatrix}.
\end{aligned} \tag{37}$$

Finally, a direct computation yields

$$R\left(\lambda,\mathscr{A}^{V}\right)=(I-R(\lambda,\mathscr{A})\mathscr{M})^{-1}R(\lambda,\mathscr{A})$$
$$=\begin{pmatrix} H(\lambda) & H(\lambda)\delta_0 R\left(\lambda,\frac{d}{ds}\right) \\ R\left(\lambda,\frac{d}{ds}\right)MH(\lambda) & R\left(\lambda,\frac{d}{ds}\right)MH(\lambda)\delta_0 R\left(\lambda,\frac{d}{ds}\right)+R\left(\lambda,\frac{d}{ds}\right)\end{pmatrix}. \tag{38}$$

□

Here is finally the result about strong convergence of the complex inversion formula.

Proposition 8. *Let $(S(t))_{t\geq 0}$ be the exponentially bounded resolvent family for* (2) *on X. Suppose that X is a UMD space. Then there exists $\widetilde{\omega}\geq\omega_0(S)$, such that for all $x\in X$ and $\alpha>\widetilde{\omega}$ one has*

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}H(\lambda)x\,d\lambda=S(t)x, \tag{39}$$

in X, uniformly in t from compact subsets of $]0,\infty[$.

Proof. Since X is a UMD, the product space $\mathscr{X}$ is a UMD (see [45, 46]). Applying [19, Theorem 1] to the semigroup $(\mathscr{T}^{V}(t))_{t\geq 0}$, we obtain

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}R\left(\lambda,\mathscr{A}^{V}\right)\begin{pmatrix}x\\ f\end{pmatrix}d\lambda=\mathscr{T}^{V}(t)\begin{pmatrix}x\\ f\end{pmatrix}, \tag{40}$$

for all $\binom{x}{f}\in\mathscr{X}$ and $\alpha>\omega_0(\mathscr{A}^{V})$, where the limit exists uniformly with respect to t in $]0,\infty[$. Taking $f=0$ in (40) and by virtue of Lemma 7(i), a straightforward computation shows that for all $\alpha>\widetilde{\omega}:=\max(0,\omega_0(\mathscr{A}^{V}))\geq\omega^{*}$ we have

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\alpha-iN}^{\alpha+iN}e^{\lambda t}H(\lambda)x\,d\lambda=S(t)x, \tag{41}$$

for all $x\in X$, uniformly for $t\in]0,\infty[$. □

We see that the results in [22, Theorem 1], [23, Theorem 4.2], and Proposition 8 assert only strong convergence and uniformity in t from compact subsets of $]0,\infty[$. As for the semigroups' case, it would be of interest to see whether, for scalar Volterra integral systems, this convergence holds true for $t=0$ (it does not for $x\in D(A)$, (see Proposition 6)). For the semigroups, which are the special cases of scalar Volterra integral equations, an affirmative answer was given in [38, Proposition 2.3].

The next result is a version of [38, Proposition 2.3] for resolvent family.

Proposition 9. *Let $a\in L^1_{\mathrm{loc}}(R^{+})$ be exponentially bounded and let $(S(t))_{t\geq 0}$ be the exponentially bounded resolvent family for* (3). *If A generates an analytic C_0-semigroup in X, then for all $x\in X$, and $\alpha>\max(\omega_0(S),\omega_0(a))$ the integral $\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda$ converges in X as $N\to\infty$, hence, in $D(A)$ for all $x\in D(A)$.*

Proof. The proof is more or less the same for the semigroups' case. Assume first that $x\in D(A)$. Then for all $0\neq\lambda>\omega$ and using the resolvent identity we obtain

$$H(\lambda)x=\frac{x}{\lambda}+\widehat{a}(\lambda)H(\lambda)Ax. \tag{42}$$

Let $\partial\mathscr{C}_N^{+}$ be the boundary of the half-disc $\mathscr{C}_N^{+}$ defined by $\mathscr{C}_N^{+}=\{\lambda\in\mathbb{C}:|\lambda|\leq N,|\arg\lambda|\leq\pi/2\}$. Then Cauchy's theorem yields

$$\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda=-i\int_{\partial\mathscr{C}_N^{+}}H(\lambda)x\,dz=\int_{-(\pi/2)}^{\pi/2}Ne^{i\theta}H\left(Ne^{i\theta}\right)x\,d\theta. \tag{43}$$

