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Research Article

A New Iterative Scheme for Countable Families of Weak Relatively Nonexpansive Mappings and System of Generalized Mixed Equilibrium Problems

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We construct a new iterative scheme by hybrid methods and prove strong convergence theorem for approximation of a common fixed point of two countable families of weak relatively nonexpansive mappings which is also a solution to a system of generalized mixed equilibrium problems in a uniformly convex real Banach space which is also uniformly smooth using the properties of generalized f-projection operator. Using this result, we discuss strong convergence theorem concerning general H-monotone mappings and system of generalized mixed equilibrium problems in Banach spaces. Our results extend many known recent results in the literature.

1. Introduction

Let E be a real Banach space with dual E^*, and let C be nonempty, closed and convex subset of E. A mapping $T : C \to C$ is called *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

A point $x \in C$ is called *a fixed point* of T if $Tx = x$. The set of fixed points of T is denoted by $F(T) := \{x \in C : Tx = x\}$.

We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$J(x) = \left\{ f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2 \right\}. \tag{1.2}$$

The following properties of J are well known (the reader can consult [1–3] for more details).

(1) If E is uniformly smooth, then J is norm-to-norm uniformly continuous on each bounded subset of E.

(2) $J(x) \neq \emptyset$, $x \in E$.

(3) If E is reflexive, then J is a mapping from E onto E^*.

(4) If E is smooth, then J is single valued.

Throughout this paper, we denote by ϕ, the functional on $E \times E$ defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, J(y) \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{1.3}$$

From [4], in uniformly convex and uniformly smooth Banach spaces, we have

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2, \quad \forall x, y \in E. \tag{1.4}$$

Definition 1.1. Let C be a nonempty subset of E and let $\{T_n\}_{n=0}^{\infty}$ be a countable family of mappings from C into E. A point $p \in C$ is said to be an *asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(\{T_n\}_{n=0}^{\infty})$. One says that $\{T_n\}_{n=0}^{\infty}$ is *countable family of relatively nonexpansive mappings* (see, e.g., [5]) if the following conditions are satisfied:

(R1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;

(R2) $\phi(p, T_n x) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T_n)$, $n \geq 0$;

(R3) $\cap_{n=0}^{\infty} F(T_n) = \widehat{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.2. A point $p \in C$ is said to be a *strong asymptotic fixed point* of $\{T_n\}_{n=0}^{\infty}$ if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty}\|x_n - T_n x_n\| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(\{T_n\}_{n=0}^{\infty})$. One says that a mapping $\{T_n\}_{n=0}^{\infty}$ is *countable family of weak relatively nonexpansive mappings* (see, e.g., [5]) if the following conditions are satisfied:

(R1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;

(R2) $\phi(p, T_n x) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T_n)$, $n \geq 0$;

(R3) $\cap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Definition 1.3. Let C be a nonempty subset of E and let T be a mapping from C into E. A point $p \in C$ is said to be an *asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges weakly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$. We say that a mapping T is *relatively nonexpansive* (see, e.g., [6–11]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$;

(R3) $F(T) = \widehat{F}(T)$.

Definition 1.4. A point $p \in C$ is said to be an *strong asymptotic fixed point* of T if C contains a sequence $\{x_n\}_{n=0}^{\infty}$ which converges strongly to p and $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. The set of strong asymptotic fixed points of T is denoted by $\widetilde{F}(T)$. We say that a mapping T is *weak relatively nonexpansive* (see, e.g., [12, 13]) if the following conditions are satisfied:

(R1) $F(T) \neq \emptyset$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$, for all $x \in C$, $p \in F(T)$;

(R3) $F(T) = \widetilde{F}(T)$.

Definition 1.3 (Definition 1.4) is a special form of Definition 1.1 (Definition 1.2) as $T_n \equiv T$, for all $n \geq 0$. Furthermore, Su et al. [5] gave an example which is a countable family of weak relatively nonexpansive mappings but not a countable family of relatively nonexpansive mappings. It is obvious that relatively nonexpansive mapping is weak relatively nonexpansive mapping. In fact, for any mapping $T : C \to C$, we have $F(T) \subset \widetilde{F}(T) \subset \widehat{F}(T)$. Therefore, if T is relatively nonexpansive mapping, then $F(T) = \widetilde{F}(T) = \widehat{F}(T)$. Kang et al. [12] gave an example of a weak relatively nonexpansive mapping which is not relatively nonexpansive.

Let $F : C\times C \to \mathbb{R}$ be a bifunction, $A : C \to E^*$ a mapping and $\varphi : C \to \mathbb{R}$ a real-valued function. The generalized mixed equilibrium problem is to find $x \in C$ (see, e.g., [14–19]) such that

$$F(x, y) + \langle Ax, y - x\rangle + \varphi(y) - \varphi(x) \geq 0, \tag{1.5}$$

for all $y \in C$. We will denote the solutions set of (1.5) by GMEP(F, φ). Thus

$$\text{GMEP}(F, A, \varphi) := \{x^* \in C : F(x^*, y) + \langle Ax^*, y - x^*\rangle + \varphi(y) - \varphi(x^*) \geq 0,\ \forall y \in C\}. \tag{1.6}$$

If $\varphi = 0$ and $A = 0$, then problem (1.5) reduces to an equilibrium problem studied by many authors (see, e.g., [20–28]), which is to find $x^* \in C$ such that

$$F(x^*, y) \geq 0 \tag{1.7}$$

for all $y \in C$. We shall denote the solutions set of (1.7) by EP(F).

If $\varphi = 0$ and $E = H$ (a real Hilbert space), then problem (1.5) reduces to a generalized equilibrium problem studied by many authors (see, e.g., [29–31]), which is to find $x^* \in C$ such that

$$F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0 \tag{1.8}$$

for all $y \in C$.

If $A = 0$ and $E = H$, then problem (1.5) reduces to mixed equilibrium problem considered by many authors (see, e.g., [32–34]), which is to find $x^* \in C$ such that

$$F(x^*, y) + \varphi(y) - \varphi(x^*) \geq 0 \tag{1.9}$$

for all $y \in C$.

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and equilibrium problems as special cases (see, e.g., [35]). Some methods have been proposed to solve the mixed equilibrium problem (see, e.g., [33, 34, 36]). Numerous problems in physics, optimization and economics reduce to find a solution of problem (1.8).

In [9], Matsushita and Takahashi introduced a hybrid iterative scheme for approximation of fixed points of relatively nonexpansive mapping in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$,

$$\begin{aligned} y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ H_n &= \{w \in C : \phi(w, y_n) \leq \phi(w, x_n)\}, \\ W_n &= \{w \in C : \langle x_n - w, J x_0 - J x_n \rangle\}, \\ x_{n+1} &= \Pi_{H_n \cap W_n} x_0, \quad n \geq 0 \end{aligned} \tag{1.10}$$

They proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)} x_0$, where $F(T) \neq \emptyset$.

