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Research Article

A Less Conservative Stability Criterion for Delayed Stochastic Genetic Regulatory Networks

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This paper concerns the problem of stability analysis for delayed stochastic genetic regulatory networks. By introducing an appropriate Lyapunov-Krasovskii functional and employing delay-range partition approach, a new stability criterion is given to ensure the mean square stability of genetic regulatory networks with time-varying delays and stochastic disturbances. The stability criterion is given in the form of linear matrix inequalities, which can be easily tested by the LMI Toolbox of MATLAB. Moreover, it is theoretically shown that the obtained stability criterion is less conservative than the one in W. Zhang et al., 2012. Finally, a numerical example is presented to illustrate our theory.

1. Introduction

With the further progress of gene expression, researchers find that a gene expression is affected by other genes; conversely, it also influences others. Based on this reciprocal impact relation, gene expression forms a complex network—genetic regulatory network (GRN). GRNs are dynamical systems, which consist of an interaction of genes, proteins, and small molecules. In the past two decades, scholars have established mathematical models to represent GRNs. Basically, there are four types of GRN models, that is, Petri net model [1], Bayesian network model [2], Boolean model [3, 4], and (functional) differential equation model [5, 6]. The concentrations of mRNA and protein are described as the state variables in the functional differential equation model.

As dynamical systems, stability analysis is the first priority to explore GRNs. On the one hand, time delay inevitably occurs in GRNs due to the slow process of transcription, translation, and translocation [7]. On the other hand, internal noises of cells caused by random birth and death of the individual molecules and external noises from environmental fluctuations make the gene expression be best viewed as a stochastic process [8, 9]. So, it is very necessary to analyze the stability of GRNs with time-varying delays and stochastic disturbances [6, 10–18].

Recently, for a class of GRNs with interval time-varying delays and stochastic disturbances (see (17a), (17b) below), Wu et al. [18] established several delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria in terms of linear matrix inequalities (LMIs) by using the stochastic analysis approach, employing some free-weighting matrices and introducing a type of Lyapunov-Krasovskii functional which includes the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ and $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$, where P and Q are real symmetric positive definite matrices. Furthermore, in [18], the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ in the stochastic differential of $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ was first estimated by employing Leibniz-Newton formula and the inequality

$$\begin{aligned}
& -2\xi^T(t)M\int_{t-\tau(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\
& \leq (\tau(t)-\tau_1)\,\xi^T(t)MP^{-1}M^T\xi(t) \\
& \quad + \int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\,\mathrm{d}s,
\end{aligned} \tag{1}$$

where M is a free-weighting matrix, and then $\tau(t) - \tau_1$ was enlarged to $\tau_2 - \tau_1$. Clearly, the conservatism would be produced because $\tau(t) - \tau_1$ is enlarged to $\tau_2 - \tau_1$. In order to reduce the conservatism, Zhang et al. [17] first estimated the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ by using Leibniz-Newton formula and the inequality

$$\begin{aligned} & -2\xi^T(t)S\int_{t-\tau(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\ & \leq (1-\alpha)(\tau(t)-\tau_1)\xi^T(t)SP^{-1}S^T\xi(t) \\ & \quad + \alpha(\tau(t)-\tau_1)\xi^T(t)NP^{-1}N^T\xi(t) \\ & \quad + \int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\,\mathrm{d}s, \end{aligned} \tag{2}$$

where S and N are free-weighting matrices and α is an adjusting parameter with $0 < \alpha < 1$, and then a so-called convex combination technique was employed to obtain less conservative delay-range-dependent and/or rate-dependent global stochastic asymptotical stability criteria. It should be emphasized that the Lyapunov-Krasovskii functional used in [17] includes not only the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_1^T(s)Ph_1(s)\mathrm{d}s\,\mathrm{d}\theta$ and $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$, but also the items like $\int_{t-\alpha\tau_1}^{t} x^T(s)Q_4x(s)\mathrm{d}s$ and $\int_{t-\tau_1}^{t-\alpha\tau_1} x^T(s)Q_4x(s)\mathrm{d}s$ involved in the adjustable parameter α, which can be viewed as a generalized delay-partition approach due to the adjustable parameter α. Generally, an appropriate delay-partition approach can bring less conservative stability criteria (see [19] and the references therein).

Note that free-weighting matrices (e.g., matrices M, S, and N above) have been employed in both [17, 18] to obtain less conservative stability criteria. However, it is emphasized in [18, Remark 2] that free-weighting matrices may produce a super-high amount of computation for the feasible solutions of LMIs. In order to overcome the disadvantage, in this paper we will propose a delay-range partition (DRP) approach to estimate accurately the item $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ (see (27) and (30)), where no free-weighting matrix is involved. By employing an appropriate Lyapunov-Krasovskii functional and introducing a DRP approach, a mean square stability criterion for GRNs with time-varying delays and stochastic disturbances is first established. Then it is theoretically shown that the proposed stability criterion is less conservative than [17, Theorem 1]. Finally, a numerical example is given to illustrate the theoretical results proposed here. The main contribution of this paper can be listed as follows: (i) the Lyapunov-Krasovskii functional employed in this paper does not include the items like $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} h_3^T(s)Qh_3(s)\mathrm{d}s\,\mathrm{d}\theta$ which are required in [17, 18]; (ii) the items like $-\int_{t-\tau(t)}^{t-\tau_1} h_1^T(s)Ph_1(s)\mathrm{d}s$ in the stochastic differential of Lyapunov-Krasovskii functional are estimated accurately by proposing a DRP approach; (iii) theoretical comparison of the stability criterion [17, Theorem 1] and the one proposed in this paper is given; and (iv) there is no free-weighting matrix involved, which reduces the computational complexity.

The rest of the paper is organized as follows. In Section 2, the model of GRNs to be studied is described. A DRP-based mean square stability criterion (Theorem 3 below) for GRNs with time-varying delays and stochastic disturbances is established in Section 3. The theoretical comparison of Theorem 3 and [17, Theorem 1] is presented in Section 4. In Section 5, an example is given to show the validity of the obtained results. Finally, in Section 6, the conclusions are drawn.

Notation. For a positive integer n, set $\langle n\rangle = \{1, 2, \ldots, n\}$. R^n denotes the n-dimensional Euclidean space. We denote by $R^{m\times n}$ the set of $m \times n$ matrices over R. A^T and A^{-1} represent the transpose and inverse of a matrix A, respectively. For real symmetric matrices X and Y, the notation $X \geq Y$ $(X > Y)$ means that the matrix $X - Y$ is positive semidefinite (positive definite). I_n is the $n\times n$ identity matrix. In a symmetric matrix, $\star$ denotes the entries implied by symmetry.

2. Model Description

The following differential equations have been used recently to describe GRNs [7]:

$$\begin{aligned} \dot{m}_i(t) = & -a_i m_i(t) + b_i(p_1(t-\sigma(t)), p_2(t-\sigma(t)), \ldots, \\ & p_n(t-\sigma(t))), \end{aligned} \tag{3a}$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t-\tau(t)), \quad i \in \langle n\rangle, \tag{3b}$$

where $m_i(t)$ and $p_i(t)$ are the concentrations of the ith mRNA and protein at time t, respectively; $a_i > 0$, $c_i > 0$, and $d_i > 0$ are constants, representing the degradation rate of the ith mRNA, the degradation rate of the ith protein, and the translation rate of the ith mRNA to ith protein, respectively; both $\sigma(t)$ and $\tau(t)$ are transcriptional and translational delays, respectively; b_i is the regulatory function of the ith gene, which is generally a nonlinear function of the variables $p_1(t), p_2(t), \ldots, p_n(t)$, but it is monotonic with each variable.

For convenience, we give the following assumptions throughout the paper.

Assumption 1. The delays $\sigma(t)$ and $\tau(t)$ are differentiable functions satisfying

$$0 \leq \sigma_1 \leq \sigma(t) \leq \sigma_2, \qquad 0 \leq \tau_1 \leq \tau(t) \leq \tau_2, \tag{4a}$$

$$\dot{\sigma}(t) \leq \sigma_d < +\infty, \qquad \dot{\tau}(t) \leq \tau_d < +\infty, \tag{4b}$$

where $\sigma_1, \sigma_2, \sigma_d, \tau_1, \tau_2$, and τ_d are constants.