Since A generates an analytic semigroup on X, the set $\{(1/\widehat{a}(Ne^{i\theta}))R((1/\widehat{a}(Ne^{i\theta})),A)\}$ is uniformly bounded in $\mathscr{C}_N^{+}$ (see [47]) and, hence, $\{Ne^{i\theta}H(Ne^{i\theta})\}$ is uniformly bounded in $\mathscr{C}_N^{+}$. It follows that $\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda$ is uniformly bounded in $\mathscr{C}_N^{+}$. Using once again the resolvent identity we obtain

$$\int_{\partial\mathscr{C}_N^{+}}H(\lambda)x\,dz=\int_{\partial\mathscr{C}_N^{+}}\frac{d\lambda}{\lambda}x+\int_{\partial\mathscr{C}_N^{+}}\widehat{a}(\lambda)H(\lambda)Ax\,d\lambda=\pi ix+i\int_{-(\pi/2)}^{\pi/2}\widehat{a}\left(Ne^{i\theta}\right)Ne^{i\theta}H\left(Ne^{i\theta}\right)Ax\,d\theta, \tag{44}$$

for all $x\in D(A)$. By the fact that A generates an analytic semigroup and by virtue of Riemann-Lebesgue's theorem (see [20, Theorem 1.8.1.c]), the integral on the right-hand side of the above equality converges to zero as $N\to\infty$. Using the last equality, the density of $D(A)$ in X, and the fact that the integral $\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda$ is uniformly bounded in $\mathscr{C}_N^{+}$, an elementary equicontinuity argument guarantees the convergence in X for every $x\in X$ thanks to Proposition 6. Now, let $x\in D(A)$ and $y_0:=(\lambda_0 I-A)x$ for some $\lambda_0\in\rho(A)$. Then we get

$$\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda=R\left(\lambda_0,A\right)\int_{\alpha-iN}^{\alpha+iN}H(\lambda)y_0\,d\lambda. \tag{45}$$

Since $AR(\lambda_0,A)\in\mathscr{L}(X)$, we conclude that $\int_{\alpha-iN}^{\alpha+iN}H(\lambda)x\,d\lambda$ converges in $D(A)$. □

Finally, combining Propositions 8 and 9 leads to the following corollary.

Corollary 10. *Let A be the generator of an analytic C_0-semigroup in X and let $(S(t))_{t\geq 0}$ be the resolvent family for* (2). *Suppose that X is a UMD space. Then, for all $x \in X$ (resp., $x \in D(A)$) and $\alpha > \omega^*$, one has*

$$\lim_{N\to\infty} \frac{1}{2i\pi} \int_{\alpha-iN}^{\alpha+iN} e^{\lambda t} H(\lambda) x d\lambda = \begin{cases} S(t)x & \forall t > 0, \\ \frac{x}{2} & \text{for } t = 0, \end{cases} \quad (46)$$

in X (resp., in $D(A)$), which is uniform on t for any compact interval of $[0,\infty[$.

5. Characterization of Admissibility

We now turn our attention back to the notion of admissibility. In this section we present sufficient and necessary conditions for the L^p-admissibility of control operators for integrodifferential Volterra control systems (1) which has the following equivalent form:

$$x(t) = x_0 + \int_0^t (1 + 1 * k)(t-s) Ax(s) ds + \int_0^t Bu(s) ds, \quad t \geq 0, \qquad x(0) = x_0 \in X, \quad (47)$$

extending the result for the semigroups [38, Proposition 3.2]. Here we assume that the control operator $B \in \mathscr{L}(U; X_{-1})$ where X_{-1} is the extrapolation space with respect to A (see, e.g., [15]) and U is another Banach space. It is further assumed that $k \in L^1_{loc}(\mathbb{R}^+)$ is exponentially bounded and that the uncontrolled system (i.e., (21)) admits an exponentially bounded resolvent family $(S(t))_{t\geq 0}$.

Since the resolvent of (21) commutes with the operator A, then it can be easily seen that the restriction $(S_1(t))_{t\geq 0}$ to X_1 of $(S(t))_{t\geq 0}$, the solution of (21), is strongly continuous. Moreover, since $\rho(A) \neq \varnothing$ (see Proposition 5) $(S_1(t))_{t\geq 0}$ solves for each $x_0 \in X$ and A_1 replacing A. Likewise, $S(t)$ has a unique bounded extension to X_{-1} for each $t \geq 0$ and $t \mapsto S_{-1}(t)$ is also strongly continuous, and it solves (21) in X_{-1} with A_{-1} replacing A.

