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Research Article

A Note on Stability of an Operator Linear Equation of the Second Order

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We prove some Hyers-Ulam stability results for an operator linear equation of the second order that is patterned on the difference equation, which defines the Lucas sequences (and in particular the Fibonacci numbers). In this way, we obtain several results on stability of some linear functional and differential and integral equations of the second order and some fixed point results for a particular (not necessarily linear) operator.

1. Introduction

Let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}_0$, and $\mathbb{N}$ stand, as usual, for the sets of complex numbers, real numbers, integers, nonnegative integers, and positive integers, respectively. Let S be a nonempty set, X a Banach space over a field $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, $p, q \in \mathbb{K}$, $q \neq 0$ and $p^2 - 4q \neq 0$, and a_1, a_2 denote the complex roots of the equation

$$qx^2 - px + 1 = 0. \tag{1.1}$$

Clearly we have $a_1 \neq a_2$,

$$p = \frac{1}{a_1} + \frac{1}{a_2}, \qquad q = \frac{1}{a_1 a_2}. \tag{1.2}$$

In what follows, X^S denotes the family of all functions mapping S into X and X^S is a linear space over $\mathbb{K}$ with the operations given by

$$(f+h)(x) := f(x) + h(x), \qquad (\alpha f)(x) := \alpha f(x) \tag{1.3}$$

for all $f, h \in X^S$, $\alpha \in \mathbb{K}$, $x \in S$. Throughout this paper, we assume that

(H) $\mathcal{C}$ is a nontrivial subgroup of the group $(X^S, +)$ and $\mathcal{L} : \mathcal{C} \to X^S$ is an additive operator (i.e., $\mathcal{L}(f+h) = \mathcal{L}f + \mathcal{L}h$ for $f, h \in X^S$).

We investigate the Hyers-Ulam stability of the operator equation

$$F = p\mathcal{L}F - q\mathcal{L}^2 F \tag{1.4}$$

for functions $F \in \mathcal{C}$ with $\mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}$. Namely, we show under suitable assumptions that for every function $f \in \mathcal{C}$ satisfying (1.4) approximately, that is,

$$\sup_{x \in S} \left\| p\mathcal{L}f(x) - q\mathcal{L}^2 f(x) - f(x) \right\| < \infty, \tag{1.5}$$

there exists a unique solution of the equation that is "near" to f. This kind of issues arise during study of the real-world phenomena, where we very often apply equations; however, in general, those equations are satisfied only with some error. Sometimes that error is neglected and it is believed that this will have only a minor influence on the final outcome. Since it is not always the case, it seems to be of interest to investigate when we can neglect the error, why, and to what extent.

One of the tools for systematic treatment of the problem described above seems to be the notion of Hyers-Ulam stability and some ideas inspired by it. That notion of stability was motivated by a question of Ulam (cf. [1, 2]), and a solution to it published by Hyers in [3]. At the moment, it is a very popular subject of investigation in the areas of, for example, functional, differential, integral equations, but also in other fields of mathematics (for information on this kind of stability and further references see, e.g., [4–8]). Also, the Hyers-Ulam stability is related to the notions of shadowing and controlled chaos (see, e.g., [9–12]).

If $\mathcal{L}f = f \circ \xi$ for $f \in \mathcal{C}$ (with $\mathcal{C} = X^S$ and a fixed mapping $\xi : S \to S$), then (1.4) takes the form

$$f(x) = pf(\xi(x)) - qf\left(\xi^2(x)\right), \tag{1.6}$$

which is a linear functional equation in a single variable of second order (for some information and further references on the functional equations in single variable, we refer to [13–15]). Stability of (1.6) has been already investigated in [16–23]. A particular case of (1.6), with $S = \mathbb{Z}$ and $\xi(x) = x + 1$, is the difference equation

$$f(x) = pf(x+1) - qf(x+2). \tag{1.7}$$

If $p, 1/q \in \mathbb{Z}$, then solutions $f : \mathbb{N}_0 \to \mathbb{Z}$ of the difference equation (1.7) are called the Lucas sequences (see, e.g., [24]); in some special cases they are given specific names; that is, the Fibonacci numbers ($p = -1$, $q = -1$, $f(0) = 0$, and $f(1) = 1$), the Lucas numbers ($p = -1$, $q = -1$, $f(0) = 2$, and $f(1) = 1$), the Pell numbers ($p = -2$, $q = -1$, $f(0) = 0$, and $f(1) = 1$), the Pell-Lucas (or companion Lucas) numbers ($p = -2$, $q = -1$, $f(0) = 2$, and $f(1) = 2$), and the Jacobsthall numbers ($p = -1$, $q = -2$, $f(0) = 0$, and $f(1) = 1$).

2. The Main Result

Now we will present a theorem that is the main result of this paper. In this section, we consider only the case

$$a_1, a_2 \in \mathbb{K}. \tag{2.1}$$

Some complementary results for the case where $a_1 \notin \mathbb{K}$ or $a_2 \notin \mathbb{K}$ will be given in the fourth section.

For simplicity, we write in the sequel

$$\|f\| := \sup_{x \in S} \|f(x)\|, \quad f \in X^S. \tag{2.2}$$

Next, for a given $g \in X^S$ and $\{g_n\}_{n\in\mathbb{N}} \subset X^S$, the equality

$$g = \lim_{n \to \infty} g_n \tag{2.3}$$

means that $g(x) = \lim_{n\to\infty} g_n(x)$ for every $x \in S$.

