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Research Article

Blow-Up Phenomena for Porous Medium Equation with Nonlinear Flux on the Boundary

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We investigate the blow-up phenomena for nonnegative solutions of porous medium equation with Neumann boundary conditions. We find that the absorption and the nonlinear flux on the boundary have some competitions in the blow-up phenomena.

1. Introduction

In this paper, we are concerned with the blow-up of solutions of porous medium equations with nonlinear flux on the boundary. Consider

$$u_t = \Delta u^m - f(u), \quad (x,t) \in \Omega \times (0,t^*), \tag{1}$$

$$\frac{\partial u^m}{\partial \nu} = g(u), \quad (x,t) \in \partial\Omega \times (0,t^*), \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3}$$

where $m > 1$, the nonnegative initial value $u_0(x) \in C(\Omega)\bigcap L^\infty(\Omega)$, Ω is a bounded region in $\mathbb{R}^N$ $(N \geq 2)$ with the sufficiently smooth boundary $\partial\Omega$, ν is the unit normal vector on $\partial\Omega$, t^* is the blow-up time if blow-up occurs, or else $t^* = \infty$.

The blow-up phenomena for the nonnegative solutions of the heat equation with nonlinear sources ($m = 1$ and $f(u) = -u^p$ in (1)) in the whole space $\mathbb{R}^N$ was first found by Fujita in 1966, see [1]. He proved the following results:

(a) if $1 < p < 1 + (2/N)$, then (1) has no global positive solutions;
(b) if $p > 1 + (2/N)$, then there exist global positive solutions.

The critical case $p = 1 + (2/N)$ was proved to belong to the blow-up case in 1970's by several authors [2–4]. In 1980, Galaktionov and others [5] considered the nonnegative solutions of (1) (with $m > 1$ and $f(u) = -u^p$) in whole space $\mathbb{R}^N$. They found some results similar to those for the heat equation ($m = 1$) as follows

(a) if $1 < p < m+(2/N)$, then (1) has no global solutions;
(b) if $p > m + (2/N)$, then there exist global positive solutions that decay like $t^{-1/(p-1)}$.

In [6, 7], Galaktionov, Mochizuki and Suzuki, had also revealed that the critical case $p = m + (2/N)$ belongs to the blow-up case, see also [8, 9].

In 2010, Payne et al. [10] considered a semilinear heat equation with nonlinear boundary condition ($m = 1$ in (1)) and established conditions on nonlinearities sufficient to guarantee that $u(x,t)$ exists for all time $t > 0$ as well as conditions on data forcing the solution $u(x,t)$ to blow up at some finite time t^*. When $N = 1$, the blow-up phenomena for the solutions of the porous medium equation with nonlinear flux on the boundary had also been studied by several authors [11, 12]. For other interesting results on the large time behavior on the solutions of the porous medium equation, we refer the reader to papers [13–16].

Inspired by the above papers, we will study the blow-up phenomena for the solutions of the porous medium equation with nonlinear flux on the boundary in higher dimensional space ($N \geq 2$). In fact, we find that if the absorption is more powerful than the boundary flux, then the solutions of the problem (1)–(3) exist for all time on a bounded star-shaped region. On the other hand, if the boundary flux is more powerful, then the solutions of the problem (1)–(3) blow-up at a finite time. Moreover, we will give the upper-bound estimates for the blow-up time.

The paper is organized as follows. In Section 2, we concentrate our attention on the conditions of the global existence for the solutions of the problem (1)–(3). Section 3 is devoted to the investigation of the blow-up phenomena for the solutions of the problem (1)–(3).

2. Criterion for Global Existence

In this section, we investigate the global solutions of problem (1)–(3). The main result of this section is the following theorem.