The mild solution of (47) is formally given by the variation of the constant formula

$$x(t) = S(t) x_0 + \int_0^t S_{-1}(t-s) Bu(s) ds, \quad (48)$$

which is actually the classical solution if $B \in \mathscr{L}(U, X)$, $x_0 \in D(A)$, and u is sufficiently smooth. In general however, B is not a bounded operator from U into X and so an additional assumption on B will be needed to ensure that $x(t) \in X$ for every $x_0 \in X$ and every $u \in L^p(\mathbb{R}^+; U)$ or $L^p_{loc}(\mathbb{R}^+; U)$.

In the same spirit of the semigroups' case, the following are the most natural definitions of the L^p-admissibility for resolvent families.

Definition 11. Let $B \in \mathscr{L}(U; X_{-1})$ and $p \in [1, \infty[$.

(i) B is called (infinite-time) an L^p-admissible operator for $(S(t)_{t\geq 0})$, if there exists a constant $M > 0$, such that

$$\|S_{-1} * Bu(t)\|_X \leq M \|u\|_{L^p([0,\infty[;U)} \qquad \forall u \in L^p([0,\infty[;U), \ t > 0. \quad (49)$$

(ii) B is called a finite-time L^p-admissible operator for $(S(t))_{t\geq 0}$ if there exist $t_0 > 0$ and a constant $M(t_0) > 0$, such that

$$\|S_{-1} * Bu(t_0)\|_X \leq M(t_0) \|u\|_{L^p([0,t_0];U)} \qquad \forall u \in L^p([0,t_0];U). \quad (50)$$

Note that the definition of (infinite-time) L^p-admissible control operator for $(S(t)_{t\geq 0})$ was introduced in [34] when $p = 2$ and implies the finite-time L^2-admissibility condition considered in [33]. Our definitions of finite- and infinite-time L^p-admissible control operator for $(S(t)_{t\geq 0})$ correspond to that of the semigroups and also imply that of [33] when $p = 2$. It is well known that (P_1): finite-time L^p-admissibility and the uniform finite-time L^p-admissibility, which means that for all $t > 0$ there exists a constant $M(t) > 0$, such that $\|S_{-1} * Bu(t)\|_X \leq M(t)\|u\|_{L^p([0,t];U)}$ for all $u \in L^p([0,t];U)$, are equivalent for semigroups and (P_2): finite-time L^p-admissibility and the infinite-time L^p-admissibility are equivalent for exponentially stable semigroups. We emphasize that, in [35], the authors have found an example (i.e., (2) with A generator of exponentially stable semigroup) for which finite-time L^2-admissibility and infinite-time L^2-admissibility are not equivalent, but, in their example, we can see that the associated resolvent family is not exponentially stable. Thus, a question that remains open to our knowledge is, whether for Volterra integral systems, these problems (i.e., (P_1)-(P_2)) are still true for resolvent families. In [37, Corollary 5.4 and Proposition 5.6], partial answers were given to these problems when $p = 1$.

It has been observed in [36, for $p = 2$] (resp., [37, for $p \geq 1$]) that L^2- (resp., L^p) admissibility of $B \in \mathscr{L}(U; X_{-1})$ is equivalent to the fact that there exists a constant $M > 0$, such that

$$\left\| \int_0^\infty S_{-1}(t) Bu(t) dt \right\|_X \leq M \|u\|_{L^p(\mathbb{R}^+;U)}, \quad (51)$$

for all $u \in L^p_c(\mathbb{R}^+; U)$ (the space of functions in $L^p(\mathbb{R}^+; U)$ with compact support).

Of course, L^p-admissibility of B guarantees that the operator $\mathfrak{B}_\infty : L^p_c(\mathbb{R}^+; U) \to X$, given by

$$\mathfrak{B}_\infty u := \int_0^\infty S_{-1}(t) Bu(t) dt \quad (52)$$

possesses an extension to a linear bounded operator from $L^p(\mathbb{R}^+; U)$ to X. We denote this extension again by $\mathfrak{B}_\infty$.

As for $p = 2$ (see [36]), it is easy to verify that under $\omega_0(S) < 0$ formula (52) holds for every $u \in L^2(\mathbb{R}^+; U)$.

