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Existence of extended Nash equilibriums of nonmonetized noncooperative games

Chunxiang Zhao, Shihuang Hong* and Chunhong Li

*Correspondence:
hongshh@hotmail.com
Department of Mathematics,
Hangzhou Dianzi University, Xuelin
Street, Hangzhou, 310018, China

Abstract

In this paper, we consider extended Nash equilibriums of nonmonetized noncooperative games. By using a modified fixed point theorem of set-valued mappings on partially ordered sets, we prove an existence theorem of extended Nash equilibriums of the nonmonetized noncooperative game. Finally, an example is given to illustrate the advantages of our results.

MSC: 06A06; 47H10; 58J20; 91A06; 91A10

Keywords: partially ordered set; fixed point; nonmonetized noncooperative game; extended Nash equilibrium

1 Introduction

The generalized Nash equilibrium problem is an extension of the standard Nash equilibrium problem, in which the strategy set of each player depends on the strategies of all the other players as well as on his own. This problem has recently attracted much attention due to its applications in various fields like mathematics, management, economics and engineering. For details, we refer the reader to a recent survey paper by Facchinei and Kanzow [1] and the references therein. Very recently, a special class of generalized Nash equilibriums in which the noncooperative games possess strategies as subsets of vector spaces and the ranges of the payoff functions as partially ordered sets (posets) was examined by many researchers, for example, Carl *et al.* [2], Li *et al.* [3–5] and Xie *et al.* [6, 7]. Such games are said to be nonmonetized noncooperative (NNGs for short).

The existence of generalized Nash equilibriums has become the focus of research in NNGs. For instance, in [2, 4] the authors proved the existence via different fixed point theorems on Banach lattices. Since a Banach lattice is equipped with some metric topology and algebraic structures, the proofs of the existence follow the same idea as the traditional games, applying fixed point theorems in topological vector spaces. To avoid this, very recently, in [5] the authors obtained the existence results on lattices, which are equipped with neither a topological structure nor an algebraic structure, but only with a special partial order, *i.e.*, a lattice order. Moreover, in [6] the authors extended the concept of generalized Nash equilibrium to extended Nash equilibrium and proved an existence theorem of extended Nash equilibriums of the NNG by applying a fixed point theorem on posets.

Motivated by the works mentioned above, in this paper we also consider extended Nash equilibriums of the NNG and establish an existence result due to an improved fixed point

theorem corresponding to [2] for set-valued mappings on posets without the consideration of a topological structure nor an algebraic structure. We relax the assumptions of the order compactness and the chain-completeness and hence our result compares favorably with that of [6]. Finally, we will give an example to show the advantages of our results.

2 Preliminaries

In this section, we recall briefly some definitions and properties of posets. For more details, we refer to [2–6] and [8–10].

Let $\mathcal{P} = (\mathcal{P}, \preceq)$ be a poset with the partial order $\preceq$. An element $u \in \mathcal{P}$ is called an upper bound of a subset A of $\mathcal{P}$ if $x \preceq u$ for each $x \in A$. If $u \in A$, we say that u is the greatest element of A and denote $u = \max A$. The supremum of A, denoted by $\sup A$, is an element u_0 which is an upper bound of A and $u_0 \preceq v$ as long as v is another upper bound of A. It is clear that $\max A = \sup A$ if they exist. The maximal element of A is an element $y \in A$ which satisfies $y = z$ whenever $z \in A$ and $y \preceq z$. Obviously, if $u \in A$ is an upper bound of A, then u is a maximal element of A. The lower bound, the smallest element ($\min A$), the infimum ($\inf A$), and the minimal element of A can be similarly defined.

A subset A is called a chain if any two elements of A are comparable, *i.e.*, $x \preceq y$ or $y \preceq x$ for all $x, y \in A$. A is said to be a countable chain if A is a chain and is countable. By *c.c.* we denote a countable chain. For any $z, w \in \mathcal{P}$, we define the order intervals as follows:

$$[z) = \{x \in \mathcal{P} : z \preceq x\}, \qquad (w] = \{x \in \mathcal{P} : x \preceq w\} \quad \text{and}$$

$$[z, w] = [z) \cap (w] = \{x \in \mathcal{P} : z \preceq x \preceq w\}.$$

For given posets $(X, \preceq^X)$ and $(U, \preceq^U)$, a single-valued mapping F from $(X, \preceq^X)$ into $(U, \preceq^U)$ is said to be order increasing if $F(x) \preceq^U F(y)$ whenever $x \preceq^X y$.