In [37], Plubtieng and Ungchittrakool introduced the following hybrid projection algorithm for a pair of relatively nonexpansive mappings: $x_0 \in C$,

$$\begin{aligned} z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T x_n + \beta_n^{(3)} J S x_n\right), \\ y_n &= J^{-1}(\alpha_n J x_0 + (1-\alpha_n) J z_n), \\ C_n &= \left\{z \in C : \phi(z, y_n) \leq \phi(z, x_n) + \alpha_n\left(\|x_0\|^2 + 2\langle w, J x_n - J x_0 \rangle\right)\right\}, \\ Q_n &= \{z \in C : \langle x_n - z, J x_n - J x_0 \rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x_0, \end{aligned} \tag{1.11}$$

where $\{\alpha_n\}$, $\{\beta_n^{(1)}\}$, $\{\beta_n^{(2)}\}$, and $\{\beta_n^{(3)}\}$ are sequences in $(0, 1)$ satisfying $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(3)} = 1$ and T and S are relatively nonexpansive mappings and J is the single-valued duality mapping on E. They proved under the appropriate conditions on the parameters that the sequence $\{x_n\}$ generated by (1.11) converges strongly to a common fixed point of T and S.

In [10], Takahashi and Zembayashi introduced the following hybrid iterative scheme for approximation of fixed point of relatively nonexpansive mapping which is also a solution to an equilibrium problem in a uniformly convex real Banach space which is also uniformly smooth: $x_0 \in C$, $C_1 = C$, $x_1 = \Pi_{C_1} x_0$,

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\\
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, J u_n - J y_n\rangle &\geq 0, \quad \forall y \in C,\\
C_{n+1} &= \{w \in C_n : \phi(w, u_n) \leq \phi(w, x_n)\},\\
x_{n+1} &= \Pi_{C_n+1} x_0, \quad n \geq 1,
\end{aligned} \tag{1.12}$$

where J is the duality mapping on E. Then, they proved that $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega x_0$, where $\Omega = \mathrm{EP}(F) \cap F(T) \neq \emptyset$.

Recently, Li et al. [38] introduced the following hybrid iterative scheme for approximation of fixed points of a relatively nonexpansive mapping using the properties of generalized f-projection operator in a uniformly smooth real Banach space which is also uniformly convex: $x_0 \in C$,

$$\begin{aligned}
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n),\\
C_{n+1} &= \{w \in C_n : G(w, J y_n) \leq G(w, J x_n)\},\\
x_{n+1} &= \Pi^f_{C_n+1} x_0, \quad n \geq 1.
\end{aligned} \tag{1.13}$$

They proved a strong convergence theorem for finding an element in the fixed points set of T. We remark here that the results of Li et al. [38] extended and improved on the results of Matsushita and Takahashi [9].

Quite recently, motivated by the results of Matsushita and Takahashi [9] and Plubtieng and Ungchittrakool [37], Su et al. [5] proved the following strong convergence theorem by hybrid iterative scheme for approximation of common fixed point of two countable families of weak relatively nonexpansive mappings in uniformly convex and uniformly smooth Banach space.

Theorem 1.5. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. Suppose $\{T_n\}_{n=1}^{\infty}$ and $\{S_n\}_{n=1}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := (\cap_{n=1}^{\infty} F(T_n)) \cap (\cap_{n=1}^{\infty} F(S_n)) \neq \emptyset$. Suppose that $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$,*

$$\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right),\\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n),\\
C_n &= \{w \in C_{n-1} \cap Q_{n-1} : \phi(w, y_n) \leq \phi(w, x_n)\},\\
C_0 &= \{w \in C : \phi(w, y_0) \leq \phi(w, x_0)\},\\
Q_n &= \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, J x_0 - J x_n\rangle \geq 0\},\\
Q_0 &= C,\\
x_{n+1} &= \Pi_{C_n \cap Q_n} x_0, \quad n \geq 1,
\end{aligned} \tag{1.14}$$

with the conditions

(i) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(3)} > 0$;

(iii) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$.

Then, $\{x_n\}_{n=0}^{\infty}$ *converges strongly to* $\Pi_\Omega x_0$.

Motivated by the above-mentioned results and the ongoing research, it is our purpose in this paper to prove a strong convergence theorem for two countable families of weak relatively nonexpansive mappings in a uniformly convex real Banach space which is also uniformly smooth using the properties of generalized f-projection operator. Our results extend the results of Li et al. [38], Su et al. [5] and many other recent known results in the literature.

2. Preliminaries

Let E be a real Banach space. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(t) := \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \le 1, \|y\| \le t\right\}. \tag{2.1}$$

E is uniformly smooth if and only if

$$\lim_{t\to 0}\frac{\rho_E(t)}{t} = 0. \tag{2.2}$$

Let $\dim E \ge 2$. The *modulus of convexity* of E is the function $\delta_E : (0,2] \to [0,1]$ defined by

$$\delta_E(\epsilon) := \inf\left\{1 - \left\|\frac{x+y}{2}\right\| : \|x\| = \|y\| = 1; \epsilon = \|x-y\|\right\}. \tag{2.3}$$

E is *uniformly convex* if for any $\epsilon \in (0,2]$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $x, y \in E$ with $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| \ge \epsilon$, then $\|(1/2)(x+y)\| \le 1-\delta$. Equivalently, E is uniformly convex if and only if $\delta_E(\epsilon) > 0$ for all $\epsilon \in (0,2]$. A normed space E is called *strictly convex* if for all $x, y \in E$, $x \ne y$, $\|x\| = \|y\| = 1$, we have $\|\lambda x + (1-\lambda)y\| < 1$, for all $\lambda \in (0,1)$.

Let E be a smooth, strictly convex and reflexive real Banach space and let C be a nonempty, closed and convex subset of E. Following Alber [39], the generalized projection Π_C from E onto C is defined by

$$\Pi_C x := \arg\min_{y\in C}\phi(y,x), \quad \forall x \in E. \tag{2.4}$$

The existence and uniqueness of Π_C follows from the property of the functional $\phi(x,y)$ and strict monotonicity of the mapping J (see, e.g., [3, 4, 39–41]). If E is a Hilbert space, then Π_C is the metric projection of H onto C.

Next, we recall the concept of generalized f-projector operator, together with its properties. Let $G : C \times E^* \to \mathbb{R} \cup \{+\infty\}$ be a functional defined as follows:

$$G(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi\rangle + \|\varphi\|^2 + 2\rho f(\xi), \tag{2.5}$$

where $\xi \in C$, $\varphi \in E^*$, ρ is a positive number and $f : C \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semi-continuous. From the definitions of G and f, it is easy to see the following properties:

(i) $G(\xi, \varphi)$ is convex and continuous with respect to φ when ξ is fixed;

(ii) $G(\xi, \varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed.

Definition 2.1 (Wu and Huang [42]). Let E be a real Banach space with its dual E^*. Let C be a nonempty, closed and convex subset of E. One says that $\Pi_C^f : E^* \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f \varphi = \left\{ u \in C : G(u, \varphi) = \inf_{\xi \in C} G(\xi, \varphi) \right\}, \quad \forall \varphi \in E^*. \tag{2.6}$$

For the generalized f-projection operator, Wu and Huang [42] proved the following theorem basic properties.