We say that $\mathcal{E} \subset X^S$ is closed with respect to the uniform convergence (abbreviated in the sequel to *c.u.c.*) provided the following holds true:

(E) if $f_n \in \mathcal{E}$ for $n \in \mathbb{N}$, $f \in X^S$ and $f_n \rightrightarrows f$, then $f \in \mathcal{E}$,
where the symbol $f_n \rightrightarrows f$ means that the sequence $\{f_n\}_{n\in\mathbb{N}}$ tends uniformly to f.
Moreover, we use in the sequel the following two hypotheses:

($C1$) $\mathcal{C} \subset (a_1 - a_2)\mathcal{C}$ and $a_j\mathcal{C} = \mathcal{C}$ for $j \in \{1, 2\}$;

($L1$) $\mathcal{L}(a_j f) = a_j \mathcal{L} f$ for $f \in \mathcal{C}$, $j \in \{1, 2\}$.

Now, we are in a position to formulate the main result of this paper.

Theorem 2.1. *Let $\varepsilon > 0$, $L_1 > 0$, $L_2 > 0$, and let $g \in \mathcal{C}$ with $\mathcal{L}g, \mathcal{L}^2 g \in \mathcal{C}$ satisfying the inequality*

$$\left\| g - p\mathcal{L}g + q\mathcal{L}^2 g \right\| \le \varepsilon. \tag{2.4}$$

Suppose that (2.1), ($C1$), *and* ($L1$) *are valid and one of the following three collections of hypotheses is fulfilled.*

(α) *$L_1 < |a_j|$ for $j \in \{1,2\}$, $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$, $\mathcal{C}$ is c.u.c., and*

$$\|\mathcal{L}f - \mathcal{L}h\| \le L_1 \|f - h\|, \quad f, h \in \mathcal{C}. \tag{2.5}$$

(β) *$\mathcal{L}$ is injective, $L_2 > |a_j|$ for $j \in \{1,2\}$, $\mathcal{L}(\mathcal{C})$ is c.u.c., $\mathcal{C} \subset \mathcal{L}(\mathcal{C})$, and*

$$\left\|\mathcal{L}^{-1} f - \mathcal{L}^{-1} h\right\| \le L_2^{-1} \|f - h\|, \quad f, h \in \mathcal{L}(\mathcal{C}). \tag{2.6}$$

(γ) *$\mathcal{L}$ is injective, $L_1 < |a_1|$, $L_2 > |a_2|$, $\mathcal{C}$ is c.u.c., $\mathcal{L}(\mathcal{C}) = \mathcal{C}$, and conditions* (2.5) *and* (2.6) *hold true.*

Then there exists a unique function $F \in \mathcal{C}$ with $\mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}$, that satisfies (1.4) *and*

$$\|g - F\| < \infty; \tag{2.7}$$

moreover, F is given by (3.30) *and*

$$\|g - F\| \le \frac{\varepsilon}{|q||a_2 - a_1|} \left(\frac{1}{\left|L_1' - |a_1|\right|} + \frac{1}{\left|L_2' - |a_2|\right|} \right), \tag{2.8}$$

where

$$L_i' := \begin{cases} L_1 & \text{if } (\alpha) \text{ holds}, \\ L_2 & \text{if } (\beta) \text{ holds}, \quad i \in \{1,2\}. \\ L_i & \text{if } (\gamma) \text{ holds}, \end{cases} \tag{2.9}$$

Remark 2.2. Clearly, if $\mathcal{C}$ is a linear subspace of X^S and $\mathcal{L}$ is linear (over $\mathbb{K}$), then ($C1$) and ($L1$) are valid. However, if $\mathcal{C}$ is "only" additive, $\mathcal{C}$ is a linear subspace of X^S but over $\mathbb{Q}$ (i.e., actually a divisible subgroup of X^S), and $a_1, a_2 \in \mathbb{Q}$, then ($C1$) and ($L1$) hold, as well. This shows that it makes sense to assume only ($L1$) instead of linearity of $\mathcal{L}$.

Below, before the proof of Theorem 2.1, we provide simple and natural examples of linear operators $\mathcal{L}$ that satisfy the assumptions of Theorem 2.1 (with suitable a_1, a_2).

(i) Let $\mathcal{C} = X^S$, $n \in \mathbb{N}$, and $\mathcal{L}f = \sum_{i=1}^n \Psi_i \circ f \circ \xi_i$, where $\Psi_i : X \to X$ is linear and bounded and $\xi_i : S \to S$ is fixed for each $i \in \{1, \ldots, n\}$. Then

$$\|\mathcal{L}f(x) - \mathcal{L}h(x)\| \le \sum_{i=1}^n \lambda_i \|f(\xi_i(x)) - h(\xi_i(x))\|, \quad f, h \in X^S, \ x \in S, \tag{2.10}$$

with $\lambda_i := \inf\{L \in \mathbb{R} : \|\Psi_i(w)\| \le L\|w\|, w \in X\}$. Hence (2.5) holds with $L_1 := \sum_{i=1}^n \lambda_i$.