Assumption 2. The function b_i is taken as

$$b_i(p_1(t), p_2(t), \ldots, p_n(t)) = \sum_{j=1}^{n} b_{ij}(p_j(t)), \tag{5}$$

which is called SUM logic. Here, b_{ij} is a monotonic function of the Hill form; that is,

$$b_{ij}(x) = \begin{cases} \alpha_{ij}\dfrac{(x/\beta_j)^{H_j}}{1+(x/\beta_j)^{H_j}} & \text{if transcription factor } j \text{ is an activator of gene } i, \\ \alpha_{ij}\dfrac{1}{1+(x/\beta_j)^{H_j}} & \text{if transcription factor } j \text{ is a represssor of gene } i, \end{cases} \quad (6)$$

where H_j is the Hill coefficient, β_j is a scalar, and α_{ij} is a bounded constant, which denotes the dimensionless transcriptional rate of transcription factor j to gene i.

Clearly, GRN ((3a), (3b)) can be rewritten as

$$\dot{m}_i(t) = -a_i m_i(t) + \sum_{j=1}^{n} w_{ij} h_j(p_j(t-\sigma(t))) + v_i, \quad (7a)$$

$$\dot{p}_i(t) = -c_i p_i(t) + d_i m_i(t-\tau(t)), \quad i \in \langle n\rangle, \quad (7b)$$

where

$$w_{ij} = \begin{cases} \alpha_{ij} & \text{if transcription factor } j \text{ is an activator of gene } i, \\ 0 & \text{if there is no connection between } j \text{ and } i, \\ -\alpha_{ij} & \text{if transcription factor } j \text{ is a represssor of gene } i, \end{cases} \quad (8)$$

$$h_j(x) = \frac{(x/\beta_j)^{H_j}}{1+(x/\beta_j)^{H_j}}, \qquad v_i = \sum_{j\in\mathscr{V}_i} \alpha_{ij},$$

and $\mathscr{V}_i$ is the set of all the transcription factors j which is a repressor of gene i.

Rewriting GRN ((7a), (7b)) into compact matrix form, we obtain

$$\dot{m}(t) = -Am(t) + Wh(p(t-\sigma(t))) + v, \quad (9a)$$

$$\dot{p}(t) = -Cp(t) + Dm(t-\tau(t)), \quad (9b)$$

where

$$\begin{aligned} A &= \operatorname{diag}(a_1, a_2, \ldots, a_n), \\ W &= [w_{ij}]_{n\times n}, \\ C &= \operatorname{diag}(c_1, c_2, \ldots, c_n), \\ D &= \operatorname{diag}(d_1, d_2, \ldots, d_n), \\ v &= \operatorname{col}(v_1, v_2, \ldots, v_n), \\ m(t) &= \operatorname{col}(m_1(t), m_2(t), \ldots, m_n(t)), \\ p(t) &= \operatorname{col}(p_1(t), p_2(t), \ldots, p_n(t)), \\ h(p(t)) &= \operatorname{col}(h_1(p_1(t)), h_2(p_2(t)), \ldots, h_n(p_n(t))). \end{aligned} \quad (10)$$

Let (m^*, p^*) be an equilibrium point of ((9a), (9b)); that is, it is a solution of the following equation:

$$-Am^* + Wh(p^*) + v = 0, \qquad -Cp^* + Dm^* = 0. \quad (11)$$

For convenience, we shift the equilibrium point (m^*, p^*) to the origin by using the transformations $x(t) = m(t) - m^*$ and $y(t) = p(t) - p^*$; then we have

$$\dot{x}(t) = -Ax(t) + Wf(y(t-\sigma(t))), \quad (12a)$$

$$\dot{y}(t) = -Cy(t) + Dx(t-\tau(t)), \quad (12b)$$

where $f(y(t)) = h(y(t)+p^*) - h(p^*)$.

From the relationship between h and f, one can easily find that f satisfies the following sector condition:

$$f_i(0) = 0, \qquad l_i^- \le \frac{f_i(s)}{s} \le l_i^+, \quad i \in \langle n\rangle, \ \forall 0 \ne s \in R, \quad (13)$$

where l_i^- and l_i^+ are a pair of nonnegative scalars and $f_i(s)$ is the ith entry of $f(s)$. Since h_i is a monotonically increasing and differentiable function with saturation, we have to choose l_i^- as zero or a small positive number. Let $L^- = \operatorname{diag}(l_1^-, l_2^-, \ldots, l_n^-)$ and $L^+ = \operatorname{diag}(l_1^+, l_2^+, \ldots, l_n^+)$.

As shown in [15–17] the gene regulation is an intrinsically noisy process. For this reason, in this paper, we consider a class of GRNs with both time delays and noise disturbances by the following model:

$$\begin{aligned} dx(t) &= [-Ax(t) + Wf(y(t-\sigma(t)))]\,dt \\ &\quad + H(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t)))\,d\omega(t), \end{aligned} \quad (14a)$$

$$dy(t) = [-Cy(t) + Dx(t-\tau(t))]\,dt, \quad (14b)$$

where $\omega(t)$ is an m-dimensional Brown motion, $m \ge 1$, and $H(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t)))$ is the noise intensity matrix at time t such that

$$\begin{aligned} \operatorname{trace}(HH^T) &\le x^T(t)H_1x(t) + y^T(t)H_2y(t) \\ &\quad + x^T(t-\tau(t))H_3x(t-\tau(t)) \\ &\quad + y^T(t-\sigma(t))H_4y(t-\sigma(t)), \end{aligned} \quad (15)$$

where H_i $(i = 1,2,3,4)$ are real symmetric positive semidefinite matrices.

For simplicity, set

$$\begin{aligned} h_1(t) &= -Ax(t) + Wf(y(t-\sigma(t))), \\ h_2(t) &= -Cy(t) + Dx(t-\tau(t)), \\ h_3(t) &= H(x(t), x(t-\tau(t)), y(t), y(t-\sigma(t))). \end{aligned} \quad (16)$$

Then, GRN ((14a), (14b)) can be represented as

$$dx(t) = h_1(t)\,dt + h_3(t)\,d\omega(t), \quad (17a)$$

$$dy(t) = h_2(t)\,dt. \quad (17b)$$

3. Stability Criterion

In the following theorem, we will propose a DRP approach to present an asymptotical stability criterion in the mean square sense for GRNs with time-varying delays and stochastic disturbance.

Theorem 3. *For given scalars $\alpha \in (0,1)$, $\tau_2 > \tau_1 > 0$, $\sigma_2 > \sigma_1 > 0$, τ_d and σ_d, and positive integers s_1 and s_2, under the conditions* (15) *and* ((4a), (4b)), *we can conclude that GRN* ((14a), (14b)) *is asymptotically stable in the sense of mean square, if there exist a scalar $\rho > 0$ and matrices $U_i^T = U_i > 0$ $(i = 1,2)$, $V_j^T = V_j > 0$, $W_j^T = W_j > 0$ $(j = 1,2,\ldots,7)$, and $\Lambda := \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n) > 0$ such that the following LMIs hold:*

$$U_1 \le \rho I, \tag{18}$$

$$\begin{aligned}\Psi_{k,l}: &= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6 \\ &\quad + \Psi_k + \Psi_l < 0, \quad \forall k \in \langle s_1\rangle,\ l \in \langle s_2\rangle,\end{aligned} \tag{19}$$