Thanks to Proposition 5, the Laplace-transform of $S_{-1}(\cdot)$ is well defined and it is given by

$$\widehat{S_{-1}}(\lambda) = \left(\lambda I - \left(1 + \hat{k}(\lambda)\right) A_{-1}\right)^{-1} =: H_{-1}(\lambda) \quad \forall \operatorname{Re}(\lambda) > \max\left(\omega_0(k), \omega_0(S)\right). \tag{53}$$

In the sequel we use the following notations.

For $N \geq 0$, $B \in \mathscr{L}(U; X_{-1})$, $u \in L^p([0,\tau]; U)$, and $\tau \in [0, \infty]$ we set

$$\varphi_N^\tau(u) = \int_0^\tau S_{-N}(\sigma) Bu(\sigma)\, d\sigma, \qquad \varphi_N(u) := \varphi_N^\infty(u),$$
$$\varphi^\tau(u) = \int_0^\tau S_{-1}(\sigma) Bu(\sigma)\, d\sigma, \qquad \varphi(u) := \varphi^\infty(u), \tag{54}$$

with

$$S_{-N}(t)x := \frac{1}{2i\pi}\int_{\alpha - iN}^{\alpha + iN} e^{\lambda t} H_{-1}(\lambda) x\, d\lambda \quad \forall x \in X_{-1}. \tag{55}$$

We may now formulate and prove the main result of this section by giving a necessary or/and sufficient condition for finite- (or infinite-) time L^p-admissibility of B. The necessary condition here is essentially based on a geometric property of the underlying control space U that is the UMD-property. The result encompasses Hilbert control spaces, but the proposition below yields the criterion's necessity.

Theorem 12. *Let $B \in \mathscr{L}(U; X_{-1})$. Then the following assertions hold.*

(i) *If $\varphi_N^{\tau_0}$ is uniformly bounded on $\mathscr{L}(L^p([0,\tau_0]; U); X)$ for some $\tau_0 > 0$ then B is finite-time L^p-admissible for $(S(t))_{t\geq 0}$.*

(ii) *Assume that U is a UMD space, $p > 1$, and $\omega_0(S) < 0$. Then B is L^p-admissible for $(S(t))_{t\geq 0}$ if and only if φ_N is uniformly bounded on $\mathscr{L}(L_c^p(\mathbb{R}^+; U); X)$.*

Proof. Part (i). The proof of this is similar to that of [38, Proposition 3.2]. Notice that, for any $\mu \in \rho(A)$ and constant input $u_0 \in U$, we have $\mu^2 R^2(\mu, A) B u_0 \in D(A)$. By Proposition 6, we have

$$X - \lim_{N\to\infty} S_N(\tau_0)\mu^2 R^2(\mu, A) B u_0 = S(\tau_0)\mu^2 R^2(\mu, A) B u_0. \tag{56}$$

Thanks to [48, Lemma 1] the integral $\int_\varepsilon^{\tau_0} S_{-1}(\tau) B u(\tau) d\tau$ takes value in X due to the fact that the considered input $u(t)$ is a step function. Thus by a density argument, to prove that B is finite-time L^p-admissible (for time τ_0) for $(S(t))_{t\geq 0}$, it suffices to prove that for any step function $u : [0, \tau_0] \to U$ with compact support that does not contain zero, the following uniform estimate

$$\left\| \int_0^{\tau_0} S_{-1}(\tau) Bu(\tau)\, d\tau \right\|_X \leq M_{\tau_0} \|u\|_{L^p([0,\tau_0];U)}, \tag{57}$$

holds for some $M_{\tau_0} > 0$.

So, consider a step function u with compact support that does not contain zero. Then there exists $\varepsilon > 0$ such that

$$\varphi_N^{\tau_0}(u) := \int_0^{\tau_0} S_{-N}(\sigma) Bu(\sigma)\, d\sigma, \quad = \int_\varepsilon^{\tau_0} S_{-N}(\sigma) Bu(\sigma)\, d\sigma. \tag{58}$$