Next, let $\mathcal{C} = X^S$, $\Psi : X \to X$, and $\xi : S \to S$ be bijective, Ψ linear, Ψ^{-1} bounded, and $\mathcal{L}f = \Psi \circ f \circ \xi$. Then

$$\left\| \mathcal{L}^{-1} f(x) - \mathcal{L}^{-1} h(x) \right\| \leq L_0 \left\| f\left(\xi^{-1}(x)\right) - h\left(\xi^{-1}(x)\right) \right\|, \quad f, h \in X^S, x \in S, \tag{2.11}$$

where $L_0 := \inf\{L \in \mathbb{R} : \|\Psi^{-1}(w)\| \leq L\|w\|, w \in X\}$. Clearly, as above, that inequality yields (2.6) with $L_2 := L_0^{-1}$. If additionally Ψ is bounded, then analogously as before we obtain that (2.5) holds, as well, with some $L_1 > 0$.

(ii) Let $a, b \in \mathbb{R}$, $a < b$, $S = [a, b]$, $\mathcal{C}$ the family of all continuous functions mapping the interval $[a, b]$ into $\mathbb{R}$, $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathbb{R}$, $\xi_1, \ldots, \xi_n : S \to S$ continuous, and $\mathcal{L}f(x) = \sum_{i=1}^{n} \int_a^x A_i f \circ \xi_i(t) dt$ for $f \in \mathcal{C}$, $x \in S$. Then it is easily seen that (α) is fulfilled with $|a_i| > L_1 := (b - a) \sum_{j=1}^{n} |A_j|$, $i \in \{1, 2\}$.

(iii) Let $a, b \in \mathbb{R}$, $a < b$, $\mathcal{C}$ be the family of all continuously differentiable functions $f : [a, b] \to \mathbb{R}$ with $f(a) = 0$ and $\mathcal{L} = d/dt$. Then (β) is satisfied with $|a_i| < L_2 := 1/(b - a)$, $i \in \{1, 2\}$.

3. Proof of Theorem 2.1

The subsequent lemma will be useful in the proof of Theorem 2.1.

Lemma 3.1. *Assume that* (2.1), (C1), *and* (L1) *are valid and one of the collections of hypotheses* (α)–(γ) *is fulfilled with some* $L_1, L_2 \in (0, \infty)$. *Let* $f_1, f_2 \in \mathcal{C}$, *with* $\mathcal{L}^i(f_j) \in \mathcal{C}$ *for* $i, j \in \{1, 2\}$, *be solutions of* (1.4) *and* $\|f_1 - f_2\| < \infty$. *Then* $f_1 = f_2$.

Proof. Let $h_i := \mathcal{L}f_i - a_2 f_i$ for $i \in \{1, 2\}$. Then, by (1.2) and (1.4),

$$\mathcal{L}h_i = \mathcal{L}^2 f_i - a_2 \mathcal{L} f_i = a_1 a_2 (p\mathcal{L} f_i - f_i) - a_2 \mathcal{L} f_i = a_1 h_i \tag{3.1}$$

for $i \in \{1, 2\}$. Consequently, for each $k \in \mathbb{N}$,

$$\|h_1 - h_2\| = |a_1|^{-k} \left\| \mathcal{L}^k h_1 - \mathcal{L}^k h_2 \right\| \leq |a_1|^{-k} L_1^k \|h_1 - h_2\| \tag{3.2}$$

if (α) or (γ) holds, and

$$\|h_1 - h_2\| = |a_1|^k \left\| \mathcal{L}^{-k} h_1 - \mathcal{L}^{-k} h_2 \right\| \leq |a_1|^k L_2^{-k} \|h_1 - h_2\| \tag{3.3}$$

if (β) holds. This means that $h_1 = h_2$.

Now, in view of the definition of h_i,

$$\mathcal{L} f_1 - a_2 f_1 - (\mathcal{L} f_2 - a_2 f_2) = h_1 - h_2 = 0, \tag{3.4}$$

which means that

$$\mathcal{L} f_1 - \mathcal{L} f_2 = a_2 (f_1 - f_2). \tag{3.5}$$

So, analogously as before, for each $k \in \mathbb{N}$, in the case of (α), we have

$$\|f_1 - f_2\| = |a_2|^{-k} \left\| \mathcal{L}^k f_1 - \mathcal{L}^k f_2 \right\| \leq |a_2|^{-k} L_1^k \|f_1 - f_2\|, \tag{3.6}$$

and, in the case of (β) or (γ),

$$\|f_1 - f_2\| = |a_2|^{k} \left\| \mathcal{L}^{-k} f_1 - \mathcal{L}^{-k} f_2 \right\| \leq |a_2|^{k} L_2^{-k} \|f_1 - f_2\|. \tag{3.7}$$

It is easily seen that in each of those cases the above two inequalities imply that $f_1 = f_2$. □

Now, we have all tools to prove Theorem 2.1.