where

$$\begin{aligned}
\Psi_1 &= \widetilde{\Psi}_1 + \widetilde{\Psi}_1^T + \rho\left(e_1^T H_1 e_1 + e_3^T H_2 e_3 + e_7^T H_3 e_7 + e_9^T H_4 e_9\right),\\
\widetilde{\Psi}_1 &= e_1^T U_1(-Ae_1 + We_{13}) + e_7^T U_2(-Ce_7 + De_3),\\
\Psi_2 &= e_1^T(V_1+V_5)e_1 + e_4^T(V_2-V_1)e_4 + e_5^T(V_3-V_2)e_5\\
&\quad + e_2^T\left[(1+\alpha\tau_d)V_4 - (1-\alpha\tau_d)V_3\right]e_2\\
&\quad - (1-\tau_d)e_3^T V_4 e_3 - e_6^T V_5 e_6,\\
\Psi_3 &= e_7^T(W_1+W_5)e_7 + e_{10}^T(W_2-W_1)e_{10}\\
&\quad + e_{11}^T(W_3-W_2)e_{11}\\
&\quad + e_8^T\left[(1+\alpha\sigma_d)W_4 - (1-\alpha\sigma_d)W_3\right]e_8\\
&\quad - (1-\sigma_d)e_9^T W_4 e_9 - e_{12}^T W_5 e_{12},\\
\Psi_4 &= (-Ae_1 + We_{13})^T(\tau_1 V_6 + \tau_{12}V_7)(-Ae_1 + We_{13})\\
&\quad + (-Ce_7 + De_3)^T(\sigma_1 W_6 + \sigma_{12}W_7)(-Ce_7 + De_3),\\
\Psi_5 &= \widetilde{\Psi}_5 + \widetilde{\Psi}_5^T,\\
\widetilde{\Psi}_5 &= e_{13}^T(L^+)^{-}\Lambda L^- e_9 - e_{13}^T(L^+)^{-}\Lambda e_{13} - e_9^T\Lambda L^- e_9 + e_9^T\Lambda e_{13},\\
\Psi_6 &= -\frac{1}{\alpha\tau_1}(e_4-e_1)^T V_6(e_4-e_1)\\
&\quad - \frac{1}{(1-\alpha)\tau_1}(e_5-e_4)^T V_6(e_5-e_4)\\
&\quad - \frac{1}{\alpha\sigma_1}(e_{10}-e_7)W_6(e_{10}-e_7)\\
&\quad - \frac{1}{(1-\alpha)\sigma_1}(e_{11}-e_{10})^T W_6(e_{11}-e_{10}),\\
\Psi_k &= -\frac{s_1}{k\alpha\tau_{12}}(e_2-e_5)^T V_7(e_2-e_5)\\
&\quad - \frac{s_1}{(1-\alpha)k\tau_{12}}(e_3-e_2)^T V_7(e_3-e_2)\\
&\quad - \frac{s_1}{(s_1-k+1)\tau_{12}}(e_6-e_3)^T V_7(e_6-e_3),\\
\Psi_l &= -\frac{s_2}{l\alpha\sigma_{12}}(e_8-e_{11})^T W_7(e_8-e_{11})\\
&\quad - \frac{s_2}{(1-\alpha)l\sigma_{12}}(e_9-e_8)^T W_7(e_9-e_8)\\
&\quad - \frac{s_2}{(s_2-l+1)\sigma_{12}}(e_{12}-e_9)^T W_7(e_{12}-e_9),\\
e_i &= \Big[\underbrace{0\ \cdots\ 0}_{\text{number } i-1}\ \ I_n\ \ \underbrace{0\ \cdots\ 0}_{\text{number } n-i}\Big], \qquad i = 1,2,\ldots,n,\\
\sigma_{12} &= \sigma_2 - \sigma_1, \qquad \tau_{12} = \tau_2 - \tau_1.
\end{aligned} \tag{20}$$

Proof. Let $\beta(t) = \tau_1 + \alpha(\tau(t) - \tau_1)$ and $\gamma(t) = \sigma_1 + \alpha(\sigma(t) - \sigma_1)$. Choose a Lyapunov-Krasovskii functional candidate as

$$V(t) = \sum_{i=1}^{4} V_i(t), \tag{21}$$

where

$$\begin{aligned}
V_1(t) &= x^T(t)U_1x(t) + y^T(t)U_2y(t),\\
V_2(t) &= \int_{t-\alpha\tau_1}^{t} x^T(s)V_1x(s)\,ds\\
&\quad + \int_{t-\tau_1}^{t-\alpha\tau_1} x^T(s)V_2x(s)\,ds\\
&\quad + \int_{t-\beta(t)}^{t-\tau_1} x^T(s)V_3x(s)\,ds\\
&\quad + \int_{t-\tau(t)}^{t-\beta(t)} x^T(s)V_4x(s)\,ds\\
&\quad + \int_{t-\tau_2}^{t} x^T(s)V_5x(s)\,ds,\\
V_3(t) &= \int_{t-\alpha\sigma_1}^{t} y^T(s)W_1y(s)\,ds\\
&\quad + \int_{t-\sigma_1}^{t-\alpha\sigma_1} y^T(s)W_2y(s)\,ds\\
&\quad + \int_{t-\gamma(t)}^{t-\sigma_1} y^T(s)W_3y(s)\,ds\\
&\quad + \int_{t-\sigma(t)}^{t-\gamma(t)} y^T(s)W_4y(s)\,ds\\
&\quad + \int_{t-\sigma_2}^{t} y^T(s)W_5y(s)\,ds,
\end{aligned}$$

$$
\begin{aligned}
V_4(t) = & \int_{-\tau_1}^{0} \int_{t+\theta}^{t} h_1^T(s) V_6 h_1(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\tau_2}^{-\tau_1} \int_{t+\theta}^{t} h_1^T(s) V_7 h_1(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\sigma_1}^{0} \int_{t+\theta}^{t} h_2^T(s) W_6 h_2(s)\,\mathrm{d}s\,\mathrm{d}\theta \\
& + \int_{-\sigma_2}^{-\sigma_1} \int_{t+\theta}^{t} h_2^T(s) W_7 h_2(s)\,\mathrm{d}s\,\mathrm{d}\theta,
\end{aligned} \tag{22}
$$

and the matrices $U_i^T = U_i > 0$ $(i = 1, 2)$, $V_j^T = V_j > 0$, and $W_j^T = W_j > 0$ $(j = 1, 2, \ldots, 7)$ are taken from a feasible solution to (18) and (19). By Itô's formula, we can obtain the following stochastic differential:

$$
\mathrm{d}V(t) = \sum_{i=1}^{4} \mathscr{L} V_i(t)\,\mathrm{d}t + 2x^T(t) U_1 h_3(t)\,\mathrm{d}\omega(t), \tag{23}
$$

where $\mathscr{L}$ is the weak infinitesimal operator and

$$
\begin{aligned}
\mathscr{L} V_1(t) = & 2x^T(t) U_1 h_1(t) + 2y^T(t) U_2 h_2(t) \\
& + h_3^T(t) U_1 h_3(t) \\
\leq & \eta^T(t) \Psi_1 \eta(t),
\end{aligned} \tag{24}
$$

$$
\begin{aligned}
\mathscr{L} V_2(t) = & x^T(t)(V_1 + V_5) x(t) \\
& + x^T(t - \alpha\tau_1)(V_2 - V_1) x(t - \alpha\tau_1) \\
& + x^T(t - \tau_1)(V_3 - V_2) x(t - \tau_1) \\
& + \left(1 - \dot{\beta}(t)\right) x^T(t - \beta(t))(V_4 - V_3) x(t - \beta(t)) \\
& - (1 - \dot{\tau}(t)) x^T(t - \tau(t)) V_4 x(t - \tau(t)) \\
& - x^T(t - \tau_2) V_5 x(t - \tau_2) \\
\leq & \eta^T(t) \Psi_2 \eta(t),
\end{aligned} \tag{25}
$$

$$
\begin{aligned}
\mathscr{L} V_3(t) = & y^T(t)(W_1 + W_5) y(t) \\
& + y^T(t - \alpha\sigma_1)(W_2 - W_1) y(t - \alpha\sigma_1) \\
& + y^T(t - \sigma_1)(W_3 - W_2) y(t - \sigma_1) \\
& + (1 - \dot{\gamma}(t)) y^T(t - \gamma(t))(W_4 - W_3) y(t - \gamma(t)) \\
& - (1 - \dot{\sigma}(t)) y^T(t - \sigma(t)) W_4 y(t - \sigma(t)) \\
& - y^T(t - \sigma_2) W_5 y(t - \sigma_2) \\
\leq & \eta^T(t) \Psi_3 \eta(t),
\end{aligned} \tag{26}
$$