In the following definition, we require some weaker conditions compared with Definitions 2.1 and 2.2 of [6].

Definition 2.1 A poset $(\mathcal{P}, \preceq)$ is said to be
  (i) inductive if every *c.c.* $C \subset \mathcal{P}$ has an upper bound in $\mathcal{P}$ and strongly inductive if $\sup C$ exists in $\mathcal{P}$ for every *c.c.* $C \subset \mathcal{P}$;
 (ii) inversely inductive if every *c.c.* in $\mathcal{P}$ has a lower bound in $\mathcal{P}$ and strongly inversely inductive whenever every *c.c.* $C \subset \mathcal{P}$ has the infimum in $\mathcal{P}$;
(iii) bi-inductive whenever it is both inductive and inversely inductive and strongly bi-inductive whenever it is both strongly inductive and strongly inversely inductive.

Lemma 2.2 *If $\mathcal{P}$ is a bi-inductive poset, then every subset $A \subset \mathcal{P}$ is also a bi-inductive poset.*

The following fixed point theorem on posets, which improves the corresponding result of [2], will play an essential role in our main results.

Theorem 2.3 *Let $(X, \preceq)$ be a poset. For a given nonempty subset $\mathcal{P}$ of X and a set-valued function $F : X \to 2^X$, we assume that*
  (H0) *there exist $u_0, v_0 \in \mathcal{P}$ with $u_0 \preceq v_0$ such that $F[X] = \bigcup_{x \in X} F(x) \subset [u_0, v_0]$;*
  (H1) *if $p \in \mathcal{P}$, then $\min F(p)$ and $\max F(p)$ exist and belong to $\mathcal{P}$. Moreover, $\min F(p)$ is a lower bound of $F[X \cap [p]]$ and $\max F(p)$ is an upper bound of $F[X \cap (p]]$;*

(H2) *the set $\{\max F(p) : p \in \mathcal{P}\}$ is a strongly inversely inductive subset of X and the set $\{\min F(p) : p \in \mathcal{P}\}$ is a strongly inductive subset of X.*

Then F has the greatest fixed point u^* and the smallest fixed point u_* in $\mathcal{P}$, that is, $u^* = \max \mathscr{F}(X)$ and $u_* = \min \mathscr{F}(X)$, where $\mathscr{F}(X) = \{x \in X : x \in F(x)\}$.

Proof Let $G(p) = \min F(p)$ for each $p \in \mathcal{P}$. By virtue of (H1), G is well defined and an increasing mapping from $\mathcal{P}$ into itself. From (H2) it follows that every *c.c.* of $G[\mathcal{P}]$ has a supremum in $\mathcal{P}$. Moreover, $u_0 \preceq G(u_0)$ with u_0 given in (H0). Hence, $G[[u_0)] \subset [u_0)$ by the fact that G is increasing.

In what follows, we prove that G has a fixed point in $[u_0)$. Let $u_{i+1} = G(u_i)$ for $i = 0, 1, \ldots$. Then $\{u_i\}$ is a *c.c.* and hence has a supremum by (H2). In addition, $x \preceq Gx$ for any $x \in \{u_i\}$. Let $u_0^1 = \sup\{u_i\}$ and $\mathcal{P}_1 = \{u_i\}_{i=0}^{\infty} \cup \{u_0^1\}$. If $u_0^1 = G(u_0^1)$, then G has a fixed point. Otherwise, take $u_{i+1}^1 = G(u_i^1)$ for $i = 0, 1, \ldots$. Again, the set $\{u_0^1, u_1^1, \ldots\}$ has the supremum $u_0^2 = \sup u_i^1$ by (H2). Denote $\mathcal{P}_2 = \{u_0^1, u_1^1, \ldots\} \cup \{u_0^2\}$. If $u_0^2 = G(u_0^2)$, then G has a fixed point. Otherwise, repeating this process, either G has a fixed point, or we can obtain a set sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots$ satisfying