To this end, fix $i \in \{1,2\}$. Then $|a_i| > L_1$ or $|a_i| < L_2$. First consider the situation: $L_1 < |a_i|$. Clearly this means that (α) or (γ) must be valid, which yields $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$. Write

$$\mathcal{A}_k^i := a_i^{-k}\left[\mathcal{L}^k g - \left(p - a_i^{-1}\right)\mathcal{L}^{k+1} g\right], \quad k \in \mathbb{N}_0. \tag{3.8}$$

Note that, by $(C1)$ and $(L1)$, $A_k^i \in \mathcal{C}$ for $k \in \mathbb{N}_0$. Further, for each $k \in \mathbb{N}_0$, from (1.2) we get

$$\begin{aligned} \mathcal{A}_k^i - \mathcal{A}_{k+1}^i &= a_i^{-k}\left[\mathcal{L}^k g - \left(p - a_i^{-1}\right)\mathcal{L}^{k+1} g\right] - a_i^{-k-1}\left[\mathcal{L}^{k+1} g - \left(p - a_i^{-1}\right)\mathcal{L}^{k+2} g\right] \\ &= a_i^{-k}\mathcal{L}^k\left(g - p\mathcal{L}g + q\mathcal{L}^2 g\right), \end{aligned} \tag{3.9}$$

whence, according to (2.4) and (2.5),

$$\left\|\mathcal{A}_k^i - \mathcal{A}_{k+1}^i\right\| \leq |a_i|^{-k} L_1^k \left\|g - p\mathcal{L}g + q\mathcal{L}^2 g\right\| \leq |a_i|^{-k} L_1^k \varepsilon, \tag{3.10}$$

and consequently

$$\left\|\mathcal{A}_k^i - \mathcal{A}_{k+n}^i\right\| \leq \sum_{j=k}^{k+n-1} |a_i|^{-j} L_1^j \varepsilon, \quad n \in \mathbb{N}. \tag{3.11}$$

This means that, for each $x \in S$, $\{\mathcal{A}_n^i(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and therefore, there exists the limit $F_i(x) = \lim_{n\to\infty}\mathcal{A}_n^i(x)$. Moreover, (3.11) yields $\mathcal{A}_n^i \rightrightarrows F_i$, whence

$$F_i, \mathcal{L}F_i, \mathcal{L}^2 F_i \in \mathcal{C}, \tag{3.12}$$

because $\mathcal{C}$ is c.u.c. and $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$.

Observe that, for every $n, k \in \mathbb{N}_0$,

$$\mathcal{L}^n \mathcal{A}_k^i = a_i^n \mathcal{A}_{n+k}^i. \tag{3.13}$$

Further, by (2.5), for each $n \in \mathbb{N}$,

$$\left\| \mathcal{L}F_i - \mathcal{L}\mathcal{A}_n^i \right\| \leq L_1 \left\| F_i - \mathcal{A}_n^i \right\|, \tag{3.14}$$

which yields

$$\mathcal{L}^k F_i = \lim_{n \to \infty} \mathcal{L}^k \mathcal{A}_n^i, \qquad k \in \{1, 2\}. \tag{3.15}$$

So, in view of (1.2) and (3.13), we have

$$\begin{aligned} p\mathcal{L}F_i - q\mathcal{L}^2 F_i &= p \lim_{n \to \infty} \mathcal{L}\mathcal{A}_n^i - q \lim_{n \to \infty} \mathcal{L}^2 \mathcal{A}_n^i \\ &= pa_i \lim_{n \to \infty} \mathcal{A}_{n+1}^i - qa_i^2 \lim_{n \to \infty} \mathcal{A}_{n+2}^i \\ &= pa_i F_i - qa_i^2 F_i \\ &= F_i, \end{aligned} \tag{3.16}$$

and, by (3.11) with $k = 0$ and $n \to \infty$,

$$\left\| g - \left(p - a_i^{-1}\right)\mathcal{L}g - F_i \right\| \leq \frac{|a_i|\varepsilon}{|a_i| - L_1}. \tag{3.17}$$

Now, consider the case when $|a_i| < L_2$. Then, according to the assumptions, $\mathcal{L}$ is injective, (2.6) holds, and $\mathcal{C} \subset \mathcal{L}(\mathcal{C})$, that is,

$$\mathcal{L}^{-k}(\mathcal{C}) \subset \mathcal{C} \subset \mathcal{L}(\mathcal{C}), \quad k \in \mathbb{N}. \tag{3.18}$$

Write

$$\mathcal{A}_k^i := a_i^k \left[\mathcal{L}^{-k} g - \left(p - a_i^{-1}\right) \mathcal{L}^{-k+1} g \right], \quad k \in \mathbb{N}_0. \tag{3.19}$$

Then, for each $k \in \mathbb{N}$, we have $\mathcal{A}_k^i \in \mathcal{C}$ (because g is such that $\mathcal{L}g \in \mathcal{C}$),

$$\begin{aligned} \mathcal{A}_k^i - \mathcal{A}_{k-1}^i &= a_i^k \left[\mathcal{L}^{-k} g - \left(p - a_i^{-1}\right) \mathcal{L}^{-k+1} g \right] - a_i^{k-1} \left[\mathcal{L}^{-k+1} g - \left(p - a_i^{-1}\right) \mathcal{L}^{-k+2} g \right] \\ &= a_i^k \mathcal{L}^{-k} \left(g - p\mathcal{L}g + q\mathcal{L}^2 g \right), \end{aligned} \tag{3.20}$$

and next, by (2.6),

$$\left\| \mathcal{A}_k^i - \mathcal{A}_{k-1}^i \right\| \leq |a_i|^k L_2^{-k} \left\| g - p\mathcal{L}g + q\mathcal{L}^2 g \right\| \leq |a_i|^k L_2^{-k} \varepsilon. \tag{3.21}$$

This yields

$$\left\|\mathcal{A}_k^i - \mathcal{A}_{k+n}^i\right\| \leq \sum_{j=k+1}^{k+n} |a_i|^j L_2^{-j} \varepsilon, \quad k, n \in \mathbb{N}_0,\ n > 0. \tag{3.22}$$