Theorem 1. *Let Ω be a bounded star-shaped region and assume that $q > m$ satisfy*

$$2q < m + p. \tag{4}$$

If f and g satisfy the following conditions:

$$f(\xi) \geq k_1 \xi^p, \quad \xi \geq 0, \tag{5}$$

$$0 \leq g(\xi) \leq k_2 \xi^q, \quad \xi \geq 0, \tag{6}$$

where k_1, k_2 are nonnegative constants, then the nonnegative solutions $u(x,t)$ of the problem (1)–(3) do not blow up.

Proof. Let

$$\Phi(t) = \int_\Omega u^2 dx. \tag{7}$$

Differentiating (7) and making use of (1), we obtain that

$$\Phi'(t) = 2\int_\Omega u u_t dx = 2\int_\Omega u\left[\Delta u^m - f(u)\right] dx. \tag{8}$$

From the hypothesis (5), we get

$$\Phi'(t) \leq 2\int_\Omega u\left(\Delta u^m - k_1 u^p\right) dx. \tag{9}$$

By (2), (6) and the divergence theorem, we have

$$\begin{aligned}\int_\Omega u\Delta u^m dx &= \int_{\partial\Omega} u\nabla u^m \cdot \nu ds - \int_\Omega \nabla u \cdot \nabla u^m dx \\ &= \int_{\partial\Omega} u\frac{\partial u^m}{\partial \nu} ds - \int_\Omega \nabla u \cdot \nabla u^m dx \\ &\leq k_2 \int_{\partial\Omega} u^{q+1} ds - \int_\Omega \nabla u \cdot \nabla u^m dx.\end{aligned} \tag{10}$$

Here we used the identity $\operatorname{div}(u\nabla u^m) = u\Delta u^m + \nabla u \cdot \nabla u^m$. By the divergence theorem again, we get

$$\int_{\partial\Omega} \left(u^{q+1}x\right)\cdot \nu\, ds = \int_\Omega \operatorname{div}\left(u^{q+1}x\right) dx. \tag{11}$$

Let

$$\rho_0 = \min_{\partial\Omega}(x\cdot\nu), \qquad d = \max_{\partial\Omega}|x|. \tag{12}$$

Point out that ρ_0 is positive because Ω is star-shaped by hypothesis. Notice also that

$$\begin{aligned}\operatorname{div}\left(u^{q+1}x\right) &= \operatorname{div}\left[\left(u^{(m+1)/2}\right)^{2(q+1)/(m+1)} x\right] \\ &= Nu^{q+1} + \frac{2(q+1)}{m+1} u^{(2q-m+1)/2}\left(x\cdot\nabla u^{(m+1)/2}\right).\end{aligned} \tag{13}$$

We thus have

$$\begin{aligned}\int_{\partial\Omega} u^{q+1} ds \leq \frac{N}{\rho_0}\int_\Omega u^{q+1} dx &+ \frac{2d(q+1)}{\rho_0(m+1)} \\ &\times \int_\Omega u^{(2q-m+1)/2}\left|\nabla u^{(m+1)/2}\right| dx.\end{aligned} \tag{14}$$

On the another hand

$$\begin{aligned}\nabla u \cdot \nabla u^m &= \nabla u \cdot \left(m u^{m-1}\nabla u\right) \\ &= m\left(u^{(m-1)/2}\nabla u\right)\left(u^{(m-1)/2}\nabla u\right) \\ &= \frac{4m}{(m+1)^2}\left|\nabla u^{(m+1)/2}\right|^2.\end{aligned} \tag{15}$$

Therefore, from (10)–(15), we have

$$\begin{aligned}\Phi'(t) &\leq 2k_2\int_{\partial\Omega} u^{q+1} ds - 2\int_\Omega \nabla u\cdot\nabla u^m dx \\ &\quad - 2k_1\int_\Omega u^{p+1} dx \\ &\leq \frac{2k_2 N}{\rho_0}\int_\Omega u^{q+1} dx \\ &\quad + \frac{4k_2 d(q+1)}{\rho_0(m+1)}\int_\Omega u^{(2q-m+1)/2}\left|\nabla u^{(m+1)/2}\right| dx \\ &\quad - \frac{8m}{(m+1)^2}\int_\Omega \left|\nabla u^{(m+1)/2}\right|^2 dx - 2k_1\int_\Omega u^{p+1} dx.\end{aligned} \tag{16}$$