Since $(\varphi_N^{\tau_0})_N$ is uniformly bounded with $\|\varphi_N^{\tau_0}\| \leq K_{\tau_0}$ for some $K_{\tau_0} > 0$ and for all $N > 0$, then $\mu^2 R^2(\mu, A)\varphi_N^{\tau_0}$ is uniformly bounded as $\mu \to \infty$ (see [48, Corollary 3]) and we have for some $K, \mu_0 > 0$ and for all $\mu > \mu_0$

$$\left\|\mu^2 R^2(\mu, A)\varphi_N^{\tau_0}(u)\right\|_X = \left\| \int_\varepsilon^{\tau_0} S_N(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\sigma \right\|_X, \quad \leq K_{\tau_0} K \|u\|_{L^p([0,\tau_0];U)}. \tag{59}$$

Invoking (56), an elementary equicontinuity argument, and vector-valued dominated convergence theorem in X, respectively, we obtain

$$X - \lim_{N\to\infty}\int_\varepsilon^{\tau_0} S_N(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\sigma = \int_\varepsilon^{\tau_0} S(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\sigma, \tag{60}$$

which implies via (59) that for all $\mu > \mu_0$

$$\begin{aligned} &\left\| \int_\varepsilon^{\tau_0} S(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\sigma \right\|_X \\ &\leq \sup_{N\geq 0} \left\| \int_\varepsilon^{\tau_0} S_N(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\tau \right\|_X, \\ &\leq K_{\tau_0} K \|u\|_{L^p([0,\tau_0];U)}. \end{aligned} \tag{61}$$

In other words, we have

$$\left\| \int_\varepsilon^{\tau_0} S(\sigma)\mu^2 R^2(\mu, A) Bu(\sigma)\, d\sigma \right\|_X = \left\| \mu^2 R^2(\mu, A) \int_\varepsilon^{\tau_0} S_{-1}(\sigma) Bu(\sigma)\, d\sigma \right\|_X, \tag{62}$$

due to the fact that $S(t)$ commute with $R(\mu, A)$ (see [48, Theorem 7]).

Passing to the limit (i.e., $\mu \to \infty$) in the above inequality, we deduce

$$\left\| \int_{\varepsilon}^{\tau_0} S_{-1}(\sigma) Bu(\sigma) d \right\|_X \leq K_{\tau_0} K \|u\|_{L^p([0,\tau_0];U)}. \tag{63}$$

The same argument also shows that

$$\left\| \int_{\varepsilon}^{\varepsilon'} S_{-1}(\sigma) Bu(\sigma) d\sigma \right\|_X \leq K_{\tau_0} K \|u\chi_{[\varepsilon,\varepsilon']}\|_{L^p([0,\tau_0];U)} \tag{64}$$

which implies that the sequence $(\int_{\varepsilon}^{\tau_0} S_{-1}(\sigma)Bu(\sigma)d\sigma)_{\varepsilon}$ is a Cauchy sequence in X. Moreover,

$$X_{-1} - \lim_{\varepsilon \to 0} \int_{\varepsilon}^{\tau_0} S_{-1}(\sigma) Bu(\sigma) d\sigma = \int_0^{\tau_0} S_{-1}(\sigma) Bu(\sigma) d\sigma \in X. \tag{65}$$

By virtue of $X \hookrightarrow X_{-1}$, and the fact that $\int_{\varepsilon}^{\tau_0} S_{-1}(\sigma)Bu(\sigma)d\sigma$ satisfies (63), we find

$$\left\| \int_0^{\tau_0} S_{-1}(\sigma) Bu(\sigma) d\sigma \right\|_X \leq M(\tau_0) \|u\|_{L^p([0,\tau_0];U)}, \tag{66}$$

which completes the proof of (i).

Part (ii): ($\Rightarrow$) Assume that B is L^p-admissible for $(S(t))_{t\geq 0}$. Let $u \in L^p_c(\mathbb{R}^+;U)$, with $\mathrm{Supp}(u) = [a,b]$. By Fubini's theorem we have