So, for each $x \in S$, $\{\mathcal{A}_n^i(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence, and consequently there exists the limit $F_i(x) = \lim_{n\to\infty}\mathcal{A}_n^i(x)$. Note that, by (3.22), $\mathcal{A}_n^i \rightrightarrows F_i$, whence

$$F_i \in \mathcal{L}(\mathcal{C}) \tag{3.23}$$

(because $\mathcal{A}_n^i \in \mathcal{C} \subset \mathcal{L}(\mathcal{C})$ for $n \in \mathbb{N}$ and $\mathcal{L}(\mathcal{C})$ is c.u.c.), and again by (3.22), with $k = 0$ and $n \to \infty$,

$$\left\|g - \left(p - a_i^{-1}\right)\mathcal{L}g - F_i\right\| \leq \frac{\varepsilon L_2^{-1}|a_i|}{1 - L_2^{-1}|a_i|} = \frac{|a_i|\varepsilon}{L_2 - |a_i|}. \tag{3.24}$$

It is easy to observe that

$$\mathcal{L}^{-n}\mathcal{A}_k^i = a_i^{-n}\mathcal{A}_{k+n}^i, \quad k, n \in \mathbb{N}_0. \tag{3.25}$$

Further, by (2.6), for each $k \in \mathbb{N}$,

$$\mathcal{L}^{-k}F_i = \lim_{n\to\infty}\mathcal{L}^{-k}\mathcal{A}_n^i. \tag{3.26}$$

So, by (3.18), (3.23), and (3.25), we have

$$\begin{aligned}
p\mathcal{L}^{-1}F_i - qF_i &= p\lim_{n\to\infty}\mathcal{L}^{-1}\mathcal{A}_n^i - q\lim_{n\to\infty}\mathcal{A}_n^i \\
&= pa_i^{-1}\lim_{n\to\infty}\mathcal{A}_{n+1}^i - q\lim_{n\to\infty}\mathcal{A}_n^i \\
&= pa_i^{-1}F_i - qF_i = a_i^{-2}F_i \\
&= \lim_{n\to\infty}a_i^{-2}\mathcal{A}_n^i = \lim_{n\to\infty}\mathcal{L}^{-2}\mathcal{A}_{n-2}^i \\
&= \mathcal{L}^{-2}F_i.
\end{aligned} \tag{3.27}$$

This and (3.23) yield $qF_i = p\mathcal{L}^{-1}F_i - \mathcal{L}^{-2}F_i \in \mathcal{C}$, that is, $F_i \in \mathcal{C}$. Repeating yet that reasoning twice, we get

$$F_i \in \mathcal{L}^{-2}(\mathcal{C}) \cap \mathcal{L}^{-1}(\mathcal{C}) \tag{3.28}$$

(i.e., (3.12) holds) and consequently

$$p\mathcal{L}F_i - q\mathcal{L}^2F_i = \mathcal{L}^2\left(p\mathcal{L}^{-1}F_i - qF_i\right) = \mathcal{L}^2\left(\mathcal{L}^{-2}F_i\right) = F_i. \tag{3.29}$$

Thus we have proved that, for $i \in \{1,2\}$, in either case inequalities (3.17) or (3.24), respectively, hold and F_i is a solution to (1.4), with (3.12) fulfilled. Define $F : S \to X$ by

$$F := \frac{a_2}{a_2 - a_1} F_1 - \frac{a_1}{a_2 - a_1} F_2. \tag{3.30}$$

Then, by (3.12),

$$F, \mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}, \tag{3.31}$$

and it follows from (3.16) and (3.29), respectively, that

$$p\mathcal{L}F - q\mathcal{L}^2 F = \frac{a_2}{a_2 - a_1}\left[p\mathcal{L}F_1 - q\mathcal{L}^2 F_1\right] - \frac{a_1}{a_2 - a_1}\left[p\mathcal{L}F_2 - q\mathcal{L}^2 F_2\right] = F. \tag{3.32}$$

Moreover,

$$a_2\left(p - a_1^{-1}\right) - a_1\left(p - a_2^{-1}\right) = 0, \tag{3.33}$$

and consequently

$$\begin{aligned} \|g - F\| &= \frac{1}{|a_2 - a_1|}\|(a_2 - a_1)g - a_2 F_1 + a_1 F_2\| \\ &= \frac{1}{|a_2 - a_1|}\Big\|(a_2 - a_1)g - a_2 F_1 + a_1 F_2 \\ &\quad - \left[a_2\left(p - a_1^{-1}\right) - a_1\left(p - a_2^{-1}\right)\right]\mathcal{L}g\Big\| \\ &\le \frac{|a_2|}{|a_2 - a_1|}\left\|g - F_1 - \left(p - a_1^{-1}\right)\mathcal{L}g\right\| \\ &\quad + \frac{|a_1|}{|a_2 - a_1|}\left\|g - F_2 - \left(p - a_2^{-1}\right)\mathcal{L}g\right\|, \end{aligned} \tag{3.34}$$

whence, by (3.17) and (3.24), respectively, we obtain (2.8).