Lemma 2.2 (Wu and Huang [42]). *Let E be a real reflexive Banach space with its dual E^*. Let C be a nonempty, closed, and convex subset of E. Then the following statements hold:*

(i) *Π_C^f is a nonempty closed convex subset of C for all $\varphi \in E^*$;*

(ii) *if E is smooth, then for all $\varphi \in E^*$, $x \in \Pi_C^f$ if and only if*

$$\langle x - y, \varphi - Jx\rangle + \rho f(y) - \rho f(x) \geq 0, \quad \forall y \in C; \tag{2.7}$$

(iii) *if E is strictly convex and $f : C \to \mathbb{R} \cup \{+\infty\}$ is positive homogeneous (i.e., $f(tx) = tf(x)$ for all $t > 0$ such that $tx \in C$ where $x \in C$), then Π_C^f is a single valued mapping.*

Fan et al. [43] showed that the condition f is positive homogeneous which appeared in Lemma 2.2 and can be removed.

Lemma 2.3 (Fan et al. [43]). *Let E be a real reflexive Banach space with its dual E^* and C a nonempty, closed and convex subset of E. Then if E is strictly convex, then Π_C^f is a single-valued mapping.*

Recall that J is a single valued mapping when E is a smooth Banach space. There exists a unique element $\varphi \in E^*$ such that $\varphi = Jx$ for each $x \in E$. This substitution in (4.3) gives

$$G(\xi, Jx) = \|\xi\|^2 - 2\langle \xi, Jx\rangle + \|x\|^2 + 2\rho f(\xi). \tag{2.8}$$

Now, we consider the second generalized f-projection operator in a Banach space.

Definition 2.4. Let E be a real Banach space and C a nonempty, closed, and convex subset of E. One says that $\Pi_C^f : E \to 2^C$ is a generalized f-projection operator if

$$\Pi_C^f x = \left\{ u \in C : G(u, Jx) = \inf_{\xi \in C} G(\xi, Jx) \right\}, \quad \forall x \in E. \tag{2.9}$$

Obviously, the definition of $\{T_n\}_{n=0}^{\infty}$ is a countably family of weak relatively nonexpansive mappings is equivalent to

(R′1) $F(\{T_n\}_{n=0}^{\infty}) \neq \emptyset$;

(R′2) $G(p, JT_n x) \leq G(p, Jx)$, for all $x \in C,\ p \in F(T_n),\ n \geq 0$;

(R′3) $\cap_{n=0}^{\infty} F(T_n) = \widetilde{F}(\{T_n\}_{n=0}^{\infty})$.

Lemma 2.5 (Li et al. [38]). *Let E be a Banach space and $f : E \to \mathbb{R} \cup \{+\infty\}$ a lower semi-continuous convex functional. Then there exists $x^* \in E^*$ and $\alpha \in \mathbb{R}$ such that*

$$f(x) \geq \langle x, x^* \rangle + \alpha, \quad \forall x \in E. \tag{2.10}$$

We know that the following lemmas hold for operator Π_C^f.

Lemma 2.6 (Li et al. [38]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Then the following statements hold:*

(i) $\Pi_C^f x$ *is a nonempty closed and convex subset of C for all $x \in E$;*

(ii) *for all $x \in E$, $\widehat{x} \in \Pi_C^f x$ if and only if*

$$\left\langle \widehat{x} - y, Jx - J\widehat{x} + \rho f(y) - \rho f(x) \right\rangle \geq 0, \quad \forall y \in C; \tag{2.11}$$

(iii) *if E is strictly convex, then $\Pi_C^f x$ is a single-valued mapping.*

Lemma 2.7 (Li et al. [38]). *Let C be a nonempty, closed, and convex subset of a smooth and reflexive Banach space E. Let $x \in E$ and $\widehat{x} \in \Pi_C^f$. Then*

$$\phi(y, \widehat{x}) + G(\widehat{x}, Jx) \leq G(y, Jx), \quad \forall y \in C. \tag{2.12}$$

Lemma 2.8 (Su et al. [5]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex Banach space E. Let T be a weak relatively nonexpansive mapping of C into itself. Then $F(T)$ is closed and convex.*

Also, this following lemma will be used in the sequel.

Lemma 2.9 (Kamimura and Takahashi [4]). *Let C be a nonempty, closed, and convex subset of a smooth, uniformly convex Banach space E. Let $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ be sequences in E such that either $\{x_n\}_{n=1}^{\infty}$ or $\{y_n\}_{n=1}^{\infty}$ is bounded. If $\lim_{n\to\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\to\infty}\|x_n - y_n\| = 0$.*

Lemma 2.10 (Cho et al. [44]). *Let E be a uniformly convex real Banach space. For arbitrary $r > 0$, let $B_r(0) := \{x \in E : \|x\| \leq r\}$ and $\lambda, \mu, \gamma \in [0,1]$ such that $\lambda + \mu + \gamma = 1$. Then, there exists a continuous strictly increasing convex function*

$$g : [0, 2r] \longrightarrow \mathbb{R}, \quad g(0) = 0 \tag{2.13}$$

such that for every $x, y, z \in B_r(0)$, the following inequality holds:

$$\|\lambda x + \mu y + \gamma z\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 - \lambda\mu g(\|x - y\|). \tag{2.14}$$

For solving the equilibrium problem for a bifunction $F : C \times C \rightarrow \mathbb{R}$, let us assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 2.11 (Liu et al. [14] and Zhang [19]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex and reflexive Banach space E. Assume that $F : C \times C \rightarrow \mathbb{R}$ satisfies (A1)–(A4), $A : C \rightarrow E^*$ a continuous and monotone mapping, and $\varphi : C \rightarrow \mathbb{R}$ a lower semicontinuous and convex functional. For $r > 0$ and $x \in E$, there exists $z \in C$ such that*

$$Q(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0, \quad \forall y \in C, \tag{2.15}$$

where $Q(z, y) = F(z, y) + \langle Az, y - z\rangle + \varphi(y) - \varphi(z)$, $z, y \in C$. Furthermore, define a mapping $T_r : E \rightarrow C$ as follows:

$$T_r(x) = \left\{ z \in C : Q(z, y) + \frac{1}{r}\langle y - z, Jz - Jx\rangle \geq 0,\ \forall y \in C \right\}. \tag{2.16}$$

Then, the following hold:

(i) *T_r is single-valued;*

(ii) *for any $x, y \in E$,*

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle; \tag{2.17}$$

(iii) *$F(T_r) = \mathrm{GMEP}(F, A, \varphi)$;*

(iv) *$\mathrm{GMEP}(F, A, \varphi)$ is closed and convex.*

Lemma 2.12 (Zhang [19]). *Let C be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E. Assume that $F : C \times C \to \mathbb{R}$ satisfies (A1)–(A4), and let $r > 0$. Then for each $x \in E$ and $q \in F(T_r)$,*

$$\phi(q, T_r x) + \phi(T_r x, x) \leq \phi(q, x). \tag{2.18}$$

For the rest of this paper, the sequence $\{x_n\}_{n=0}^{\infty}$ converges strongly to p shall be denoted by $x_n \to p$ as $n \to \infty$, and we shall assume that $\beta_n^{(1)}, \beta_n^{(2)}, \beta_n^{(1)} \in [0,1]$ such that $\beta_n^{(1)} + \beta_n^{(2)} + \beta_n^{(1)} = 1$, for all $n \geq 0$.