$$
\begin{aligned}
\mathscr{L} V_4(t) = & h_1^T(t)(\tau_1 V_6 + \tau_{12} V_7) h_1(t) \\
& - \int_{t-\tau_1}^{t} h_1^T(s) V_6 h_1(s)\,\mathrm{d}s - \int_{t-\tau_2}^{t-\tau_1} h_1^T(s) V_7 h_1(s)\,\mathrm{d}s \\
& + h_2^T(t)(\sigma_1 W_6 + \sigma_{12} W_7) h_2(t) \\
& - \int_{t-\sigma_1}^{t} h_2^T(s) W_6 h_2(s)\,\mathrm{d}s \\
& - \int_{t-\sigma_2}^{t-\sigma_1} h_2^T(s) W_7 h_2(s)\,\mathrm{d}s \\
\leq & \eta^T(t) \Psi_4 \eta(t) \\
& - \frac{1}{\alpha\tau_1} \int_{t-\alpha\tau_1}^{t} h_1^T(s)\,\mathrm{d}s\, V_6 \int_{t-\alpha\tau_1}^{t} h_1(s)\,\mathrm{d}s \\
& - \frac{1}{(1-\alpha)\tau_1} \int_{t-\tau_1}^{t-\alpha\tau_1} h_1^T(s)\,\mathrm{d}s\, V_6 \int_{t-\tau_1}^{t-\alpha\tau_1} h_1(s)\,\mathrm{d}s \\
& - \frac{1}{\beta(t) - \tau_1} \int_{t-\beta(t)}^{t-\tau_1} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\beta(t)}^{t-\tau_1} h_1(s)\,\mathrm{d}s \\
& - \frac{1}{\tau(t) - \beta(t)} \int_{t-\tau(t)}^{t-\beta(t)} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\tau(t)}^{t-\beta(t)} h_1(s)\,\mathrm{d}s \\
& - \frac{1}{\tau_2 - \tau(t)} \int_{t-\tau_2}^{t-\tau(t)} h_1^T(s)\,\mathrm{d}s\, V_7 \int_{t-\tau_2}^{t-\tau(t)} h_1(s)\,\mathrm{d}s \\
& - \frac{1}{\alpha\sigma_1} \int_{t-\alpha\sigma_1}^{t} h_2^T(s)\,\mathrm{d}s\, W_6 \int_{t-\alpha\sigma_1}^{t} h_2(s)\,\mathrm{d}s \\
& - \frac{1}{(1-\alpha)\sigma_1} \int_{t-\sigma_1}^{t-\alpha\sigma_1} h_2^T(s)\,\mathrm{d}s\, W_6 \int_{t-\sigma_1}^{t-\alpha\sigma_1} h_2(s)\,\mathrm{d}s \\
& - \frac{1}{\gamma(t) - \sigma_1} \int_{t-\gamma(t)}^{t-\sigma_1} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\gamma(t)}^{t-\sigma_1} h_2(s)\,\mathrm{d}s \\
& - \frac{1}{\sigma(t) - \gamma(t)} \int_{t-\sigma(t)}^{t-\gamma(t)} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\sigma(t)}^{t-\gamma(t)} h_2(s)\,\mathrm{d}s \\
& - \frac{1}{\sigma_2 - \sigma(t)} \int_{t-\sigma_2}^{t-\sigma(t)} h_2^T(s)\,\mathrm{d}s\, W_7 \int_{t-\sigma_2}^{t-\sigma(t)} h_2(s)\,\mathrm{d}s.
\end{aligned} \tag{27}
$$

For any scalars a, b with $a < b$, it follows from (16) that $\int_a^b h_1(t)\mathrm{d}t = x(b) - x(a) - \int_a^b h_3(t)\mathrm{d}\omega(t)$ and $\int_a^b h_2(t)\mathrm{d}t = y(b) - y(a)$, and hence

$$
\begin{aligned}
& \mathscr{E}\left(\int_a^b h_1^T(t)\,\mathrm{d}t V_j \int_a^b h_1(t)\,\mathrm{d}t \right) \\
& \quad \geq [x(b) - x(a)]^T V_j [x(b) - x(a)], \quad j = 6, 7,
\end{aligned} \tag{28a}
$$

$$
\begin{aligned}
& \int_a^b h_2^T(t)\,\mathrm{d}t W_j \int_a^b h_2(t)\,\mathrm{d}t \\
& \quad = [y(b) - y(a)]^T W_j [y(b) - y(a)], \quad j = 6, 7,
\end{aligned} \tag{28b}
$$

where $\mathscr{E}$ represents the mathematical expectation operator. Next, from the sector condition (13), we can obtain that

$$\begin{aligned} 0 \le\ & -2\left[\left(L^{+}\right)^{-} f\left(y\left(t-\sigma\left(t\right)\right)\right)-y\left(t-\sigma\left(t\right)\right)\right]^{T} \\ & \times \Lambda\left[f\left(y\left(t-\sigma\left(t\right)\right)\right)-L^{-} y\left(t-\sigma\left(t\right)\right)\right] \\ =\ & \eta^{T}\left(t\right) \Psi_{5} \eta\left(t\right). \end{aligned} \tag{29}$$

When $\tau(t) \in [\tau_1 + ((k-1)/s_1)\tau_{12},\ \tau_1 + (k/s_1)\tau_{12}]$ for some positive integer $k \in \langle s_1 \rangle$ and $\sigma(t) \in [\sigma_1 + ((l-1)/s_2)\sigma_{12},\ \sigma_1 + (l/s_2)\sigma_{12}]$ for some positive integer $l \in \langle s_2 \rangle$, it is easy to see that

$$\begin{aligned} & \frac{1}{\beta\left(t\right)-\tau_{1}} \ge \frac{s_{1}}{k\alpha\tau_{12}}, \frac{1}{\tau\left(t\right)-\beta\left(t\right)} \ge \frac{s_{1}}{\left(1-\alpha\right)k\tau_{12}}, \frac{1}{\tau_{2}-\tau\left(t\right)} \\ & \quad \ge \frac{s_{1}}{\left(s_{1}-k+1\right)\tau_{12}}, \end{aligned} \tag{30}$$

$$\begin{aligned} & \frac{1}{\gamma\left(t\right)-\sigma_{1}} \ge \frac{s_{2}}{l\alpha\sigma_{12}}, \frac{1}{\sigma\left(t\right)-\gamma\left(t\right)} \ge \frac{s_{2}}{\left(1-\alpha\right)l\sigma_{12}}, \frac{1}{\sigma_{2}-\sigma\left(t\right)} \\ & \quad \ge \frac{s_{2}}{\left(s_{2}-l+1\right)\sigma_{12}}. \end{aligned} \tag{31}$$

Then, the combination of (21)–(31) gives

$$\mathscr{E}\mathscr{L}V\left(t\right) \le \eta^{T}\left(t\right)\Psi_{k,l}\eta\left(t\right), \tag{32}$$

where

$$\begin{aligned} \eta\left(t\right) = \operatorname{col}\big(& x\left(t\right), x\left(t-\beta\left(t\right)\right), x\left(t-\tau\left(t\right)\right), \\ & x\left(t-\alpha\tau_{1}\right), x\left(t-\tau_{1}\right), x\left(t-\tau_{2}\right), \\ & y\left(t\right), y\left(t-\gamma\left(t\right)\right), y\left(t-\sigma\left(t\right)\right), y\left(t-\alpha\sigma_{1}\right), \\ & y\left(t-\sigma_{1}\right), y\left(t-\sigma_{2}\right), f\left(y\left(t-\sigma\left(t\right)\right)\right)\big). \end{aligned} \tag{33}$$

Due to (19), we have $\mathscr{E}\mathscr{L}V(t) < 0$, and hence GRN ((14a), (14b)) is asymptotically stable in the mean square sense. □

Remark 4. In the above theorem a DRP approach has been proposed to establish an asymptotic mean square stability criterion for GRN ((14a), (14b)). Both DRP approach and the so-called piecewise analysis method (see, e.g., [20]) divide the delay-varying intervals into some parts with equal length. Then DRP approach enlarges the expectation of weak infinitesimal operator of the same Lyapunov-Krasovskii functional in every subinterval, while the piecewise analysis method constructs different Lyapunov-Krasovskii functional in every subinterval.

Remark 5. Comparing with the Lyapunov-Krasovskii functionals employed in Theorem 3 and [17, Theorem 1], we remove the items $\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} \operatorname{trace}[f_1^T(s)Z_2f_1^T(s)]\mathrm{d}s\,\mathrm{d}\theta$ and $\int_{-\sigma_2}^{-\sigma_1}\int_{t+\theta}^{t} \operatorname{trace}[f_2^T(s)Z_4f_2^T(s)]\mathrm{d}s\,\mathrm{d}\theta$, which is required in [17]. This will reduce the number of LMI variables to be solved, and hence Theorem 3 requires less computer time than [17, Theorem 1]. Furthermore, it will be shown in the next section that Theorem 3 is certainly less conservative than [17, Theorem 1].