We obtain from the Young inequality that

$$\begin{aligned}&\int_\Omega u^{(2q-m+1)/2}\left|\nabla u^{(m+1)/2}\right| dx \\ &\quad\leq \frac{\sigma}{2}\int_\Omega u^{2q-m+1} dx + \frac{1}{2\sigma}\int_\Omega \left|\nabla u^{(m+1)/2}\right|^2 dx,\end{aligned} \tag{17}$$

where

$$\sigma = \frac{k_2 d(q+1)(m+1)}{4m\rho_0}. \tag{18}$$

This σ leads to

$$\frac{4k_2 d(q+1)}{\rho_0(m+1)} \int_\Omega u^{(2q-m+1)/2} \left|\nabla u^{(m+1)/2}\right| dx \leq \frac{8m}{(m+1)^2}\sigma^2 \int_\Omega u^{2q-m+1} dx + \frac{8m}{(m+1)^2} \int_\Omega \left|\nabla u^{(m+1)/2}\right|^2 dx. \tag{19}$$

Combining this with (16), we get

$$\Phi'(t) \leq \frac{2k_2 N}{\rho_0} \int_\Omega u^{q+1} dx + \frac{8m}{(m+1)^2}\sigma^2 \times \int_\Omega u^{2q-m+1} dx - 2k_1 \int_\Omega u^{p+1} dx. \tag{20}$$

Let

$$\alpha = \frac{q-m}{p-q}. \tag{21}$$

Therefore, the hypotheses that $q > m$ and $2q < m + p$ imply that

$$0 < \alpha < 1. \tag{22}$$

So, by Hölder's inequality, we have

$$\int_\Omega u^{2q-m+1} dx \leq \left(\int_\Omega u^{q+1} dx\right)^\alpha \left(\int_\Omega u^{p+1} dx\right)^{1-\alpha}. \tag{23}$$

For $\epsilon > 0$, we obtain from (23) that

$$\int_\Omega u^{2q-m+1} dx \leq \left(\epsilon \int_\Omega u^{p+1} dx\right)^{1-\alpha} \left(\epsilon^{(\alpha-1)/\alpha} \int_\Omega u^{q+1} dx\right)^\alpha \leq (1-\alpha)\epsilon \int_\Omega u^{p+1} dx + \alpha\epsilon^{(\alpha-1)/\alpha} \int_\Omega u^{q+1} dx. \tag{24}$$

Thus, inserting (24) in (20), we obtain

$$\begin{aligned} \Phi'(t) &\leq \frac{2k_2 N}{\rho_0} \int_\Omega u^{q+1} dx + \frac{8m}{(m+1)^2}\sigma^2 \\ &\quad \times \left\{(1-\alpha)\epsilon \int_\Omega u^{p+1} dx + \alpha\epsilon^{(\alpha-1)/\alpha} \int_\Omega u^{q+1} dx\right\} \\ &\quad - 2k_1 \int_\Omega u^{p+1} dx \\ &= \left(\frac{2k_{2N}}{\rho_0} + \frac{8m\sigma^2\alpha}{(m+1)^2}\epsilon^{(\alpha-1)/\alpha}\right) \\ &\quad \times \int_\Omega u^{q+1} dx + \left(\frac{8m\sigma^2\epsilon(1-\alpha)}{(m+1)^2} - 2k_1\right) \\ &\quad \times \int_\Omega u^{p+1} dx = M_1 \int_\Omega u^{q+1} dx - M_2 \int_\Omega u^{p+1} dx, \end{aligned} \tag{25}$$

where

$$M_1 = \frac{2k_{2N}}{\rho_0} + \frac{8m\sigma^2\alpha}{(m+1)^2}\epsilon^{(\alpha-1)/\alpha} > 0, \qquad M_2 = 2k_1 - \frac{8m\sigma^2\epsilon(1-\alpha)}{(m+1)^2}, \tag{26}$$