3. Main Results

Theorem 3.1. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed, and convex subset of E. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $C \times C$ satisfying (A1)–(A4), $A_k : C \to E^*$ a continuous and monotone mapping and $\varphi_k : C \to \mathbb{R}$ a lower semicontinuous and convex functional. Suppose that $\{T_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := \cap_{k=1}^{m} \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Let $f : E \to \mathbb{R}$ be a convex and lower semicontinuous mapping with $C \subset \mathrm{int}(D(f))$, and suppose that $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C$, $C_0 = C$,*

$$\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right), \\
y_n &= J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J z_n), \\
u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\
C_{n+1} &= \{w \in C_n : G(w, J u_n) \leq G(w, J x_n)\}, \\
x_{n+1} &= \Pi_{C_{n+1}}^{f} x_0, \quad n \geq 0,
\end{aligned} \tag{3.1}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iii) $0 \leq \alpha_n \leq \alpha < 1$ *for some* $\alpha \in (0,1)$;

(iv) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, \ldots, m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, \ldots, m)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{\Omega}^{f} x_0$.

Proof. By Lemma 2.8, we know that $(\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$ is closed and convex. We also know from Lemma 2.11(iv) that $\cap_{k=1}^{m} \mathrm{GMEP}(F_k, A_k, \varphi_k)$ is closed and convex. Hence, $\Omega = \cap_{k=1}^{m} \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n))$ is a nonempty, closed and convex subset of C. Consequently, $\Pi_{\Omega}^{f} x_0$ is well defined. We first show that C_n, for all $n \geq 0$ is closed and convex. It is obvious that $C_0 = C$ is closed and convex. Thus, we only need to show that C_n is closed and convex for each $n \geq 1$. Since $G(z, Ju_n) \leq G(z, Jx_n)$ is equivalent to

$$2(\langle z, Jx_n \rangle - \langle z, Ju_n \rangle) \leq \|x_n\|^2 - \|u_n\|^2, \tag{3.2}$$

this implies that C_{n+1} is closed and convex for all $n \geq 0$. This shows that $\Pi_{C_{n+1}}^{f} x_0$ is well defined for all $n \geq 0$. By taking $\theta_n^k = T_{r_k,n}^{Q_k} T_{r_{k-1},n}^{Q_{k-1}} \cdots T_{r_2,n}^{Q_2} T_{r_1,n}^{Q_1}$, $k = 1, 2, \ldots, m$ and $\theta_n^0 = I$ for all $n \geq 1$, we obtain $u_n = \theta_n^m y_n$. We next show that $\Omega \subset C_n$, for all $n \geq 0$. For $n = 0$, we have $\Omega \subset C = C_0$. Then for each $x^* \in \Omega$, we obtain

$$\begin{aligned}
G(x^*, Ju_n) &\leq G\big(x^*, J\theta_n^m y_n\big) \leq G\big(x^*, Jy_n\big) \\
&= G(x^*, (\alpha_n Jx_n + (1-\alpha_n) Jz_n)) \\
&= \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1-\alpha_n)\langle x^*, Jz_n \rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n) Jz_n\|^2 + 2\rho f(x^*) \\
&\leq \|x^*\|^2 - 2\alpha_n \langle x^*, Jx_n \rangle - 2(1-\alpha_n)\langle x^*, Jz_n \rangle \\
&\quad + \alpha_n \|Jx_n\|^2 + (1-\alpha_n)\|Jz_n\|^2 + 2\rho f(x^*) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n) G(x^*, Jz_n) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n) G\Big(x^*, \beta_n^{(1)} Jx_n + \beta_n^{(2)} JT_n x_n + \beta_n^{(3)} JS_n x_n\Big) \\
&\leq \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(\|x^*\|^2 - 2\beta_n^{(1)} \langle x^*, Jx_n \rangle - 2\beta_n^{(2)} \langle x^*, JT_n x_n \rangle \\
&\qquad\qquad - 2\beta_n^{(3)} \langle x^*, JS_n x_n \rangle \\
&\qquad\qquad + \beta_n^{(1)} \|x_n\|^2 + \beta_n^{(2)} \|T_n x_n\|^2 + \beta_n^{(3)} \|S_n x_n\|^2 + 2\rho f(x^*)\Big) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(\beta_n^{(1)} G(x^*, Jx_n) + \beta_n^{(2)} G(x^*, JT_n x_n) + \beta_n^{(3)} G(x^*, JS_n x_n)\Big) \\
&\leq G(x^*, Jx_n).
\end{aligned} \tag{3.3}$$

So, $x^* \in C_n$. This implies that $\emptyset \neq \Omega \subset C_n$, for all $n \geq 0$. It follows that $\{x_n\}_{n=0}^{\infty}$ is well defined for all $n \geq 0$.

We now show that $\lim_{n\to\infty} G(x_n, Jx_0)$ exists. Since $f : E \to \mathbb{R}$ is a convex and lower semi-continuous, applying Lemma 2.5, we see that there exists $u^* \in E^*$ and $\alpha \in \mathbb{R}$ such that

$$f(y) \geq \langle y, u^* \rangle + \alpha, \quad \forall y \in E. \tag{3.4}$$

It follows that

$$\begin{aligned} G(x_n, Jx_0) &= \|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 + 2\rho f(x_n) \\ &\geq \|x_n\|^2 - 2\langle x_n, Jx_0 \rangle + \|x_0\|^2 + 2\rho\langle x_n, u^* \rangle + 2\rho\alpha \\ &= \|x_n\|^2 - 2\langle x_n, Jx_0 - \rho u^* \rangle + \|x_0\|^2 + 2\rho\alpha \\ &\geq \|x_n\|^2 - 2\|x_n\| \left\| Jx_0 - \rho u^* \right\| + \|x_0\|^2 + 2\rho\alpha \\ &= \left(\|x_n\| - \left\| Jx_0 - \rho u^* \right\| \right)^2 + \|x_0\|^2 - \left\| Jx_0 - \rho u^* \right\|^2 + 2\rho\alpha. \end{aligned} \tag{3.5}$$

Since $x_n = \Pi_{C_n}^f x_0$, it follows from (3.5) that

$$G(x^*, Jx_0) \geq G(x_n, Jx_0) \geq \left(\|x_n\| - \left\| Jx_0 - \rho u^* \right\| \right)^2 + \|x_0\|^2 - \left\| Jx_0 - \rho u^* \right\|^2 + 2\rho\alpha \tag{3.6}$$

for each $x^* \in \Omega$. This implies that $\{x_n\}_{n=0}^\infty$ is bounded and so is $\{G(x_n, Jx_0)\}_{n=0}^\infty$. By the construction of C_n, we have that $C_m \subset C_n$ and $x_m = \Pi_{C_m}^f x_0 \in C_n$ for any positive integer $m \geq n$. It then follows from Lemma 2.7 that

$$\phi(x_m, x_n) + G(x_n, Jx_0) \leq G(x_m, Jx_0). \tag{3.7}$$

It is obvious that

$$\phi(x_m, x_n) \geq (\|x_m\| - \|x_n\|)^2 \geq 0. \tag{3.8}$$

In particular,

$$\begin{gathered} \phi(x_{n+1}, x_n) + G(x_n, Jx_0) \leq G(x_{n+1}, Jx_0), \\ \phi(x_{n+1}, x_n) \geq (\|x_{n+1}\| - \|x_n\|)^2 \geq 0, \end{gathered} \tag{3.9}$$

and so $\{G(x_n, Jx_0)\}_{n=0}^\infty$ is nondecreasing. It follows that the limit of $\{G(x_n, Jx_0)\}_{n=0}^\infty$ exists. By the fact that $C_m \subset C_n$ and $x_m = \Pi_{C_m}^f x_0 \in C_n$ for any positive integer $m \geq n$, we obtain

$$\phi(x_m, u_n) \leq \phi(x_m, x_n). \tag{3.10}$$

Now, (3.7) implies that

$$\phi(x_m, u_n) \leq \phi(x_m, x_n) \leq G(x_m, Jx_0) - G(x_n, Jx_0). \tag{3.11}$$

Taking the limit as $m, n \to \infty$ in (3.11), we obtain

$$\lim_{n\to\infty} \phi(x_m, x_n) = 0. \tag{3.12}$$

It then follows from Lemma 2.9 that $\|x_m - x_n\| \to 0$ as $m, n \to \infty$. Hence, $\{x_n\}_{n=0}^{\infty}$ is Cauchy. Since E is a Banach space and C is closed and convex, then there exists $p \in C$ such that $x_n \to p$ as $n \to \infty$.