4. Theoretical Comparisons

In this part we will offer a theoretical comparison on conservativeness of Theorem 3 and [17, Theorem 1]. For this reason, we introduce [17, Theorem 1] as follows.

Lemma 6 (see [17, Theorem 1]). *When $L^- = 0$ and $L^+ = K$, GRN ((14a), (14b)) subject to ((4a), (4b)) is asymptotically stable in the mean square sense, if there exist positive definite matrices P_i, R_i $(i = 1, 2)$, Q_j $(j = 1, 2, \ldots, 8)$, and Z_k $(k = 1, 2, \ldots, 6)$, a diagonal positive matrix Λ, matrices S_i, J_i, N_i, M_i, U_i, V_i, L_i, and T_i $(i = 1, 2)$ of appropriate sizes, and positive scalars ρ_1, ρ_2, and ρ_3 such that the following LMIs hold:*

$$\begin{gathered} P_1 \le \rho_1 I, \qquad Z_3 \le \rho_2 I, \qquad Z_4 \le \rho_3 I, \\ \begin{bmatrix} \Omega_{11} & \Omega_{12} & \Omega_{13} & \Psi_{i14} \\ \star & -\Omega_{22} & 0 & 0 \\ \star & \star & -\Omega_{33} & 0 \\ \star & \star & \star & -\Psi_{i44} \end{bmatrix} < 0, \qquad i = 1, 2, 3, 4, \end{gathered} \tag{34}$$

where

$$\begin{gathered} \Psi_{114} = \begin{bmatrix} \tau_{12}\left(1-\alpha\right)S & \tau_{12}\alpha N & \sigma_{12}\left(1-\alpha\right)L & \sigma_{12}\alpha U \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix}, \\ \Psi_{214} = \begin{bmatrix} \tau_{12}\left(1-\alpha\right)S & \tau_{12}\alpha N & \sigma_{12}T \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix}, \\ \Psi_{314} = \begin{bmatrix} \tau_{12}J & \sigma_{12}\left(1-\alpha\right)L & \sigma_{12}\alpha U \\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} \end{bmatrix}, \\ \Psi_{414} = \begin{bmatrix} \tau_{12}J & \sigma_{12}T \\ 0_{2n\times n} & 0_{2n\times n} \end{bmatrix}, \\ S = \begin{bmatrix} 0 & S_1^T & S_2^T & 0_{n\times 10n} \end{bmatrix}^T, \\ N = \begin{bmatrix} N_1^T & N_2^T & 0_{n\times 11n} \end{bmatrix}^T, \\ L = \begin{bmatrix} 0_{n\times 7n} & L_1^T & L_2^T & 0_{n\times 4n} \end{bmatrix}^T, \\ U = \begin{bmatrix} 0_{n\times 6n} & U_1^T & U_2^T & 0_{n\times 5n} \end{bmatrix}^T, \\ T = \begin{bmatrix} 0_{n\times 6n} & T_1^T & 0 & T_2^T & 0_{n\times 4n} \end{bmatrix}^T, \\ J = \begin{bmatrix} J_1^T & 0 & J_2^T & 0_{n\times 10n} \end{bmatrix}^T, \\ \Psi_{144} = \operatorname{diag}\left(\tau_{12}\left(1-\alpha\right)Z_2, \tau_{12}\alpha Z_2, \sigma_{12}\left(1-\alpha\right)Z_6, \sigma_{12}\alpha Z_6\right), \\ \Psi_{244} = \operatorname{diag}\left(\tau_{12}\left(1-\alpha\right)Z_2, \tau_{12}\alpha Z_2, \sigma_{12}Z_6\right), \\ \Psi_{344} = \operatorname{diag}\left(\tau_{12}Z_2, \sigma_{12}\left(1-\alpha\right)Z_6, \sigma_{12}\alpha Z_6\right), \\ \Psi_{444} = \operatorname{diag}\left(\tau_{12}Z_2, \sigma_{12}Z_6\right), \end{gathered}$$

$$\Omega_{11}=\begin{bmatrix}\Sigma_{11} & \Gamma_1^T\phi_1 & \Gamma_2^T\phi_2\\ \star & -\phi_1 & 0\\ \star & \star & -\phi_2\end{bmatrix},$$

$$\Sigma_{11}=\begin{bmatrix}\Theta_1 & \Theta_3\\ \star & \Theta_2\end{bmatrix},$$

$$\Omega_{22}=\operatorname{diag}\left(\tau_1 Z_1,\sigma_1 Z_5\right),$$

$$\Omega_{12}=\begin{bmatrix}\tau_1 M & \sigma_1 V\\ 0_{2n\times n} & 0_{2n\times n}\end{bmatrix},$$

$$M=\begin{bmatrix}M_1^T & 0 & M_2^T & 0_{n\times 10n}\end{bmatrix}^T,$$

$$V=\begin{bmatrix}0_{n\times 6n} & V_1^T & 0 & V_2^T & 0_{n\times 4n}\end{bmatrix}^T,$$

$$\Omega_{13}=\begin{bmatrix}S & N & M & J\\ 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n} & 0_{2n\times n}\end{bmatrix},$$

$$\Omega_{33}=\operatorname{diag}\left(Z_4,Z_4,Z_3,Z_4\right),$$

$$\Theta_1=\begin{bmatrix}\Pi_{11} & -N_1 & M_2^T+J_1 & 0 & N_1-M_1 & -J_1\\ \star & \Pi_{22} & -S_1+S_2^T & 0 & N_2 & 0\\ \star & \star & \Pi_{33} & 0 & -M_2 & -J_2\\ \star & \star & \star & \Pi_{44} & 0 & 0\\ \star & \star & \star & \star & \Pi_{55} & 0\\ \star & \star & \star & \star & \star & -R_1\end{bmatrix},$$

$$\Theta_3=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & P_1W\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ D^TP_2 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Pi_{11}=-P_1A-A^TP_1+Q_4+\left(\rho_1+\tau_1\rho_2+\tau_{12}\rho_3\right)H_1 + M_1+M_1^T+R_1,$$

$$\Pi_{22}=\left(1+\alpha\tau_d\right)Q_1-\left(1-\alpha\tau_d\right)Q_2+S_1+S_1^T-N_2-N_2^T,$$

$$\Pi_{33}=-\left(1-\tau_d\right)Q_1+\left(\rho_1+\tau_1\rho_2+\tau_{12}\rho_3\right)H_2 + J_2+J_2^T-S_2-S_2^T,$$

$$\Pi_{44}=Q_3-Q_4,\qquad \Pi_{55}=Q_2-Q_3,$$

$$\Gamma_1=\begin{bmatrix}0 & 0 & D & 0 & 0 & 0 & -C & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix},$$

$$\Gamma_2=\begin{bmatrix}-A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W\end{bmatrix},$$

$$\phi_1=\sigma_1Z_5+\sigma_{12}Z_6,\qquad \phi_2=\tau_1Z_1+\tau_{12}Z_2,$$

$$\Theta_2=\begin{bmatrix}\Xi_{11} & -U_1 & V_2^T+T_1 & 0 & U_1-V_1 & -T_1 & 0\\ \star & \Xi_{22} & -L_1+L_2^T & 0 & U_2 & 0 & 0\\ \star & \star & \Xi_{33} & 0 & -V_2 & -T_2 & \Lambda\\ \star & \star & \star & \Xi_{44} & 0 & 0 & 0\\ \star & \star & \star & \star & \Xi_{55} & 0 & 0\\ \star & \star & \star & \star & \star & -R_2 & 0\\ \star & \star & \star & \star & \star & \star & -2\Lambda K^{-1}\end{bmatrix},$$

$$\begin{aligned}\Xi_{11}&=-P_2C-C^TP_2+Q_8+\left(\rho_1+\tau_1\rho_2+\tau_{12}\rho_3\right)H_3 + V_1+V_1^T+R_2,\\ \Xi_{22}&=\left(1+\alpha\sigma_d\right)Q_5-\left(1-\alpha\sigma_d\right)Q_6+L_1+L_1^T-U_2-U_2^T,\\ \Xi_{33}&=-\left(1-\sigma_d\right)Q_5+\left(\rho_1+\tau_1\rho_2+\tau_{12}\rho_3\right)H_4-L_2 - L_2^T+T_2+T_2^T,\\ \Xi_{44}&=Q_7-Q_8,\\ \Xi_{55}&=Q_6-Q_7.\end{aligned} \tag{35}$$

In order to show that Theorem 3 is less conservative than [17, Theorem 1], the following propositions are required.