$$\begin{aligned}
&\varphi_N(u) \\
&= \int_a^b S_{-N}(\sigma) Bu(\sigma) d\sigma \\
&= \frac{1}{2\pi} \int_a^b \int_{-N}^{N} e^{(\alpha+i\lambda)\sigma} H_{-1}(\alpha+i\lambda) Bu(\sigma) d\lambda\, d\sigma \\
&= \frac{1}{2\pi} \int_{-N}^{+N} \int_a^b e^{(\alpha+i\lambda)\sigma} H_{-1}(\alpha+i\lambda) Bu(\sigma) d\sigma\, d\lambda \\
&= \frac{1}{2\pi} \int_{-N}^{+N} \int_a^b \int_0^{\infty} e^{-(\alpha+i\lambda)t} S_{-1}(t) e^{(\alpha+i\lambda)\sigma} Bu(\sigma) dt\, d\sigma\, d\lambda \\
&= \frac{1}{2\pi} \int_a^b \int_0^{\infty} \int_{-N}^{N} e^{(\alpha+i\lambda)(\sigma-t)} S_{-1}(t) Bu(\sigma) d\lambda\, dt\, d\sigma \\
&= \frac{1}{2\pi} \int_a^b \int_0^{\infty} \left[\int_{-N}^{+N} e^{(\alpha+i\lambda)(\sigma-t)} d\lambda \right] S_{-1}(t) Bu(\sigma) dt\, d\sigma \\
&= \frac{1}{2i\pi} \int_0^{\infty} e^{-\alpha t} \\
&\quad \cdot \int_a^b e^{\alpha\sigma} . \frac{e^{iN(\sigma-t)} - e^{-iN(\sigma-t)}}{t-\sigma} S_{-1}(t) Bu(\sigma) d\sigma\, dt \\
&= \int_0^{\infty} S_{-1}(t) Be^{-\alpha t} \\
&\quad \times \int_{\mathbb{R}^+} e^{\alpha\sigma} . \frac{\sin N(\sigma-t)}{\pi(\sigma-t)} \chi_{[a,b]} u(\sigma)(\sigma) d\sigma\, dt, \\
&= \int_0^{\infty} S_{-1}(t) Be^{-\alpha t} u_N^{\alpha}(t) dt.
\end{aligned} \tag{67}$$

The use of Fubini theorem in this chain of equalities is justified by the fact that the maps $(\sigma,\lambda) \to e^{\lambda\sigma} H_{-1}(\lambda)Bu(\sigma)$ and $(t,\sigma,\lambda) \to e^{\lambda(\sigma-t)} S_{-1}(t)Bu(\sigma)$ belong to $L^1([a,b] \times [\alpha - iN, \alpha + iN]; X)$, and to $L^1(\mathbb{R}^+ \times [a,b] \times [\alpha - iN, \alpha + iN]; X_{-1})$, respectively.

Here the new input u_N^{α} is given by

$$\begin{aligned}
&u_N^{\alpha}(t) \\
&:= \int_0^{\infty} \frac{\sin N(\sigma-t)}{\pi(\sigma-t)} \left(e^{\alpha\cdot} \chi_{[a,b]}\right)(\sigma) u(\sigma) d\sigma \\
&= \left(D_N * f_{\alpha}\right)(t), \quad \text{with } f_{\alpha}(\cdot) = e^{\alpha\cdot} \chi_{[a,b]}(\cdot) u \in L^p(\mathbb{R}^+;U),
\end{aligned} \tag{68}$$

and D_N denotes the Dirichlet kernel given by

$$D_N(t) = \frac{\sin(Nt)}{\pi t} \quad (t \in \mathbb{R}). \tag{69}$$

Following the lines of the proof of [23, Lemma 3.2], one establishes that for a UMD space Z and $p \in]1,\infty[$ we have

$$\begin{aligned}
&D_N * g \longrightarrow g \quad \text{in } L^p(\mathbb{R}^+;Z) \\
&\text{as } N \longrightarrow \infty \quad \forall g \in L^p(\mathbb{R}^+;Z).
\end{aligned} \tag{70}$$

Thus, making use of the fact that U is a UMD space, appealing (70), we obtain

$$D_N * f_{\alpha} \longrightarrow f_{\alpha} \quad \text{in } L^p(\mathbb{R}^+;U), \quad \text{as } N \longrightarrow \infty. \tag{71}$$

Then, using the L^p-admissibility of B and the fact that $\omega_0(S) < 0$, we obtain

$$\begin{aligned}
&\left\|\varphi_{N_1}(u) - \varphi_{N_2}(u)\right\|_X \\
&\quad = \left\| \int_0^{\infty} S_{-1}(t) Be^{-\alpha t} \left[u_{N_1}^{\alpha}(t) - u_{N_2}^{\alpha}(t)\right] dt \right\|_X \\
&\quad = \left\| \mathfrak{B}_{\infty} \left(e^{-\alpha\cdot} \left(u_{N_1}^{\alpha} - u_{N_2}^{\alpha}\right)\right)\right\|_X \\
&\quad \leq M \left\|u_{N_1}^{\alpha} - u_{N_2}^{\alpha}\right\|_{L^p(\mathbb{R}^+;U)},
\end{aligned} \tag{72}$$