Now since $\phi(x_m, x_n) \to 0$ as $m, n \to \infty$ we have in particular that $\phi(x_{n+1}, x_n) \to 0$ as $n \to \infty$ and this further implies that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Since $x_{n+1} = \Pi_{C_{n+1}} x_0 \in C_{n+1}$, we have

$$\phi(x_{n+1}, u_n) \leq \phi(x_{n+1}, x_n), \quad \forall n \geq 0. \tag{3.13}$$

Then, we obtain

$$\lim_{n\to\infty} \phi(x_{n+1}, u_n) = 0. \tag{3.14}$$

Since E is uniformly convex and smooth, we have from Lemma 2.9 that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty} \|x_{n+1} - u_n\|. \tag{3.15}$$

So,

$$\|x_n - u_n\| \leq \|x_{n+1} - x_n\| + \|x_{n+1} - u_n\|. \tag{3.16}$$

Hence,

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \tag{3.17}$$

Since J is uniformly norm-to-norm continuous on bounded sets and $\lim_{n\to\infty}\|x_n - u_n\| = 0$, we obtain

$$\lim_{n\to\infty} \|Jx_n - u_n\| = 0. \tag{3.18}$$

Since $\{x_n\}$ is bounded, so are $\{z_n\}$, $\{JT_nx_n\}$, and $\{JS_nx_n\}$. Let $r := \sup_{n\geq 0}\{\|x_n\|, \|T_nx_n\|, \|S_nx_n\|\}$. Then from Lemma 2.10, we have

$$\begin{aligned}
G(x^*, Ju_n) &\leq G\left(x^*, J\theta_n^m y_n\right) \leq G\left(x^*, Jy_n\right) \\
&= G(x^*, (\alpha_n Jx_n + (1-\alpha_n)Jz_n)) \\
&= \|x^*\|^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, Jz_n\rangle \\
&\quad + \|\alpha_n Jx_n + (1-\alpha_n)Jz_n\|^2 + 2\rho f(x^*) \\
&\leq \|x^*\|^2 - 2\alpha_n\langle x^*, Jx_n\rangle - 2(1-\alpha_n)\langle x^*, Jz_n\rangle \\
&\quad + \alpha_n\|Jx_n\|^2 + (1-\alpha_n)\|Jz_n\|^2 + 2\rho f(x^*) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)G(x^*, Jz_n) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)G\Big(x^*, \Big(\beta_n^{(1)}Jx_n + \beta_n^{(2)}JT_nx_n + \beta_n^{(3)}JS_nx_n\Big)\Big) \\
&\leq \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(\|x^*\|^2 - 2\beta_n^{(1)}\langle x^*, Jx_n\rangle - 2\beta_n^{(2)}\langle x^*, JT_nx_n\rangle \\
&\qquad\qquad - 2\beta_n^{(3)}\langle x^*, JS_nx_n\rangle + \beta_n^{(1)}\|x_n\|^2 + \beta_n^{(2)}\|T_nx_n\|^2 \\
&\qquad\qquad + \beta_n^{(3)}\|S_nx_n\|^2 - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|) + 2\rho f(x^*)\Big) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(\beta_n^{(1)}G(x^*, Jx_n) + \beta_n^{(2)}G(x^*, JT_nx_n) \\
&\qquad\qquad + \beta_n^{(3)}G(x^*, JS_nx_n) - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\Big) \\
&\leq \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(\beta_n^{(1)}G(x^*, Jx_n) + \beta_n^{(2)}G(x^*, Jx_n) \\
&\qquad\qquad + \beta_n^{(3)}G(x^*, Jx_n) - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\Big) \\
&= \alpha_n G(x^*, Jx_n) + (1-\alpha_n)\Big(G(x^*, Jx_n) - \beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|)\Big) \\
&\leq G(x^*, Jx_n) - (1-\alpha_n)\beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|).
\end{aligned} \tag{3.19}$$

It then follows that

$$(1-\alpha)\beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|) \leq (1-\alpha_n)\beta_n^{(1)}\beta_n^{(2)}g(\|Jx_n - JT_nx_n\|) \leq G(x^*, Jx_n) - G(x^*, Ju_n). \tag{3.20}$$

But

$$\begin{aligned}G(x^*, Jx_n) - G(x^*, Ju_n) &= \|x_n\|^2 - \|u_n\|^2 - 2\langle x^*, Jx_n - Ju_n\rangle \\ &\le \left|\|x_n\|^2 - \|u_n\|^2\right| + 2|\langle x^*, Jx_n - Ju_n\rangle| \\ &\le |\|x_n\| - \|u_n\||(\|x_n\| - \|u_n\|) + 2\|x^*\|\|Jx_n - Ju_n\| \\ &\le \|x_n - u_n\|(\|x_n\| + \|u_n\|) + 2\|x^*\|\|Jx_n - Ju_n\|.\end{aligned} \tag{3.21}$$

From $\lim_{n\to\infty}\|x_n - u_n\| = 0$ and $\lim_{n\to\infty}\|Jx_n - Ju_n\| = 0$, we obtain

$$G(x^*, Jx_n) - G(x^*, Ju_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.22}$$

Using the condition $\liminf_{n\to\infty}\beta_n^{(1)}\beta_n^{(2)} > 0$, we have

$$\lim_{n\to\infty} g(\|Jx_n - JT_nx_n\|) = 0. \tag{3.23}$$

By property of g, we have $\lim_{n\to\infty}\|Jx_n - JT_nx_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_n - T_nx_n\| = 0. \tag{3.24}$$

Similarly, we can show that

$$\lim_{n\to\infty}\|x_n - S_nx_n\| = 0. \tag{3.25}$$

Since $x_n \to p$ and $\{T_n\}$, $\{S_n\}$ are uniformly closed, we have $p \in (\cap_{n=0}^{\infty}F(T_n)) \cap (\cap_{n=0}^{\infty}F(S_n))$.