Proposition 7. *Let $\Sigma^T=\Sigma$, J_j, X_j $(j=1,2,3)$, and $W^T=W>0$ be given real matrices of appropriate sizes. For given scalars $\alpha\in(0,\ 1)$ and $c>0$, set*

$$\begin{aligned}S_1&=\left[(1-\alpha)J_1\quad \alpha J_2\right],\qquad T_1=\operatorname{diag}\left((1-\alpha)W,\alpha W\right),\\ S_2&=J_3,\qquad T_2=W,\\ \Sigma_0&=J_1X_1+J_2X_2+J_3X_3.\end{aligned} \tag{36}$$

If

$$\begin{bmatrix}\Sigma+\Sigma_0+\Sigma_0^T & \sqrt{c}S_i\\ \star & -T_i\end{bmatrix}<0,\quad i=1,2, \tag{37}$$

then there exists a (sufficiently large) positive integer s such that

$$\begin{aligned}&\Sigma-\frac{s}{k(1-\alpha)}c^{-1}X_1^TWX_1-\frac{s}{k\alpha}c^{-1}X_2^TWX_2\\ &-\frac{s}{s-k+1}c^{-1}X_3^TWX_3<0,\quad \forall k\in\langle s\rangle.\end{aligned} \tag{38}$$

Proof. It follows from (37) and the Schur complementary lemma [21] that

$$\Sigma+\Sigma_0+\Sigma_0^T+cS_iT_i^{-1}S_i^T<0,\quad i=1,2, \tag{39}$$

and hence there exists a (sufficiently large) positive integer s such that

$$\Sigma+\Sigma_0+\Sigma_0^T+\frac{s+1}{s}cS_iT_i^{-1}S_i^T<0,\quad i=1,2. \tag{40}$$

For an arbitrary but fixed $k\in\langle s\rangle$, one can derive from (36) and (40) that

$$\begin{aligned}&\Sigma+J_1X_1+X_1^TJ_1^T+\frac{k}{s}c(1-\alpha)J_1W^{-1}J_1^T\\ &+J_2X_2+X_2^TJ_2^T+\frac{k}{s}c\alpha J_2W^{-1}J_2^T\\ &+J_3X_3+X_3^TJ_3^T+\frac{s-k+1}{s}cJ_3W^{-1}J_3^T<0.\end{aligned} \tag{41}$$

Since

$$J_1X_1 + X_1^TJ_1^T + \frac{k(1-\alpha)}{s}cJ_1W^{-1}J_1^T + \frac{s}{k(1-\alpha)}c^{-1}X_1^TWX_1 \geq 0,$$
$$J_2X_2 + X_2^TJ_2^T + \frac{k\alpha}{s}cJ_2W^{-1}J_2^T + \frac{s}{k\alpha}c^{-1}X_2^TWX_2 \geq 0,$$
$$J_3X_3 + X_3^TJ_3^T + \frac{s-k+1}{s}cJ_3W^{-1}J_3^T + \frac{s}{s-k+1}c^{-1}X_3^TWX_3 \geq 0, \tag{42}$$

we obtain from (41) that (38) holds. The proof is completed. □

Proposition 8. *Let $\Omega^T = \Omega$, J_i, L_i, X_i, Y_i $(i = 1,2,3)$, $W^T = W > 0$, and $Z^T = Z > 0$ be given real matrices of appropriate sizes and $c_i > 0$ $(i = 1,2)$ and $\alpha \in (0, 1)$ given scalars. Set*

$$\begin{aligned}
S_{11} &= \left[c_1(1-\alpha)J_1 \;\; c_1\alpha J_2 \;\; c_2(1-\alpha)L_1 \;\; c_2\alpha L_2\right],\\
S_{12} &= \left[c_1(1-\alpha)J_1 \;\; c_1\alpha J_2 \;\; c_2L_3\right],\\
S_{21} &= \left[c_1J_3 \;\; c_2(1-\alpha)L_1 \;\; c_2\alpha L_2\right],\\
S_{22} &= \left[c_1J_3 \;\; c_2L_3\right],\\
T_{11} &= \operatorname{diag}\left(c_1(1-\alpha)W, c_1\alpha W, c_2(1-\alpha)Z, c_2\alpha Z\right),\\
T_{12} &= \operatorname{diag}\left(c_1(1-\alpha)W, c_1\alpha W, c_2Z\right),\\
T_{21} &= \operatorname{diag}\left(c_1W, c_2(1-\alpha)Z, c_2\alpha Z\right),\\
T_{22} &= \operatorname{diag}\left(c_1W, c_2Z\right),\\
\Omega_1 &= J_1X_1 + J_2X_2 + J_3X_3,\\
\Omega_2 &= L_1Y_1 + L_2Y_2 + L_3Y_3.
\end{aligned} \tag{43}$$

If

$$\begin{bmatrix} \Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T & S_{ij} \\ \star & -T_{ij} \end{bmatrix} < 0, \quad i,j = 1,2, \tag{44}$$

then there exists a pair of (sufficiently large) positive integers s_1 and s_2 such that

$$\begin{aligned}
&\Omega - \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2\\
&\quad - \frac{s_1}{s_1-k+1}c_1^{-1}X_3^TWX_3\\
&\quad - \frac{s_2}{l(1-\alpha)}c_2^{-1}Y_1^TZY_1 - \frac{s_2}{l\alpha}c_2^{-1}Y_2^TZY_2\\
&\quad - \frac{s_2}{s_2-l+1}c_2^{-1}Y_3^TZY_3 < 0, \quad \forall k \in \langle s_1\rangle,\; l \in \langle s_2\rangle.
\end{aligned} \tag{45}$$

Proof. It follows from (44) that

$$\Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + S_{ij}T_{ij}^{-1}S_{ij}^T < 0, \quad i,j = 1,2, \tag{46}$$

and hence

$$\begin{aligned}
&\Omega + \Omega_1 + \Omega_1^T + \Omega_2 + \Omega_2^T + \delta S_{i1}T_{i1}^{-1}S_{i1}^T\\
&\quad + (1-\delta)S_{i2}T_{i2}^{-1}S_{i2}^T < 0, \quad i = 1,2,\; \forall\delta \in [0,1];
\end{aligned} \tag{47}$$

that is,

$$\begin{bmatrix} \Omega_\delta + \Omega_1 + \Omega_1^T & \sqrt{c_1}\widehat{S}_i \\ \star & \widehat{T}_i \end{bmatrix} < 0, \quad i = 1,2,\; \forall\delta \in [0,1], \tag{48}$$

where

$$\begin{aligned}
&\widehat{S}_1 = \left[(1-\alpha)J_1 \;\; \alpha J_2\right], \qquad \widehat{T}_1 = \operatorname{diag}\left((1-\alpha)W, \alpha W\right),\\
&\widehat{S}_2 = J_3, \qquad \widehat{T}_2 = W,\\
&\Omega_\delta = \Omega + \Omega_2 + \Omega_2^T + \delta c_2\widetilde{S}_1\widetilde{T}_1^{-1}\widetilde{S}_1^T + (1-\delta)c_2\widetilde{S}_2\widetilde{T}_2^{-1}\widetilde{S}_2^T,\\
&\widetilde{S}_1 = \left[(1-\alpha)L_1 \;\; \alpha L_2\right], \qquad \widetilde{T}_1 = \operatorname{diag}\left((1-\alpha)Z, \alpha Z\right),\\
&\widetilde{S}_2 = L_3, \qquad \widetilde{T}_2 = Z.
\end{aligned} \tag{49}$$

Applying Proposition 7 to $\Sigma = \Omega_\delta$, $\Sigma_0 = \Omega_1$, $S_i = \widehat{S}_i$, $T_i = \widehat{T}_i$, and $c = c_1$, we obtain that

$$\begin{aligned}
&\Omega_\delta - \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2 - \frac{s_1}{s_1-k+1}\\
&\quad \times c_1^{-1}X_3^TWX_3 < 0, \quad \forall\delta \in [0,1],\; \forall k \in \langle s_1\rangle,\; \delta \in [0,1]
\end{aligned} \tag{50}$$

for some (sufficiently large) positive integer s_1.