(i) $\mathcal{P}_k = \{u_0^{k-1}, u_1^{k-1}, \ldots\} \cup \{u_0^k\}$ with $u_0^k = \sup\{u_i^{k-1}\}$ and $u_i^k = G(u_{i-1}^k)$ for $i, k = 1, 2, \ldots$;

(ii) $u_{i-1}^k \preceq u_i^k, u_j^{k-1} \preceq u_t^k$ and $u_i^0 = u_i$ for $i, k = 1, 2, \ldots$ and $j, t = 0, 1, \ldots$.

Let $\mathcal{Q} = \bigcup_{k=1}^{\infty} \mathcal{P}_k$. Then, like the proof of Lemma 2.5 in [11], we can verify that $\mathcal{Q}$ is a *c.c.*, $G(\mathcal{Q}) \subset \mathcal{Q} \subset \mathcal{P}$ and $x \preceq G(x)$ for all $x \in \mathcal{Q}$. (H2) shows that G has the greatest element $u_* = \max \mathcal{Q}$. From $G(\mathcal{Q}) \subset \mathcal{Q}$ it follows that $G(u_*) \preceq u_*$, while $u_* \in \mathcal{Q}$ implies $u_* \preceq G(u_*)$. Consequently, u_* is a fixed point of G in $[u_0)$ satisfying

$$u_* = \max \mathcal{Q} = \sup G(\mathcal{Q}) = \min\{u \in [u_0) : G(u) \preceq u\},$$

where the last equation is an immediate consequence of Proposition 2.14 in [2].

Now we prove that u_* is a lower bound of $\mathscr{F}(X)$. Suppose that this is not true. Then there exists a point $x \in \mathscr{F}(X)$ such that $u_* \not\preceq x$. Note that $x \in F(x)$, we have $\min \mathcal{Q} = u_0 \preceq x$ by (H0). On the other hand, for any $y \in \mathcal{Q}$ and $y \preceq x$, in view of the condition (H1) and the definition of G, we see that $G(y)$ is a lower bound of $F(x)$. This implies that $G(y) \preceq x$. Now the composition of $\mathcal{Q}$, combining with $u_0 \in \mathcal{Q}$, guarantees that $u_* \preceq x$, which contradicts the choice of x. Consequently, u_* is a lower bound of $\mathscr{F}(X)$.

Finally, we observe that $u_* = G(u_*) = \min F(u_*) \in F(u_*)$, which implies that $u_* \in \mathscr{F}(X)$ and hence $u_* = \min \mathscr{F}(X)$. Moreover, $u_* = \min F(u_*) \in \mathcal{P}$ by (H1).

Similar to the above discussion, we can prove the existence of the greatest fixed point u^* of F. This proof is complete. □

3 Main results

Definition 3.1 [6] Let n be a positive integer greater than 1. An n-person NNG consists of the following elements:

(1) the set of n players, which is denoted by $N = \{1, 2, \ldots, n\}$;

(2) the collection of n strategy sets $S = \{S_1, S_2, \ldots, S_n\}$, where S_i stands for the strategy set of player i for $i \in N$, which is also written as $S = S_1 \times S_2 \times \cdots \times S_n$;

(3) the set of n payoff functions $P = \{P_1, P_2, \ldots, P_n\}$, where P_i, a mapping from $S_1 \times S_2 \times \cdots \times S_n$ into the poset $(U, \preceq^U)$, is the payoff function of player i for $i \in N$;

(4) the outcome space $(U, \preceq^U)$, which is a poset.

This game is denoted by $\Gamma = \{N, S, P, U\}$.

In an n-person NNG $\Gamma = \{N, S, P, U\}$, when all the players simultaneously and independently choose their own strategies $x_1, x_2, \ldots, x_n$ to play, where $x_i \in S_i$ for $i \in N$, then the player i will receive his payoff $P_i(x_1, x_2, \ldots, x_n) \in U$. For every given $i \in N$ and any $x = (x_1, x_2, \ldots, x_n) \in S$, we adopt the following notations for convenience:

$$-i = N \backslash \{i\}, \quad i = 1, 2, 3, \ldots, n,$$

$$x_{-i} = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),$$

$$S_{-i} = S_1 \times S_2 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n.$$

Then $x_{-i} \in S_{-i}$, and we can simply write x as $x = (x_i, x_{-i})$. Moreover, we define $P_i(S_i, x_{-i}) = \{P_i(t_i, x_{-i}) : t_i \in S_i\}$.