Next, we show that $p \in \cap_{k=1}^{m}\mathrm{GMEP}(F_k, A_k, \varphi_k)$. Now, by Lemma 2.12, we obtain

$$\begin{aligned}\phi(u_n, y_n) &= \phi(\theta_n^m y_n, y_n) \\ &\le \phi(x^*, y_n) - \phi(x^*, \theta_n^m y_n) \\ &\le \phi(x^*, x_n) - \phi(x^*, u_n) \longrightarrow 0, \quad n \longrightarrow \infty.\end{aligned} \tag{3.26}$$

Using Lemma 2.9, we have $\lim_{n\to\infty}\|u_n - y_n\| = 0$. Furthermore,

$$\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - y_n\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.27}$$

Since $x_n \to p$ as $n \to \infty$ and $\|x_n - y_n\| \to 0$ as $n \to \infty$, then $y_n \to p$ as $n \to \infty$. By the fact that $\theta_n^k,\ k = 1,2,\ldots,m$ is relatively nonexpansive and using Lemma 2.12 again, we have that

$$\begin{aligned}\phi\left(\theta_n^k y_n, y_n\right) &\le \phi(x^*, y_n) - \phi\left(x^*, \theta_n^k y_n\right)\\ &\le \phi(x^*, x_n) - \phi\left(x^*, \theta_n^k y_n\right).\end{aligned} \tag{3.28}$$

Observe that

$$\begin{aligned}\phi(x^*, u_n) &= \phi(x^*, \theta_n^m y_n)\\ &= \phi\left(x^*, T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{k,n}}^{Q_k} T_{r_{k-1,n}}^{Q_{k-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n\right)\\ &= \phi\left(x^*, T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots \theta_n^k y_n\right)\\ &\le \phi\left(x^*, \theta_n^k y_n\right).\end{aligned} \tag{3.29}$$

Using (3.29) in (3.28), we obtain

$$\phi\left(\theta_n^k y_n, y_n\right) \le \phi(x^*, x_n) - \phi(x^*, u_n) \longrightarrow 0, \quad n \longrightarrow \infty. \tag{3.30}$$

Then Lemma 2.9 implies that $\lim_{n\to\infty}\|y_n - \theta_n^k y_n\| = 0,\ k = 1,2,\ldots,m$. Now

$$\left\|p - \theta_n^k y_n\right\| \le \left\|y_n - \theta_n^k y_n\right\| + \|y_n - p\| \longrightarrow 0, \quad n \longrightarrow \infty,\ k = 1,2,\ldots,m. \tag{3.31}$$

Similarly, $\lim_{n\to\infty}\|p - \theta_n^{k-1} y_n\| = 0,\ k = 1,2,\ldots,m$. This further implies that

$$\lim_{n\to\infty}\left\|\theta_n^k y_n - \theta_n^{k-1} y_n\right\| = 0. \tag{3.32}$$

Also, since J is uniformly norm-to-norm continuous on bounded sets and using (3.32), we obtain

$$\lim_{n\to\infty}\left\|J\theta_n^k y_n - J\theta_n^{k-1} y_n\right\| = 0. \tag{3.33}$$

Since $\liminf_{n\to\infty} r_{k,n} > 0,\ (k = 1,2,\ldots,m)$,

$$\lim_{n\to\infty}\frac{\|J\theta_n^k y_n - J\theta_n^{k-1} y_n\|}{r_{k,n}} = 0. \tag{3.34}$$

By Lemma 2.11, we have that for each $k = 1, 2, \ldots, m$

$$Q_k\left(\theta_n^k y_n, y\right) + \frac{1}{r_{k,n}}\left\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n\right\rangle \geq 0, \quad \forall y \in C. \tag{3.35}$$

Furthermore, using (A2) we obtain

$$\frac{1}{r_{k,n}}\left\langle y - \theta_n^k y_n, J\theta_n^k y_n - J\theta_n^{k-1} y_n\right\rangle \geq Q_k\left(y, \theta_n^k y_n\right). \tag{3.36}$$

By (A4), (3.34), and $\theta_n^k y_n \to p$, we have for each $k = 1, 2, \ldots, m$

$$Q_k(y, p) \leq 0, \quad \forall y \in C. \tag{3.37}$$

For fixed $y \in C$, let $z_{t,y} := ty + (1-t)p$ for all $t \in (0, 1]$. This implies that $z_t \in C$. This yields that $Q_k(z_t, p) \leq 0$. It follows from (A1) and (A4) that

$$\begin{aligned} 0 &= Q_k(z_t, z_t) \leq tQ_k(z_t, y) + (1-t)Q_k(z_t, p) \\ &\leq tQ_k(z_t, y) \end{aligned} \tag{3.38}$$

and hence

$$0 \leq Q_k(z_t, y). \tag{3.39}$$

From condition (A3), we obtain

$$Q_k(p, y) \geq 0, \quad \forall y \in C. \tag{3.40}$$

This implies that $p \in \mathrm{GMEP}(F_k, A_k, \varphi_k)$, $k = 1, 2, \ldots, m$. Thus, $p \in \cap_{k=1}^m \mathrm{GMEP}(F_k, A_k, \varphi_k)$. Hence, we have $p \in \Omega = \cap_{k=1}^m \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^\infty F(T_n)) \cap (\cap_{n=0}^\infty F(S_n))$.

Finally, we show that $p = \Pi_\Omega^f x_0$. Since $\Omega = \cap_{k=1}^m \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^\infty F(T_n)) \cap (\cap_{n=0}^\infty F(S_n))$ is a closed and convex set, from Lemma 2.6, we know that $\Pi_\Omega^f x_0$ is single valued and denote $w = \Pi_\Omega^f x_0$. Since $x_n = \Pi_{C_n}^f x_0$ and $w \in \Omega \subset C_n$, we have

$$G(x_n, Jx_0) \leq G(w, Jx_0), \quad \forall n \geq 0. \tag{3.41}$$

We know that $G(\xi, J\varphi)$ is convex and lower semi-continuous with respect to ξ when φ is fixed. This implies that

$$G(p, Jx_0) \leq \liminf_{n\to\infty} G(x_n, Jx_0) \leq \limsup_{n\to\infty} G(x_n, Jx_0) \leq G(w, Jx_0). \tag{3.42}$$

From the definition of $\Pi_\Omega^f x_0$ and $p \in \Omega$, we see that $p = w$. This completes the proof. $\square$

Corollary 3.2 (Li et al. [38]). *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed, and convex subset of E. Suppose that $T : C \to C$ is a relatively nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and $f : E \to \mathbb{R}$ is a convex and lower semicontinuous mapping with $C \subset \operatorname{int}(D(f))$. Suppose that $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C,\ C_0 = C$*

$$\begin{aligned} y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J T x_n), \\ C_{n+1} &= \{w \in C_n : G(w, J y_n) \le G(w, J x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}}^{f} x_0, \quad n \ge 0, \end{aligned} \tag{3.43}$$

where J is the duality mapping on E. Suppose that $\{\alpha_n\}_{n=0}^{\infty}$ is a sequence in $(0,1)$ such that $\limsup_{n\to\infty} \alpha_n < 1$. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{F(T)}^{f} x_0$.