By the Schur complementary lemma, one can easily derive from (50) that

$$\begin{bmatrix} \Omega_k + \Omega_2 + \Omega_2^T & \sqrt{c_2}\widetilde{S}_i \\ \star & -\widetilde{T}_i \end{bmatrix} < 0, \quad i = 1,2,\; \forall k \in \langle s_1\rangle, \tag{51}$$

where

$$\begin{aligned}
\Omega_k = \Omega &- \frac{s_1}{k(1-\alpha)}c_1^{-1}X_1^TWX_1 - \frac{s_1}{k\alpha}c_1^{-1}X_2^TWX_2\\
&- \frac{s_1}{s_1-k+1}c_1^{-1}X_3^TWX_3.
\end{aligned} \tag{52}$$

Again applying Proposition 7 to $\Sigma = \Omega_k$, $\Sigma_0 = \Omega_2$, $S_i = \widetilde{S}_i$, $T_i = \widetilde{T}_i$, and $c = c_2$, one can complete the proof. □

Proposition 9. *Let $p, q \in R^n$ and $M \in R^{n\times n}$ satisfying $M^T = M > 0$. Then*

$$(p+q)^TM(p+q) \leq \frac{1}{\alpha}p^TMp + \frac{1}{1-\alpha}q^TMq, \quad \forall\alpha \in (0,1). \tag{53}$$

Proof. One has

$$\begin{aligned}
&\frac{1}{\alpha}p^TMp + \frac{1}{1-\alpha}q^TMq - (p+q)^TM(p+q)\\
&\quad = \frac{1-\alpha}{\alpha}p^TMp + \frac{\alpha}{1-\alpha}q^TMq - p^TMq - q^TMp
\end{aligned}$$

$$= \left(\sqrt{\frac{1-\alpha}{\alpha}}p - \sqrt{\frac{\alpha}{1-\alpha}}q\right)^T M \left(\sqrt{\frac{1-\alpha}{\alpha}}p - \sqrt{\frac{\alpha}{1-\alpha}}q\right) \geq 0. \tag{54}$$

□

Now it is time to show that Theorem 3 is less conservative than [17, Theorem 1] in theory.

Theorem 10. *Set $L^- = 0$ and $L^+ = K$. If the LMIs in* (34) *are feasible, then the LMIs in* (18) *and* (19) *are feasible.*

Proof. Set $\rho = \rho_1$, $U_i = P_i$ $(i = 1,2)$, $V_j = Q_{5-j}$, $W_j = Q_{9-j}$ $(j = 1,2,3,4)$, $V_5 = R_1$, $W_5 = R_2$, $V_k = Z_{k-5}$, and $W_k = Z_{k-1}$ $(k = 6,7)$. Then it follows from (34) and the Schur complementary lemma that (18) holds and

$$\Sigma_{11} + \Psi_4 + \tau_1 M V_6^{-1} M^T + \sigma_1 V W_6^{-1} V^T + \widetilde{S}_{ij}\widetilde{T}_{ij}\widetilde{S}_{ij}^T < 0, \quad i,j = 1,2, \tag{55}$$

where

$$\begin{aligned}
\widetilde{S}_{11} &= \left[\tau_{12}(1-\alpha)S \quad \tau_{12}\alpha N \quad \sigma_{12}(1-\alpha)L \quad \sigma_{12}\alpha U\right],\\
\widetilde{S}_{12} &= \left[\tau_{12}(1-\alpha)S \quad \tau_{12}\alpha N \quad \sigma_{12}T\right],\\
\widetilde{S}_{21} &= \left[\tau_{12}J \quad \sigma_{12}(1-\alpha)L \quad \sigma_{12}\alpha U\right],\\
\widetilde{S}_{22} &= \left[\tau_{12}J \quad \sigma_{12}T\right],\\
\widetilde{T}_{11} &= \operatorname{diag}\left(\tau_{12}(1-\alpha)V_7, \tau_{12}\alpha V_7, \sigma_{12}(1-\alpha)W_7, \sigma_{12}\alpha W_7\right),\\
\widetilde{T}_{12} &= \operatorname{diag}\left(\tau_{12}(1-\alpha)V_7, \tau_{12}\alpha V_7, \sigma_{12}W_7\right),\\
\widetilde{T}_{21} &= \operatorname{diag}\left(\tau_{12}V_7, \sigma_{12}(1-\alpha)W_7, \sigma_{12}\alpha W_7\right),\\
\widetilde{T}_{22} &= \operatorname{diag}\left(\tau_{12}V_7, \sigma_{12}W_7\right),
\end{aligned} \tag{56}$$

and $S, N, L, U, T, J, M, V, \Psi_4$, and Σ_{11} are defined as noted previously.

By simple computation one can derive from (55) that

$$\begin{bmatrix} \widetilde{\Omega} + \widetilde{\Omega}_1 + \widetilde{\Omega}_1^T + \widetilde{\Omega}_2 + \widetilde{\Omega}_2^T & \widetilde{S}_{ij} \\ \star & -\widetilde{T}_{ij} \end{bmatrix} < 0, \quad i,j = 1,2, \tag{57}$$

where

$$\begin{aligned}
\widetilde{\Omega}_1 &= S(e_2 - e_3) + N(e_5 - e_2) + J(e_3 - e_6),\\
\widetilde{\Omega}_2 &= L(e_8 - e_9) + U(e_{11} - e_8) + T(e_9 - e_{12}),\\
\widetilde{\Omega} &= \widetilde{\Psi} + V(e_7 - e_{11}) + (e_7 - e_{11})^T V^T + M(e_1 - e_5)\\
&\quad + (e_1 - e_5)^T M^T + \tau_1 M V_6^{-1} M^T + \sigma_1 V W_6^{-1} V^T,\\
\widetilde{\Psi} &= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5,
\end{aligned} \tag{58}$$

and Ψ_i $(i = 1,2,\ldots,5)$ are defined as in (19).

By Proposition 8, there exists a pair of (sufficiently large) positive integers s_1 and s_2 such that

$$\widetilde{\Omega} + \Psi_k + \Psi_l < 0, \quad \forall k \in \langle s_1 \rangle,\ l \in \langle s_2 \rangle, \tag{59}$$

where Ψ_k and Ψ_l are defined as in (19). This, together with Proposition 9, implies that

$$\begin{aligned}
\widetilde{\Omega} &\geq \widetilde{\Psi} - \tau_1^{-1}(e_1 - e_5)^T V_6 (e_1 - e_5)\\
&\quad - \sigma_1^{-1}(e_7 - e_{11})^T W_6 (e_7 - e_{11}) \geq \widetilde{\Psi} + \Psi_6,
\end{aligned} \tag{60}$$

and hence the LMIs in (19) are feasible. The proof is completed. □

Remark 11. In Theorem 10, it has been theoretically investigated that Theorem 3 is certainly less conservative than [17, Theorem 1]. On the other hand, the numbers of LMI variables to be solved in Theorem 3 and [17, Theorem 1] are $8n^2+9n+1$ and $25n^2+10n+3$, respectively, which implies that Theorem 3 will require less computer time than [17, Theorem 1].

5. An Illustrative Example

In this section, a numerical example is given to illustrate the effectiveness and less conservativeness of our theoretical results.