Definition 3.2 [6] In an n-person NNG $\Gamma = \{N, S, P, U\}$, a selection of strategies $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n) \in S_1 \times S_2 \times \cdots \times S_n$ is called an extended Nash equilibrium of this game if the order relation

$$P_i(\tilde{x}_i, \tilde{x}_{-i}) \nprec^U P_i(x_i, \tilde{x}_{-i}), \quad \forall x_i \in S_i,$$

holds for every $i \in N$.

Lemma 3.3 [6] *Let $(S_i, \preceq_i)$ be a poset for every $i \in N$. Let $S = S_1 \times S_2 \times \cdots \times S_n$ be the Cartesian product space of $S_1, S_2, \ldots, S_n$, and let $\preceq^S$ be the coordinate partial order on S induced by the partial order $\preceq_i$, that is, for any $x, y \in S$ with $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, we have*

$$x \preceq^S y \quad \text{if and only if} \quad x_i \preceq_i y_i, \quad \forall i \in N \quad \text{and}$$

$$x \prec^S y \quad \text{if and only if} \quad x_i \preceq_i y_i \text{ and } x_{i_0} \neq y_{i_0} \quad \text{for some } i_0 \in N.$$

Then $(S, \preceq^S)$ is a poset. Furthermore, if every $(S_i, \preceq_i)$ is (strongly) inductive, then $(S, \preceq^S)$ is also (strongly) inductive. If every $(S_i, \preceq_i)$ is (strongly) bi-inductive, then $(S, \preceq^S)$ is also (strongly) bi-inductive.

Let $\mathscr{P} = \{A : A \subset S \text{ is a c.c.}\}$. We introduce a partial order on $\mathscr{P}$ as follows: $A \preceq B$ if and only if $A \subseteq B$ and $A \prec B$ if and only if $A \preceq B$ but $A \neq B$ for all $A, B \in \mathscr{P}$.

Lemma 3.4 *$(\mathscr{P}, \preceq)$ has a maximal element and a minimal element.*

Proof On the contrary, suppose that, for each $A \in \mathscr{P}$, there exists at least an element $B \in \mathscr{P}$ such that $A \prec B$. Let $f(A) = B$. Then f is a mapping from $\mathscr{P}$ into itself and satisfies $A \prec f(A)$ for each $A \in \mathscr{P}$. We assert that every countable chain of $\mathscr{P}$ has a supremum in $\mathscr{P}$. In fact, if $\mathscr{C} = \{A_1, A_2, \ldots\}$ is a countable chain of $\mathscr{P}$, then A_k is a countable subset of S for $k = 1, 2, \ldots$. Let $A = \bigcup_{k=1}^{\infty} A_k$. It is easy to see that A belongs to $\mathscr{P}$ and is a supremum of $\mathscr{C}$. In the light of Lemma 2.5 in [11], f has a fixed point $A_0 = f(A_0) \in \mathscr{P}$. On the other hand, from the definition of f we have $A_0 \prec f(A_0)$. This is a contradiction. Therefore, $\mathscr{P}$ has a maximal element. Analogously, we can prove the existence of a minimal element. This proof is complete. □

In this sequel, the maximal element (resp. minimal element) of $\mathscr{P}$ is said to be a maximal *c.c.* (resp. minimal *c.c.*) of S.