Corollary 3.3. *Let E be a uniformly convex real Banach space which is also uniformly smooth. Let C be a nonempty, closed and convex subset of E. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $C \times C$ satisfying (A1)–(A4), $A_k : C \to E^*$ a continuous and monotone mapping and $\varphi_k : C \to \mathbb{R}$ a lower semicontinuous and convex functional. Suppose $\{T_n\}_{n=0}^{\infty}$ and $\{S_n\}_{n=0}^{\infty}$ are two countable families of weak relatively nonexpansive mappings of C into itself such that $\Omega := \cap_{k=1}^{m} \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap (\cap_{n=0}^{\infty} F(T_n)) \cap (\cap_{n=0}^{\infty} F(S_n)) \neq \emptyset$. Suppose that $\{x_n\}_{n=0}^{\infty}$ is iteratively generated by $x_0 \in C,\ C_0 = C$*

$$\begin{aligned} z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_n x_n + \beta_n^{(3)} J S_n x_n\right), \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : \phi(w, u_n) \le \phi(w, x_n)\}, \\ x_{n+1} &= \Pi_{C_{n+1}} x_0, \quad n \ge 0, \end{aligned} \tag{3.44}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iii) $0 \le \alpha_n \le \alpha < 1$ *for some* $\alpha \in (0,1)$;

(iv) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, \ldots, m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, \ldots, m)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega x_0$.

Proof. Take $f(x) = 0$ for all $x \in E$ in Theorem 3.1, $G(\xi, Jx) = \phi(\xi, x)$ and $\Pi_C^f x_0 = \Pi_C x_0$. Then, the desired conclusion follows. □

Remark 3.4. Corollary 3.3 extends and improves on Theorem 1.5. In fact, the iterative procedure (3.44) is simpler than (1.14) in the following two aspects: (a) the process of computing $Q_n = \{w \in C_{n-1} \cap Q_{n-1} : \langle x_n - w, Jx_0 - Jx_n \rangle \ge 0\}$ is removed; (b) the process of computing $\Pi_{C_n \cap Q_n}$ is replaced by computing Π_{C_n}.

4. Applications

A mapping H from E to E^* is said to be

(i) monotone if $\langle Hx - Hy, x - y\rangle \geq 0$, for all $x, y \in E$;

(ii) strictly monotone if H is monotone and $\langle Hx - Hy, x - y\rangle = 0$ if and only if $x = y$;

(iii) β-Lipschitz continuous if there exists a constant $\beta \geq 0$ such that $\|Hx - Hy\| \leq \beta\|x - y\|$, for all $x, y \in E$.

Let M be a set-valued mapping from E to E^* with domain $D(M) = \{z \in E : Mz \neq \emptyset\}$ and range $R(M) = \cup\{Mz : z \in D(M)\}$. A set-valued mapping M is said to be

(i) monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(M)$ and $y_i \in Mx_i$, $i = 1, 2$;

(ii) r-strongly monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq r\|x_1 - x_2\|^2$ for each $x_i \in D(M)$ and $y_i \in Mx_i$, $i = 1, 2$;

(iii) maximal monotone if M is monotone and its graph $G(M) := \{(x, y) : y \in Mx\}$ is not properly contained in the graph of any other monotone operator;

(iv) a general H-monotone if M is monotone and $(H+\lambda M)E = E^*$ holds for every $\lambda > 0$, where H is a mapping from E to E^*.

We denote the set $\{x \in E : 0 \in Mx\}$ by $M^{-1}0$. From Li et al. [38], we know that if $H : E \to E^*$ is strictly monotone and $M : E \to 2^{E^*}$ is general H-monotone mapping, then $M^{-1}0$ is closed and convex. Furthermore, for every $\lambda > 0$ and $x^* \in E^*$, there exists a unique $x \in D(M)$ such that $x = (H + \lambda M)^{-1}x^*$. Thus, we can define a single-value mapping $T_\lambda : E \to D(M)$ by $T_\lambda x = (H + \lambda M)^{-1}Hx$. It is obvious that $M^{-1}0 = F(T_\lambda)$ for all $\lambda > 0$.

Lemma 4.1 (Alber, [39]). *If E is a uniformly convex and uniformly smooth Banach space, $\delta_E(\epsilon)$ is the modulus of convexity of E, and $\rho_E(t)$ is the modulus of smoothness of E, then the inequalities*

$$8d^2\delta_E\left(\frac{\|x - \xi\|}{4d}\right) \leq \phi(x, \xi) \leq 4d^2\rho_E\left(\frac{4\|x - \xi\|}{d}\right) \tag{4.1}$$

hold for all x and ξ in E, where $d = \sqrt{(\|x\|^2 + \|\xi\|^2)/2}$.

Lemma 4.2 (Xia and Huang [45]). *Let E be a Banach space with dual space E^*, $H : E \to E^*$ a strictly monotone mapping, and $M : E \to 2^{E^*}$ a general H-monotone mapping. Then*

(i) *$(H + \lambda M)^{-1}$ is a single-valued mapping;*

(ii) *if E is reflexive and $M : E \to 2^{E^*}$ is r-strongly monotone, $(H + \lambda M)^{-1}$ is Lipschitz continuous with constant $1/\lambda r$, where $r > 0$.*

Theorem 4.3. *Let E be a uniformly convex real Banach space which is also uniformly smooth with $\delta_E(\epsilon) \geq k\epsilon^2$ and $\rho_E(t) \leq ct^2$ for some $k, c > 0$. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $E \times E$ satisfying (A1)–(A4), $A_k : E \to E^*$ a continuous and monotone mapping, and $\varphi_k : E \to \mathbb{R}$ a lower semicontinuous and convex functional. Suppose that $H : E \to E^*$ is a strictly monotone and β-Lipschitz continuous mapping and $M_i : E \to 2^{E^*}$ is a general H-monotone mapping and r_i-strongly monotone mapping with $r_i > 0$, $i = 1, 2$ such that $\Omega := \cap_{k=1}^m \mathrm{GMEP}(F_k, A_k, \varphi_k) \cap M_1^{-1}0 \cap M_2^{-1}0 \neq \emptyset$. Let $T_\lambda^{M_i} = (H + \lambda M_i)^{-1}H$, $i = 1, 2$, let $f : E \to \mathbb{R}$ a convex and lower semicontinuous* *mapping with $D(f) = E$ and suppose for each $n \geq 0$ that there exists a $\lambda_n > 0$ such that $64c\beta^2 \leq \min\{(1/2)k\lambda_n^2 r_1^2, (1/2)k\lambda_n^2 r_2^2\}$. Let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in E,\ C_0 = E$,*

$$
\begin{aligned}
z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J T_{\lambda_n}^{M_1} x_n + \beta_n^{(3)} J T_{\lambda_n}^{M_2} x_n\right), \\
y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\
u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\
C_{n+1} &= \{w \in C_n : G(w, J y_n) \leq G(w, J x_n)\}, \\
x_{n+1} &= \Pi_{C_{n+1}}^{f} x_0, \quad n \geq 0,
\end{aligned}
\tag{4.2}
$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iii) $0 \leq \alpha_n \leq \alpha < 1$;

(iv) $\liminf_{n\to\infty} \lambda_n > 0$;

(v) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0,\infty),\ (k = 1, 2, \ldots, m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, \ldots, m)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_{\Omega}^{f} x_0$.