We consider a delayed GRN with stochastic disturbances, with the parameters described as

$$\begin{aligned}
A &= \operatorname{diag}(3,3,3), \qquad C = \operatorname{diag}(2.5,2.5,2.5),\\
D &= \operatorname{diag}(0.8,0.8,0.8),\\
W &= \begin{bmatrix} 0 & 0 & -2.5 \\ -2.5 & 0 & 0 \\ 0 & -2.5 & 0 \end{bmatrix}.
\end{aligned} \tag{61}$$

Let $G_1 = G_2 = G_3 = G_4 = 0.4I$ and $f(x) = x^2/(1+x^2)$, which means that $L^- = 0$ and $L^+ = K = 0.65I$. When $s_1 = s_2 = 10$, for $\tau_1 = \sigma_1 = 1$, $\tau_2 = \sigma_2 = 10$, $\tau_d = \sigma_d = 0.5$, and $\alpha = 0.1$, by using the MATLAB Toolbox, inequalities (18) and (19) have feasible solutions. By the theorem, the system is asymptotically stable in the mean square sense. Here, some of the solution matrices are given as follows:

$$\begin{aligned}
U_1 &= \begin{bmatrix} 3.9297 & 0.0000 & 0.0000 \\ 0.0000 & 3.9297 & 0.0000 \\ 0.0000 & 0.0000 & 3.9297 \end{bmatrix},\\
U_2 &= \begin{bmatrix} 7.8311 & -0.0000 & -0.0000 \\ -0.0000 & 7.8311 & -0.0000 \\ -0.0000 & -0.0000 & 7.8311 \end{bmatrix},
\end{aligned} \tag{62}$$

and show the trajectories of the $x(t)$ and $y(t)$ in Figure 1.

When $\tau_1 = \sigma_1 = 1$ and $s_1 = s_2 = 10$, the maximal allowable upper bounds of $\tau_2 = \sigma_2$ for different values of $\tau_d = \sigma_d$ obtained by Theorem 3 and [17, Theorem 1] are shown in Table 1. It can be seen from Table 1 that Theorem 3 is the less conservative than [17, Theorem 1].

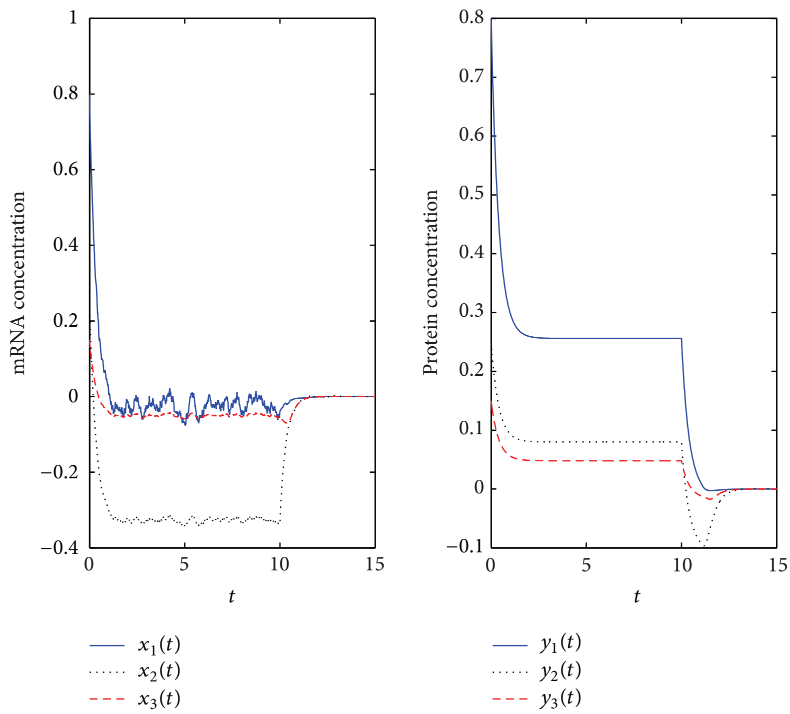

FIGURE 1: State trajectories of the GRN in the considered example.

TABLE 1: The maximal allowable bounds of $\tau_2 = \sigma_2$ for different $\tau_d = \sigma_d$.

$\tau_d = \sigma_d$	0.8	1	1.5
[17, Theorem 1]	1.5602	1.5500	1.5450
Theorem 3	2.095	2.053	2.047

6. Conclusions

In this paper, the stability problem for a class of GRNs with time-varying delays and stochastic disturbances has been investigated. By constructing an appropriate Lyapunov-Krasovskii functional and proposing a DRP approach, a mean square stability criterion is given in terms of LMIs, which can be easily tested by the LMI Toolbox of MATLAB. It is theoretically shown that the proposed result is less conservative than [17, Theorem 1]. Moreover, the number of LMI variables in this paper is more less than the one in [17, Theorem 1]. A numerical example has been provided to illustrate the theoretical results given in this paper.

Extending the idea of this paper to other system models, including singular delayed systems [19, 22–24], stochastic systems [25], Markovian jump systems [20, 26], and genetic regulatory networks [5, 10, 27], is under consideration.

Conflict of Interests

The authors declare that there is no commercial or associative interest that represents a conflict of interest in connection with the work submitted.

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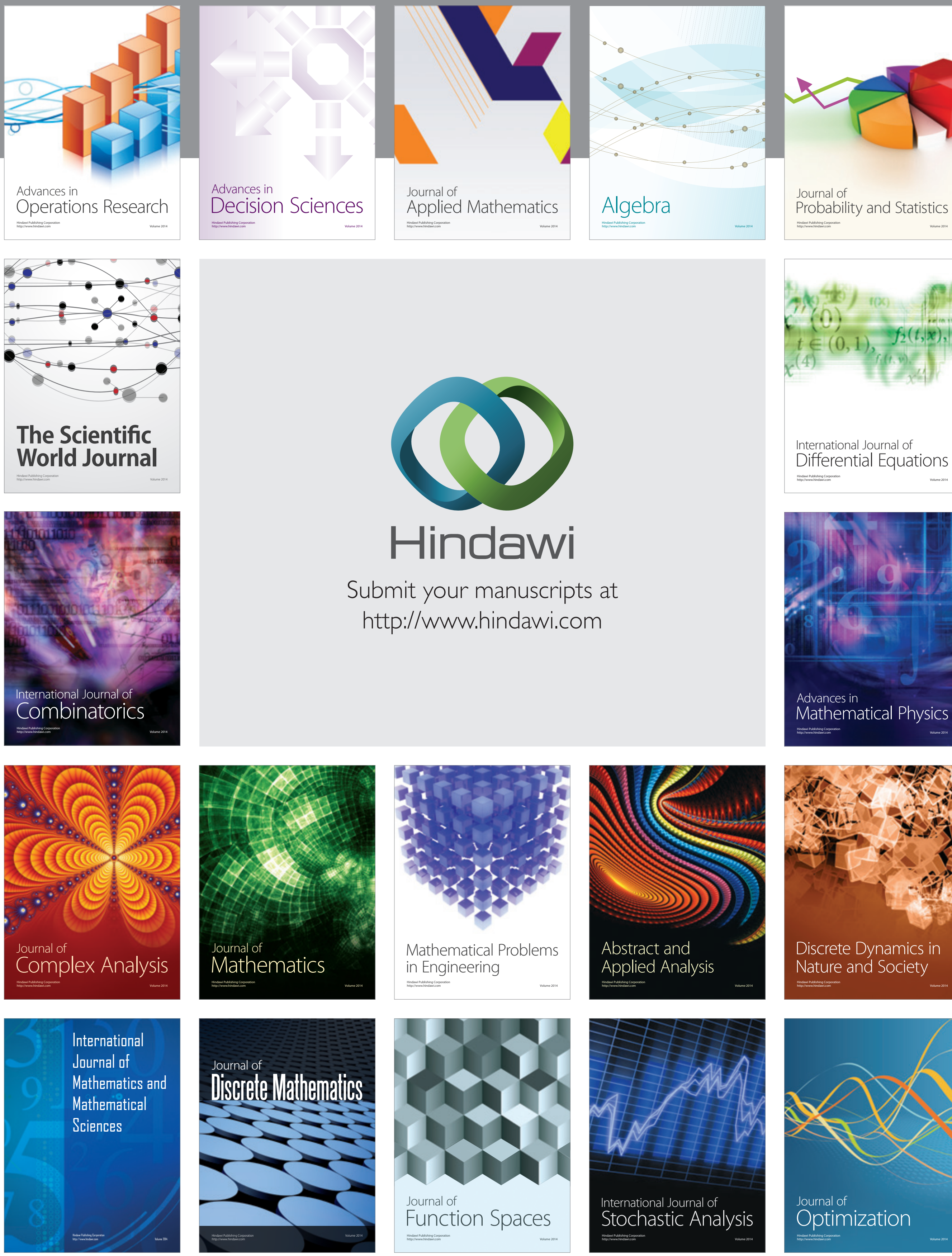

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