and let ϵ be sufficiently small to ensure $M_2 > 0$. By Hölder's inequality again, we have

$$\int_\Omega u^{q+1} dx \leq \left(\int_\Omega u^{p+1} dx\right)^{(q+1)/(p+1)} |\Omega|^{(p-q)/(p+1)}, \tag{27}$$

where we assume throughout the paper that $|\Omega| = \int_\Omega dx$ is the measure of Ω. Using (25) and (27), we obtain

$$\Phi'(t) \leq M_1 \left(\int_\Omega u^{p+1} dx\right)^{(q+1)/(p+1)} \times \left\{|\Omega|^{(p-q)/(p+1)} - \frac{M_2}{M_1}\left(\int_\Omega u^{p+1} dx\right)^{(p-q)/(p+1)}\right\}. \tag{28}$$

Moreover, using Hölder's inequality once more, we have

$$\Phi(t) = \int_\Omega u^2 dx \leq \left(\int_\Omega u^{p+1} dx\right)^{2/(p+1)} |\Omega|^{(p-1)/(p+1)}, \tag{29}$$

that is,

$$\left(\int_\Omega u^{p+1} dx\right)^{(p-q)/(p+1)} \geq \Phi(t)^{(p-q)/2} |\Omega|^{(1-p)(p-q)/2(p+1)}. \tag{30}$$

Finally, from (28) and (30), we obtain

$$\Phi'(t) \leq M_1 \left(\int_\Omega u^{p+1} dx\right)^{(q+1)/(p+1)} \times \left\{|\Omega|^{(p-q)/(p+1)} - \frac{M_2}{M_1}\Phi(t)^{(p-q)/2}|\Omega|^{(1-p)(p-q)/2(p+1)}\right\}. \tag{31}$$

We deduced from (31) that $\Phi(t) \leq \max\{\Phi(0), (M_2/M_1)^{2/(q-p)} |\Omega|\}$. On the other hand, $\Phi(t)$ is nonnegative function by assumption. So that $\Phi(t)$ keeps bounded continuously under the conditions given in Theorem 1, the solutions exsit for all time $t > 0$. That is, we find that the global solution exists when the absorption is more powerful than the nonlinear boundary flux and this accomplishes the proof of Theorem 1. □

3. Criterion for Blow-Up

In this section, we concentrate on the finite time t^* on which blow-up occurs. We construct two auxiliary functions to redefine f and g, then the nonlinear boundary-flux is more powerful than the absorption, and we obtain the following result.

Theorem 2. *Suppose*

$$0 \le \alpha \le \beta. \tag{32}$$

Let

$$F(\xi) = \int_0^\xi f(\eta)\,d\eta - \frac{m(m-1)}{2}\int_0^\xi |\nabla\eta|^2 \eta^{m-2} d\eta,$$

$$G(\xi) = \int_0^\xi g(\eta)\,d\eta,$$

$$\Psi(t) = 2\int_{\partial\Omega} G(u)\,ds - \int_\Omega \nabla u\cdot\nabla u^m dx - 2\int_\Omega F(u)\,dx. \tag{33}$$

If

$$\Psi(0) > 0,$$

$$\xi f(\xi) \le 2(1+\alpha)F(\xi), \quad \xi \ge 0, \tag{34}$$

$$\xi g(\xi) \ge 2(1+\beta)G(\xi), \quad \xi \ge 0,$$

then the solutions $u(x,t)$ *of the problem* (1)–(3) *blow up at time* $t^* < T$ *with*

$$T = \frac{\Phi(0)}{2\beta(1+\beta)\Psi(0)}. \tag{35}$$

Here $\Phi(t)$ *is defined in* (7). *Moreover, if* $\beta = 0$, *then* $T = \infty$.