Proof. We only need to prove that $\{T_{\lambda_n}^{M_1}\}$ and $\{T_{\lambda_n}^{M_2}\}$ are countable families of weak relatively nonexpansive mappings with common fixed points sets $\cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$ and $\cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_2}) = M_2^{-1}0$, respectively. Firstly, we have $\cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_1}) = M_1^{-1}0 \neq \emptyset$. Secondly, we show that $\phi(p, T_{\lambda_n}^{M_1} w) \leq \phi(p, w)$, for all $w \in E,\ p \in F(T_{\lambda_n}^{M_1}),\ n \geq 0$. Now, by Lemma 4.2 and the Lipschitz continuity of H, we have

$$
\begin{aligned}
\left\| T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w \right\| &= \left\| (H + \lambda_n M_1)^{-1} H p - (H + \lambda_n M_1)^{-1} H w \right\| \\
&\leq \frac{1}{\lambda_n r_1} \| H p - H w \| \\
&\leq \frac{\beta}{\lambda_n r_1} \| p - w \|.
\end{aligned}
\tag{4.3}
$$

By (4.3) and Lemma 4.1,

$$\begin{aligned}
\phi\left(p, T_{\lambda_n}^{M_1} w\right) &= \phi\left(T_{\lambda_n}^{M_1} p, T_{\lambda_n}^{M_1} w\right) \\
&\leq 4d^2 \rho_E \left(\frac{4\left\|T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w\right\|}{d}\right) \\
&\leq 64c \left\|T_{\lambda_n}^{M_1} p - T_{\lambda_n}^{M_1} w\right\|^2 \\
&\leq \frac{64c\beta^2}{\lambda_n^2 r_1^2} \|p - w\|^2, \\
\phi(p, w) &\geq 8d^2 \delta_E \left(\frac{\|p - w\|}{4d}\right) \geq \frac{1}{2} k \|p - w\|^2.
\end{aligned} \tag{4.4}$$

Since $64c\beta^2 \leq (1/2)k\lambda_n^2 r_1^2$, it follows from (4.4) that $\phi(p, T_{\lambda_n}^{M_1} w) \leq \phi(p, w)$, for all $w \in E$, $p \in F(T_{\lambda_n}^{M_1})$, $n \geq 0$. Thirdly, we show that $\widetilde{F}(\{T_{\lambda_n}^{M_1}\}) = \cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$. We first show that $\widetilde{F}(\{T_{\lambda_n}^{M_1}\}) \subset M_1^{-1}0$. Let $p \in \widetilde{F}(\{T_{\lambda_n}^{M_1}\})$, then there exists $\{x_n\} \subset E$ such that $x_n \to p$ and $\lim_{n\to\infty} \|x_n - T_{\lambda_n}^{M_1} x_n\| = 0$. Since H is β-Lipschitz continuous,

$$\left\|Hx_n - HT_{\lambda_n}^{M_1} x_n\right\| \leq \beta \left\|x_n - T_{\lambda_n}^{M_1} x_n\right\|. \tag{4.5}$$

Letting $n \to \infty$, we obtain

$$\frac{1}{\lambda_n}\left(Hx_n - HT_{\lambda_n}^{M_1} x_n\right) \longrightarrow 0. \tag{4.6}$$

It follows from $(1/\lambda_n)(Hx_n - HT_{\lambda_n}^{M_1} x_n) \in M_1 T_{\lambda_n}^{M_1}$ and the monotonicity of M_1 that

$$\left\langle x - T_{\lambda_n}^{M_1}, x^* - \frac{1}{\lambda_n}\left(Hx_n - HT_{\lambda_n}^{M_1} x_n\right)\right\rangle \geq 0 \tag{4.7}$$

for all $\in D(M_1)$ and $x^* \in M_1 x$. Taking the limit as $n \to \infty$, we obtain

$$\langle x - p, x^* \rangle \geq 0 \tag{4.8}$$

for all $\in D(M_1)$ and $x^* \in M_1 x$. By the maximality of M_1, we know that $p \in M_1^{-1}0$. On the other hand, we know that $F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$, $F(T_{\lambda_n}^{M_1}) \subset \widetilde{F}(T_{\lambda_n}^{M_1})$ for all $n \geq 0$, therefore, $M_1^{-1}0 = \cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_1}) = \widetilde{F}(\cap_{n=0}^{\infty} T_{\lambda_n}^{M_1})$. Thus, we have proved that $\{T_{\lambda_n}^{M_1}\}$ is a countable family of weak relatively nonexpansive mappings with common fixed points sets $\cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_1}) = M_1^{-1}0$. By following the same arguments, we can show that $\{T_{\lambda_n}^{M_2}\}$ is a countable family of weak relatively nonexpansive mappings with common fixed points sets $\cap_{n=0}^{\infty} F(T_{\lambda_n}^{M_2}) = M_2^{-1}0$. $\square$

Let E be a uniformly convex and uniformly smooth Banach space, $H = J$ and M a maximal monotone mapping. Then, we can define $J_\lambda = (J+\lambda M)^{-1}J$ for all $\lambda > 0$. We know that J_λ is relatively nonexpansive and therefore weak relatively nonexpansive and $M^{-1}0 = F(J_\lambda)$ for all $\lambda > 0$ (see, e.g., [2]), where $F(J_\lambda)$ denotes the fixed points set of J_λ. By Corollary 3.3, we obtain the following theorem.

Theorem 4.4. *Let E be a uniformly convex real Banach space which is also uniformly smooth. For each $k = 1, 2, \ldots, m$, let F_k be a bifunction from $E \times E$ satisfying (A1)–(A4), $A_k : E \to E^*$ a continuous and monotone mapping, and $\varphi_k : E \to \mathbb{R}$ a lower semicontinuous and convex functional. For each $i = 1, 2$, let $M_i \subset E \times E^*$ a maximal monotone operator, and let $J_\lambda^{M_i} = (J + \lambda M_i)^{-1}J$ for all $\lambda > 0$, and suppose C is a nonempty closed and convex subset of E such that $D(M_i) \subset C \subset J^{-1}(\cap_{\lambda>0}R(J+\lambda M_i))$, $i = 1, 2$. Assume that $\Omega := \cap_{k=1}^{m} GMEP(F_k, A_k, \varphi_k) \cap M_1^{-1}0 \cap M_2^{-1}0 \neq \emptyset$ and let $\{x_n\}_{n=0}^{\infty}$ be iteratively generated by $x_0 \in E$, $C_0 = E$,*

$$\begin{aligned} z_n &= J^{-1}\left(\beta_n^{(1)} J x_n + \beta_n^{(2)} J J T_{\lambda_n}^{M_1} x_n + \beta_n^{(3)} J J T_{\lambda_n}^{M_2} x_n\right), \\ y_n &= J^{-1}(\alpha_n J x_n + (1-\alpha_n) J z_n), \\ u_n &= T_{r_{m,n}}^{Q_m} T_{r_{m-1,n}}^{Q_{m-1}} \cdots T_{r_{2,n}}^{Q_2} T_{r_{1,n}}^{Q_1} y_n, \\ C_{n+1} &= \{w \in C_n : \phi(w, y_n) \le \phi(w, x_n)\}, \\ x_{n+1} &= \Pi_{C_n+1} x_0, \quad n \ge 0, \end{aligned} \tag{4.9}$$

with the conditions

(i) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(2)} > 0$;

(ii) $\liminf_{n\to\infty} \beta_n^{(1)} \beta_n^{(3)} > 0$;

(iii) $0 \le \alpha_n \le \alpha < 1$;

(iv) $\liminf_{n\to\infty} \lambda_n > 0$;

(v) $\{r_{k,n}\}_{n=1}^{\infty} \subset (0, \infty)$, $(k = 1, 2, \ldots, m)$ *satisfying* $\liminf_{n\to\infty} r_{k,n} > 0$, $(k = 1, 2, \ldots, m)$.

Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to $\Pi_\Omega x_0$.

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