for $N_1 > N_2 > 0$, which implies that $(\varphi_N(u))_{N\geq 0}$ is a Cauchy sequence in X. Thus $(\varphi_N(u))_{N\geq 0}$ converges in X as $N \to \infty$, for all $u \in L_c^p(\mathbb{R}^+;U)$ and that $(\varphi_N)_{N\geq 0}$ is uniformly bounded in $\mathscr{L}(L_c^p(\mathbb{R}^+;U),X)$ according to Banach-Steinhaus's theorem. Furthermore, using once again $e^{-\alpha\cdot}u_N^\alpha \to e^{-\alpha\cdot}f_\alpha = u$ in $L^p(\mathbb{R}^+;U)$, we obtain $\int_0^\infty S_{-1}(t)Be^{-\alpha t}u_N^\alpha(t)dt$ converges to $\int_0^\infty S_{-1}(t)Bu(t)dt (= \varphi(u))$ in X_{-1} as $N \to \infty$. By virtue of $X \hookrightarrow X_{-1}$ with continuous injection and the fact that $(\varphi_N(u))_{N\geq 0}$ converges in X as $N \to \infty$, we deduce that $\varphi_N(u)$ converges to $\varphi(u)$ in X as $N \to \infty$, for all $u \in L_c^p(\mathbb{R}^+;U)$.

($\Leftarrow$) Assume that φ_N is uniformly bounded on $\mathscr{L}(L_c^p(\mathbb{R}^+;U);X)$. For $\tau_0 > 0$ and $u \in L^p(\mathbb{R}^+;U)$ we have $v := u\cdot\chi_{[0,\tau_0]} \in L_c^p(\mathbb{R}^+;U)$ and

$$\|\varphi_N(v)\|_X = \|\varphi_N^{\tau_0}(u)\|_X \leq \kappa\|u\|_{L^p([0,\tau_0];U)}, \tag{73}$$

for some $\kappa > 0$.

Thanks to (i) we deduce that B is finite-time L^p-admissible (for time τ_0) for $(S(t))_{t\geq 0}$. By examining the proof of (i) we deduce that B is L^p-admissible for $(S(t))_{t\geq 0}$. This ends the proof. □

Remark 13. (1) The result of Theorem 12 may extend to systems of Volterra integral equations (3) provided that $\int_0^t S(\sigma)x d\sigma$ takes value in $D(A)$ for all $x \in X$. One may of course ask whether it is a severe restriction to consider only $a(t) = 1 + (1 * k)(t)$. This will be a subject of a forthcoming work.

(2) We can retrieve the result stated in Proposition 8 by using [38, Remark 3.5 (ii)]. Indeed, let $\mathscr{B} = Id_{\mathscr{X}}$, then $\mathscr{B}$ is infinite-time L^p-admissible for $(\mathscr{T}^{V,\omega}(t) := e^{-\omega t}\mathscr{T}^V(t))_{t\geq 0}$ with $\omega > \omega_0(\mathscr{A}^V)$. In fact, for $t_0 > 0$ and $x \in X$, consider the input $u_0(t) := \mathscr{T}^{V,\omega}(t_0 - t)\binom{x}{0}\chi_{[t_0/2,\, t_0]}$, we set:

$$\Phi_N(u_0) := \int_0^\infty \mathscr{T}_N^{V,\omega}(\sigma)u_0(\sigma)d\sigma, \tag{74}$$

with

$$\mathscr{T}_N^{V,\omega}(t)\begin{pmatrix}x\\f\end{pmatrix} := \frac{1}{2\pi}\int_{-N}^N e^{i\lambda t}R(\omega + i\lambda, \mathscr{A}^V)\begin{pmatrix}x\\f\end{pmatrix}d\lambda. \tag{75}$$

Thanks to [38, Remark 3.5] we have

$$\Phi_N(u_0) = \frac{t_0}{2}\mathscr{T}_N^{V,\omega}(t_0)\begin{pmatrix}x\\0\end{pmatrix} + L_N^\omega(t_0)\begin{pmatrix}x\\0\end{pmatrix}, \tag{76}$$

with $L_N^\omega(t_0) \in \mathscr{L}(\mathscr{X})$ and $L_N^\omega(t_0)\binom{x}{0} \to 0$ as $N \to \infty$ for all $x \in X$.