Proof. Differentiating (7) and using the hypothesis (33), we have

$$\begin{aligned}\Phi'(t) &= 2\int_\Omega u u_t dx = 2\int_\Omega u\left[\Delta u^m - f(u)\right]dx \\ &= 2\int_{\partial\Omega} u g(u)\,ds - 2\int_\Omega \nabla u\cdot\nabla u^m dx \\ &\quad - 2\int_\Omega u f(u)\,dx \ge 2(1+\beta)\Psi(t).\end{aligned} \tag{36}$$

Differentiating (33), we thus obtain from (15) that

$$\begin{aligned}\Psi'(t) &= 2\int_{\partial\Omega} g(u)u_t ds - \frac{4m}{(m+1)^2}\int_\Omega \left(\left|\nabla u^{(m+1)/2}\right|^2\right)_t dx \\ &\quad - 2\int_\Omega f(u)u_t dx + m(m-1)\int_\Omega |\nabla u|^2 u^{m-2} u_t dx.\end{aligned} \tag{37}$$

Note the identity that

$$\begin{aligned}\left(\left|\nabla u^{(m+1)/2}\right|^2\right)_t &= \frac{(m+1)^2(m-1)}{4} \\ &\quad \times |\nabla u|^2 u^{m-2} u_t + \frac{(m+1)^2}{2} u^{m-1}\nabla u\cdot\nabla u_t.\end{aligned} \tag{38}$$

So, from (37), we get

$$\begin{aligned}\Psi'(t) &= 2\int_{\partial\Omega} g(u)u_t ds - 2\int_\Omega \nabla u_t\cdot\nabla u^m dx \\ &\quad - 2\int_\Omega f(u)u_t dx.\end{aligned} \tag{39}$$

Therefore,

$$\Psi'(t) = 2\int_\Omega u_t^2 dx > 0. \tag{40}$$

Here, we have used the identities

$$\begin{aligned}\operatorname{div}(u_t\nabla u^m) &= u_t\Delta u^m + \nabla u_t\cdot\nabla u^m, \\ \int_\Omega \nabla u_t\cdot\nabla u^m dx &= \int_{\partial\Omega} u_t\nabla u^m\cdot\nu\,ds - \int_\Omega u_t\Delta u^m dx.\end{aligned} \tag{41}$$

So, the hypothesis $\Psi(0) > 0$ implies that for all $t \in (0, t^*)$, the following inequality holds $(t) > 0$:

$$\Psi(t) > 0. \tag{42}$$

By the Schwarz inequality, we have

$$\left(\Phi'(t)\right)^2 = 4\left(\int_\Omega u u_t dx\right)^2 \le 2\Phi(t)\Psi'(t). \tag{43}$$

Together with (36), we have

$$\Phi(t)\Psi'(t) \ge \frac{1}{2}\left[\Phi'(t)\right]^2 \ge (1+\beta)\Phi'(t)\Psi(t). \tag{44}$$

That is,

$$\left(\Psi\Phi^{-(1+\beta)}\right)' \ge 0. \tag{45}$$

Integrating this from 0 to t, we obtain

$$\Psi(t)(\Phi(t))^{-(1+\beta)} \ge \Psi(0)(\Phi(0))^{-(1+\beta)} = M. \tag{46}$$

Substituting (46) in (36) we obtain the differential inequality

$$\Phi'(t) \ge 2(1+\beta)\Psi \ge 2(1+\beta)M\Phi^{1+\beta}. \tag{47}$$

If $\beta > 0$, then

$$(\Phi(t))^{-\beta} \le (\Phi(0))^{-\beta} - 2\beta(1+\beta)Mt. \tag{48}$$

This leads to

$$t^* \le T = \frac{1}{2\beta(1+\beta)M}(\Phi(0))^{-\beta} = \frac{\Phi(0)}{2\beta(1+\beta)\Psi(0)}. \tag{49}$$

If $\alpha = \beta = 0$, then

$$\Phi(t) \ge \Phi(0)e^{2Mt} \tag{50}$$

holds for $t > 0$. This implies that $t^* = \infty$ and completes the proof of Theorem 2. □

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