For the proof of the statement concerning uniqueness of F, take $F_0 \in \mathcal{C}$ with $\mathcal{L}F_0, \mathcal{L}^2 F_0 \in \mathcal{C}$. Suppose that F_0 is a solution of (1.4) such that $\|g - F_0\| < \infty$. Then we have

$$\|F - F_0\| \le \|F - g\| + \|g - F_0\| < \infty, \tag{3.35}$$

and therefore, by Lemma 3.1, $F = F_0$. This completes the proof of Theorem 2.1.

4. Complementary Results

In this section, we consider the cases that are complementary to those of Theorem 2.1, that is, when $\mathbb{K} = \mathbb{R}$ and (2.1) may not be fulfilled. We will use the following assumptions:

$(C2)$ $a\mathcal{C} \subset \mathcal{C}$ for $a \in \{\Re(a_1), \Re(a_2), \Im(a_1), \Im(a_2), |a_1 - a_2|^{-2}, |a_1|^{-2}, |a_2|^{-2}\}$,

$(L2)$ $\mathcal{L}(af) = a\mathcal{L}f$ for $a \in \{\Re(a_1), \Re(a_2), \Im(a_1), \Im(a_2)\}$, $f \in \mathcal{C}$,

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z (if z is a real number, then simply $\Re(z) = z$ and $\Im(z) = 0$). Observe that, in the case $a_1, a_2 \in \mathbb{K} = \mathbb{R}$, $(C2)$ and $(L2)$ become just $(C1)$ and $(L1)$. Note also that if $\mathcal{C}$ is a real linear subspace of X^S, then $(C2)$ and $(L2)$ are fulfilled.

The next theorem complements Theorem 2.1 when (α) is valid (however, with a bit stronger assumption on L_1). The cases of (β) and (γ) are more complicated, and some results concerning them will be published separately.

Theorem 4.1. *Let $\mathbb{K} = \mathbb{R}$, $\varepsilon > 0$, $L > 0$, and $g \in \mathcal{C}$, with $\mathcal{L}g, \mathcal{L}^2 g \in \mathcal{C}$, satisfy* (2.4). *Suppose that $2L < |a_j|$ for $j \in \{1, 2\}$, $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$, and*

$$\|\mathcal{L}f - \mathcal{L}h\| \le L\|f - h\|, \quad f, h \in \mathcal{C}. \tag{4.1}$$

Then there exists a unique function $F \in \mathcal{C}$, with $\mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}$, that satisfies (1.4) *and inequality* (2.7)*; moreover,*

$$\|g - F\| \le \frac{\varepsilon}{|q||a_2 - a_1|}\left(\frac{1}{|2L - |a_1||} + \frac{1}{|2L - |a_2||}\right). \tag{4.2}$$

Proof. We apply a well-known method of complexification of the real Banach space X. Namely, (see, e.g., [25, page 39], [26], or [27, 1.9.6, page 66]) X^2 is a complex Banach space with the linear structure and the Taylor norm $\|\cdot\|_T$ given by

$$\begin{aligned} (x, y) + (z, w) &:= (x + z, y + w) \quad \text{for } x, y, z, w \in X, \\ (\alpha + i\beta)(x, y) &:= (\alpha x - \beta y, \beta x + \alpha y) \quad \text{for } x, y \in X, \alpha, \beta \in \mathbb{R}, \\ \|(x, y)\|_T &:= \sup_{0 \le \theta \le 2\pi} \|(\cos\theta)x + (\sin\theta)y\| \quad \text{for } x, y \in X. \end{aligned} \tag{4.3}$$

Note that

$$\max\{\|x\|, \|y\|\} \le \|(x, y)\|_T \le \|x\| + \|y\|, \quad x, y \in X. \tag{4.4}$$

Analogously as before we write

$$\|\mu\|_T := \sup_{x \in S} \|\mu(x)\|_T \tag{4.5}$$

for each function $\mu : S \to X^2$. Next,

$$p_i(w_1, w_2) := w_i, \quad i \in \{1, 2\}; \; w_1, w_2 \in X. \tag{4.6}$$

Let

$$\mathcal{C}_0 := \left\{\mu : S \longrightarrow X^2 : p_i \circ \mu \in \mathcal{C},\ i \in \{1,2\}\right\}, \tag{4.7}$$

$\chi : S \to X^2$ be given by $\chi(x) := (g(x), 0)$ for $x \in S$, and

$$\mathcal{L}_0\mu(x) := \left(\mathcal{L}(p_1 \circ \mu)(x), \mathcal{L}(p_2 \circ \mu)(x)\right) \tag{4.8}$$

for every $\mu \in \mathcal{C}_0$ and $x \in S$. Since $\mathcal{L}(\mathcal{C}) \subset \mathcal{C}$ and $\mathcal{C}$ is a subgroup of the group $(X^S, +)$ (i.e., the function $\mu_0 : S \to X$ defined by $\mu_0(x) = 0$ for $x \in S$, is in $\mathcal{C}$), it is easily seen that $\chi, \mathcal{L}_0\chi, \mathcal{L}_0^2\chi \in \mathcal{C}_0$ and

$$\mathcal{L}_0(\mathcal{C}_0) \subset \mathcal{C}_0. \tag{4.9}$$

Next, for each $f \in \mathcal{C}_0$, we have $f_1 := p_1 \circ f, f_2 := p_2 \circ f \in \mathcal{C}$, whence, in view of $(C2)$ and $(L2)$, for each $j \in \{1,2\}$,