Lemma 3.5 *If S is inductive (inversely inductive), then S has a maximal element s^0 (minimal element s_0). Moreover, $\sup \mathcal{P}^*$ exists and equals s^0 ($\inf \mathcal{P}_*$ exists and equals s_0) where $\mathcal{P}^*(\mathcal{P}_*)$ is a maximal c.c. (minimal c.c.) of S.*

Proof Lemma 3.4 guarantees the existence of the maximal *c.c.* $\mathcal{P}^*$ of S and the inductive hypothesis further guarantees the existence of the upper bound s^0 of $\mathcal{P}^*$. We first check that s^0 is a maximal element of S. To this end, we choose an element $x \in S$ with $s^0 \preceq^S x$. If $x \notin \mathcal{P}^*$, then $B = \mathcal{P}^* \cup \{x\}$ is also a *c.c.*, i.e., $B \in \mathscr{P}$. Obviously, $\mathcal{P}^* \prec B$, which is a contradiction since $\mathcal{P}^*$ is the maximal element of $\mathscr{P}$. Consequently, $x \in \mathcal{P}^*$. Since s^0 is the upper bound of $\mathcal{P}^*$, we have $x \preceq^S s^0$. Hence $x = s^0$ and this implies that s^0 is a maximal element of S. Note that $s^0 \in \mathcal{P}^*$, we see that s^0 is the supremum of $\mathcal{P}^*$. The proof for the existence of a minimal element of S is analogous. This proof is complete. □

We are in a position to state and prove the main result of this paper.

Theorem 3.6 *Let $\Gamma = \{N, S, P, U\}$ be an n-person NNG. If the following conditions hold*:

(I) *every player's strategy set $(S_i, \preceq_i)$ $(i \in N)$ is a strongly inductive and inversely inductive poset,*

(II) *every player's payoff function $P_i : S \to U$ $(i \in N)$ satisfies*

$$P_i(x) \preceq^U P_i(y) \quad \text{if and only if} \quad x \preceq^S y \quad \text{for any } x, y \in S \text{ and } i \in N,$$

then the game Γ has an extended Nash equilibrium. Furthermore, Γ has minimal and maximal extended Nash equilibriums.

Proof Lemma 3.4 guarantees that S has at least a maximal *c.c.* $\mathcal{P}$. Let $\mathcal{P} = \{\xi_1, \xi_2, \ldots\}$, where $\xi_k = (x_1^k, x_2^k, \ldots, x_n^k)$ and $\xi_j = (x_1^j, x_2^j, \ldots, x_n^j)$ are comparable with respect to $\preceq^S$ for $k, j = 1, 2, \ldots$. For any fixed $i \in N$, let $S_i^0 = \{x_i^k\}_{k=1}^{\infty}$. Then $S_i^0 \subset S_i$ and S_i^0 is obviously a *c.c.* of S_i for $i \in N$. Hence there exist an upper bound $\tilde{s}_i$ and a lower bound $\tilde{t}_i$ of S_i^0 since S_i is bi-inductive for $i \in N$. We will verify that S_i^0 is a maximal *c.c.* of S_i for $i \in N$. Suppose that this is not true. There exists some $i_0 \in N$ such that $S_{i_0}^0$ is not a maximal *c.c.* of S_{i_0}, i.e., there exists another *c.c.* A_{i_0} of S_{i_0} such that $S_{i_0}^0 \subset A_{i_0}$ and hence there exists a with $a \in A_{i_0}$ and $a \notin S_{i_0}^0$. We have three cases:

Case 1. $a \prec_{i_0} x_{i_0}^k$ for all $k = 1, 2, \ldots$;

Case 2. $x_{i_0}^k \prec_{i_0} a$ for all $k = 1, 2, \ldots$;

Case 3. there exists a positive integer k_1 such that $x_{i_0}^{k_1} \prec_{i_0} a \prec_{i_0} x_{i_0}^{k_1+1}$.