Passing to the limit in (76) as $N \to \infty$ and invoking [38, Corollary 4.2] we obtain

$$\Phi_N(u_0) \longrightarrow \Phi(u_0) \quad \text{as } N \longrightarrow \infty \text{ in } \mathscr{X}, \tag{77}$$

with

$$\begin{aligned}\Phi(u_0) &:= \int_0^\infty \mathscr{T}^{V,\omega}(\sigma)u_0(\sigma)d\sigma \\ &= \int_0^\infty \mathscr{T}^{V,\omega}(\sigma)\mathscr{T}^{V,\omega}(t_0-\sigma)\begin{pmatrix}x\\0\end{pmatrix}\chi_{[t_0/2,t_0]}d\sigma \\ &= \int_{t_0/2}^{t_0}\mathscr{T}^{V,\omega}(t_0)\begin{pmatrix}x\\0\end{pmatrix}d\sigma \\ &= \frac{t_0}{2}\mathscr{T}^{V,\omega}(t_0)\begin{pmatrix}x\\0\end{pmatrix} \\ &= \frac{t_0}{2}\cdot e^{-\omega t_0}\mathscr{T}^V(t_0)\begin{pmatrix}x\\0\end{pmatrix} \\ &= \frac{t_0}{2}\cdot e^{-\omega t_0}\begin{pmatrix}S(t_0) & T_{12}(t_0)\\ T_{21}(t_0) & T_{22}(t_0)\end{pmatrix}\begin{pmatrix}x\\0\end{pmatrix} \\ &= \frac{t_0}{2}\cdot e^{-\omega t_0}\begin{pmatrix}S(t_0)x\\ T_{21}(t_0)x\end{pmatrix}.\end{aligned} \tag{78}$$

In other words, for $\omega > \omega^*$ $(> \omega_0(\mathscr{A}^V))$ we have

$$\begin{aligned}\mathscr{T}_N^{V,\omega}(t_0)\begin{pmatrix}x\\0\end{pmatrix} &= \frac{1}{2\pi}\int_{-N}^N e^{i\lambda t_0}R(\omega+i\lambda,\mathscr{A}^V)\begin{pmatrix}x\\0\end{pmatrix}d\lambda \\ &= \frac{1}{2i\pi}\int_{\omega-iN}^{\omega+iN} e^{(\mu-\omega)t_0}R(\mu,\mathscr{A}^V)\begin{pmatrix}x\\0\end{pmatrix}d\mu \\ &= \frac{1}{2i\pi}\int_{\omega-iN}^{\omega+iN} e^{-\omega t_0}\cdot e^{\mu t}R(\mu,\mathscr{A}^V)\begin{pmatrix}x\\0\end{pmatrix}d\mu.\end{aligned} \tag{79}$$

Using Lemma 7(i) we obtain

$$\begin{aligned}&\mathscr{T}_N^{V,\omega}(t_0)\begin{pmatrix}x\\0\end{pmatrix} \\ &= \frac{1}{2i\pi}\int_{\omega-iN}^{\omega+iN} e^{-\omega t_0}\cdot\begin{pmatrix}H(\mu)x\\ R\left(\mu,\frac{d}{ds}\right)MH(\mu)x\end{pmatrix}d\mu \\ &= \begin{pmatrix}e^{-\omega t_0}S_N(t_0)x\\ \frac{1}{2i\pi}\int_{-N}^N e^{i\mu t_0}R\left(\omega+i\mu,\frac{d}{ds}\right)MH(\omega+i\mu)x d\mu\end{pmatrix}.\end{aligned} \tag{80}$$

Combining (76), (77), and (80), we deduce that

$$\frac{t_0}{2}\cdot e^{-\omega t_0}S_N(t_0)x \longrightarrow \frac{t_0}{2}\cdot e^{-\omega t_0}S(t_0)x \quad \text{as } N \longrightarrow \infty\ \forall x \in X, \tag{81}$$

which in turns implies that

$$\lim_{N\to\infty}\frac{1}{2i\pi}\int_{\omega-iN}^{\omega+iN} e^{\lambda t}H(\lambda)x d\lambda = S(t)x \quad \text{as } N \longrightarrow \infty\ \forall x \in X, \tag{82}$$

uniformly in t from compact subsets of $]0,\infty[$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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