$$\begin{aligned}
\mathcal{L}(\Re(a_j) f_1 - \Im(a_j) f_2) &= \Re(a_j)\mathcal{L}f_1 - \Im(a_j)\mathcal{L}f_2,\\
\mathcal{L}(\Im(a_j) f_1 + \Re(a_j) f_2) &= \Im(a_j)\mathcal{L}f_1 + \Re(a_j)\mathcal{L}f_2,
\end{aligned} \tag{4.10}$$

and consequently

$$\begin{aligned}
\mathcal{L}_0(a_j f) &= \mathcal{L}_0(\Re(a_j) f_1 - \Im(a_j) f_2, \Re(a_j) f_2 + \Im(a_j) f_1)\\
&= (\mathcal{L}(\Re(a_j) f_1 - \Im(a_j) f_2), \mathcal{L}(\Re(a_j) f_2 + \Im(a_j) f_1))\\
&= (\Re(a_j)\mathcal{L}f_1 - \Im(a_j)\mathcal{L}f_2, \Im(a_j)\mathcal{L}f_1 + \Re(a_j)\mathcal{L}f_2)\\
&= a_j(\mathcal{L}f_1,\ \mathcal{L}f_2) = a_j\mathcal{L}_0 f.
\end{aligned} \tag{4.11}$$

Thus, we have obtained that

$(L1')$ $\mathcal{L}_0(a_j f) = a_j\mathcal{L}_0 f$ for $f \in \mathcal{C}_0$ and $j \in \{1,2\}$.

Analogously, for every $\mu \in \mathcal{C}_0$, $i \in \{1,2\}$, we get

$$\begin{aligned}
&a_i\mu = (\Re(a_i)p_1 \circ \mu - \Im(a_i)p_2 \circ \mu,\ \Re(a_i)p_2 \circ \mu + \Im(a_i)p_1 \circ \mu),\\
&\frac{1}{a_i}\mu = \frac{\overline{a_i}}{|a_i|^2}\mu = \left(\frac{\Re(a_i)}{|a_i|^2}p_1 \circ \mu + \frac{\Im(a_i)}{|a_i|^2}p_2 \circ \mu,\ \frac{\Re(a_i)}{|a_i|^2}p_2 \circ \mu - \frac{\Im(a_i)}{|a_i|^2}p_1 \circ \mu\right)
\end{aligned} \tag{4.12}$$

which means that $a_i\mu,\ a_i^{-1}\mu \in \mathcal{C}_0$ (because $\mathcal{C}_0$ is a group and $(C2)$ holds). Moreover,

$$\frac{1}{a_1 - a_2}\ \mathcal{C}_0 = \frac{\overline{a_1} - \overline{a_2}}{|a_1 - a_2|^2}\ \mathcal{C}_0 \subset \mathcal{C}_0. \tag{4.13}$$

Thus we have proved that

($C1'$) $a_j\mathcal{C}_0 = \mathcal{C}_0$ and $\mathcal{C}_0 \subset (a_1 - a_2)\mathcal{C}_0$ for $j \in \{1,2\}$.

Now, we show that $\mathcal{C}_0$ is c.u.c. with regard to the Taylor norm. To this end, take $\mu \in X^S$ and $\mu_n \in \mathcal{C}_0$ for $n \in \mathbb{N}$ such that $\mu_n \rightrightarrows \mu$ (with respect to the Taylor norm). Then, by (4.4),

$$\max\{\|p_1 \circ \mu_n - p_1 \circ \mu\|,\ \|p_2 \circ \mu_n - p_2 \circ \mu\|\} \le \|\mu_n - \mu\|_T, \quad n \in \mathbb{N}, \tag{4.14}$$

which means that $p_1 \circ \mu_n \rightrightarrows p_1 \circ \mu$ and $p_2 \circ \mu_n \rightrightarrows p_2 \circ \mu$. Consequently, $p_1 \circ \mu,\ p_2 \circ \mu \in \mathcal{C}$. Hence, $\mu \in \mathcal{C}_0$.

Note yet that, according to (4.1) and (4.4), for every $\mu \in \mathcal{C}_0$, we have

$$\begin{aligned}
\|\mathcal{L}_0\mu\|_T &= \|(\mathcal{L}(p_1 \circ \mu), \mathcal{L}(p_2 \circ \mu))\|_T \le \|\mathcal{L}(p_1 \circ \mu)\| + \|\mathcal{L}(p_2 \circ \mu)\| \\
&\le L\|p_1 \circ \mu\| + L\|p_2 \circ \mu\| \le 2L \max\{\|p_1 \circ \mu\|, \|p_2 \circ \mu\|\} \\
&\le 2L\|\mu\|_T, \\
\left\|\chi - p\mathcal{L}_0\chi + q\mathcal{L}_0^2\chi\right\|_T &= \left\|\left(g - p\mathcal{L}g + q\mathcal{L}^2 g,\ 0\right)\right\|_T \\
&\le \left\|g - p\mathcal{L}g + q\mathcal{L}^2 g\right\| \le \varepsilon,
\end{aligned} \tag{4.15}$$

because $p_2 \circ \chi(x) = 0$ for each $x \in S$.

In this way, we have shown that the assumptions of Theorem 2.1(α) are satisfied (with g, $\mathcal{L}$, L_1, and $\mathcal{C}$ replaced by χ, $\mathcal{L}_0$, $2L$, and $\mathcal{C}_0$, resp.) and consequently there is a solution $H \in \mathcal{C}_0$ of the equation

$$H = p\mathcal{L}_0 H - q\mathcal{L}_0^2 H \tag{4.16}$$

such that

$$\|\chi - H\|_T \le \frac{\varepsilon}{|q||a_1 - a_2|}\left(\frac{1}{||a_1| - 2L|} + \frac{1}{||a_2| - 2L|}\right). \tag{4.17}$$

Observe that $F := p_1 \circ H$ is a solution of (1.4) and, by (4.4), (4.2) holds.

It remains to prove the statement concerning uniqueness of F. So, let $F_0 \in \mathcal{C}$, with $\mathcal{L}F_0, \mathcal{L}^2 F_0 \in \mathcal{C}$, be a solution of (1.4) such that $\|g - F_0\| < \infty$. Write $H_0(x) := (F_0(x), p_2(H(x)))$ for $x \in S$. It is easily seen that $H_0 \in \mathcal{C}_0$ and $H_0 = p\mathcal{L}_0 H_0 - q\mathcal{L}_0^2 H_0$. Moreover,

$$\|H - H_0\|_T = \|F - F_0\| \le \|F - g\| + \|g - F_0\| < \infty. \tag{4.18}$$

Hence, by Lemma 3.1 (with $\mathcal{L}$ and $\mathcal{C}$ replaced by $\mathcal{L}_0$ and $\mathcal{C}_0$, resp.), $H = H_0$, which yields $F_0 = p_1 \circ H_0 = p_1 \circ H = F$. □

5. Final Remarks on Fixed Points and Open Problems

Theorems 2.1 and 4.1 can be actually expressed in the terms of fixed points. Namely, they may be reformulated as follows.

Theorem 5.1. *Let* $\mathcal{T} := p\mathcal{L} - q\mathcal{L}^2$, $\varepsilon > 0$, $L_1 > 0$, $L_2 > 0$, *and* $g \in \mathcal{C}$, *with* $\mathcal{L}g, \mathcal{L}^2 g \in \mathcal{C}$, *satisfying the inequality*

$$\|g - \mathcal{T}g\| \leq \varepsilon. \tag{5.1}$$

Suppose that $(C1)$, $(L1)$, *and one of the following two conditions are valid:*

(a) *Condition* (2.1) *and one of the collections of hypotheses* (α)–(γ) *are fulfilled;*

(b) *the collection of hypotheses* (α) *is fulfilled and* $2L_1 < |a_j|$ *for* $j \in \{1, 2\}$.

Then there exists a unique $F \in \mathcal{C}$ *with* $\mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}$ *such that* F *is a fixed point of* $\mathcal{T}$ *and*

$$\|g - F\| < \infty; \tag{5.2}$$

moreover, if (a) *is valid, then* (2.8) *holds and if* (b) *is valid, then* (4.2) *holds with* $L = L_1$.

If $\mathcal{L}$ is linear, then Theorems 2.1 and 4.1 can also be expressed in the following way (cf. [7]).

Theorem 5.2. *Let* $\mathcal{K} := p\mathcal{L} - q\mathcal{L}^2 - \mathcal{I}$, *where* $\mathcal{I} : X^S \to X^S$ *is the identity operator (given by* $\mathcal{I}f = f$ *for* $f \in X^S$*). Suppose that* $(C1)$, $(L1)$, *and one of conditions* (a), (b) *are valid with some* $L_1 > 0$, $L_2 > 0$. *Then, for every* $g \in \mathcal{C}$ *with* $\mathcal{L}g, \mathcal{L}^2 g \in \mathcal{C}$ *and*

$$\varepsilon := \|\mathcal{K}g\| < \infty, \tag{5.3}$$

there exists a unique $F \in \mathcal{C}$ *with* $\mathcal{L}F, \mathcal{L}^2 F \in \mathcal{C}$ *and such that* $F \in \ker \mathcal{K}$ *(i.e.,* $\mathcal{K}f(x) = 0$ *for* $x \in S$*) and* $\|g - F\| < \infty$*; moreover, if* (a) *is valid, then* (2.8) *holds and if* (b) *is valid, then* (4.2) *holds with* $L = L_1$.

In connection with the results presented in this paper, there arise several natural questions (apart from those regarding the situation where (2.1) is not fulfilled). We mention here some of them.

The first one concerns optimality of estimations (2.8) and (4.2). It is known that in general they are not the best possible, and for suitable comments and examples, see [17]. It seems that this issue deserves a more systematic treatment.

Another question concerns the case where $L_1 \geq |a_i|$ for some $i \in \{1, 2\}$ when (α) is valid (and analogous situations for (β) and (γ)). In general, the assumption $L_1 \leq |a_i|$ for $i \in \{1, 2\}$ is necessary in the case of (α), as it follows from nonstability results in [18]. But maybe in some particular situation some partial stability results are possible.

One more question is whether methods similar to those used in this paper can be applied for a bit more general equation of the form

$$g = p\mathcal{L}g + q\mathcal{L}^2 g + H(x) \tag{5.4}$$

with a nontrivially given function $H : S \to X$. Also, it is interesting if these methods can be applied for a higher-order operator linear equation, for example, for the third-order equation

$$g = p\mathcal{L}g + q\mathcal{L}^2 g + r\mathcal{L}^3 g. \tag{5.5}$$

For related results, in some particular situations and obtained with different methods, we refer to [19, 28].

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