In case 1, let $\eta = (y^1, y^2, \ldots, y^n)$ with $y^i = \tilde{t}_i$ if $i \neq i_0$ and $y^{i_0} = a$. Then $\eta \notin \mathcal{P}$ and $\eta \prec^S \xi_k$ for $k = 1, 2, \ldots$. Thus $\{\eta\} \cup \mathcal{P}$ is a *c.c.* of S and $\mathcal{P} \subset \{\eta\} \cup \mathcal{P}$, which contradicts the maximality of $\mathcal{P}$. Assume that case 2 occurs. We take $y^i = \tilde{s}_i$ instead of $y^i = \tilde{t}_i$ if $i \neq i_0$ and $y^{i_0} = a$ for every component y^i of η given in case 1. Thus we have that $\eta \notin \mathcal{P}$, $\xi_k \prec^S \eta$, $\mathcal{P} \subset \{\eta\} \cup \mathcal{P}$ and $\{\eta\} \cup \mathcal{P}$ is a *c.c.* of S, which contradicts the maximality of $\mathcal{P}$ again. If case 3 occurs, then, for η given in case 1, instead of $y^i = \tilde{t}_i$, we choose $y^i = x_i^{k_1}$ for $i \neq i_0$ and $y^{i_0} = a$. Hence $\xi_{k_1} \prec^S \eta \prec^S \xi_{k_1+1}$. It is similarly able to get a contradiction. Consequently, S_i^0 is a maximal *c.c.* of S_i for $i \in N$.

From Lemma 3.5 it follows that the supremum s_i^0 and the infimum s_{0i} of S_i^0 exist for $i \in N$. We next claim that $s_i^0 = \max S_i^0$ and $s_{0i} = \min S_i^0$. In fact, if $s_i^0 \neq \max S_i^0$, then $s_i^0 \notin S_i^0$ and $S_i^0 \subset S_i^0 \cup \{s_i^0\}$ is a *c.c.* of S_i, which contradicts the maximality of S_i^0. Analogously, we can prove $s_{0i} = \min S_i^0$. Let $\xi^0 := (s_1^0, s_2^0, \ldots, s_n^0)$ and $\xi_0 := (s_{01}, s_{02}, \ldots, s_{0n})$. It is clear that $\xi^0, \xi_0 \in \mathcal{P}$. This further implies that $\xi^0 = \max \mathcal{P}$ and $\xi_0 = \min \mathcal{P}$. In addition, it is easy to see that, for any $x_{-i} \in S_{-i}$, $P_i(s_i^0, x_{-i})$ (resp. $P_i(s_{0i}, x_{-i})$) is a maximal element (resp. minimal element) of $P_i(S_i, x_{-i})$ with respect to the partial order $\preceq^U$.

For every fixed $i \in N$, we define a set-valued mapping $T_i : S \to 2^{S_i} \setminus \{\varnothing\}$ by

$$T_i(x) = \left\{ z_i \in S_i \cap [s_{0i}, s_i^0] : P_i(z_i, x_{-i}) \text{ is a maximal element of } P_i(S_i, x_{-i}) \right\}$$

for all $x = (x_1, x_2, \ldots, x_n) \in S$. We have revealed $s_i^0 \in T_i(x)$, which illustrates that $T_i(x)$ is nonempty for every $x \in S$. Apparently, $\min T_i(x)$ and $\max T_i(x)$ exist in $[s_{0i}, s_i^0]$ for every $x \in S$, moreover, $\max T_i(x) = s_i^0$. Let us define the set-valued mapping $T : S \to 2^S \setminus \{\varnothing\}$ as follows: for any $x \in S$

$$T(x) = \left\{ y = (y_1, y_2, \ldots, y_n) : y_i \in T_i(x), i \in N \right\}.$$

From the definition of T, it follows that $\max T(x) = \xi^0$ and $\min T(x) = (\min T_1(x), \min T_2(x), \ldots, \min T_n(x))$ for every $x \in S$. This implies that $T(S) \subseteq [\xi_0, \xi^0]$ and hence T satisfies the hypothesis (H0) of Theorem 2.3.

For any $\xi_k = (x_1^k, x_2^k, \ldots, x_n^k) \in \mathcal{P}$, we observe that $\max T(\xi_k) = \xi^0 = (s_1^0, s_2^0, \ldots, s_n^0)$ and $\min T(\xi_k) = (z_1', z_2', \ldots, z_n')$ with $z_i' = \min T_i(\xi_k)$ $(i \in N)$. Let $z' = (z_1', z_2', \ldots, z_n')$. Then $\min T(\xi_k) = z'$. Taking any $x'' \in S \cap (\xi_k]$ and $z_i'' \in T_i(x'')$ for $i \in N$, we have $z_i'' \preceq_i s_i^0 = \max T_i(\xi_k)$, i.e., $\max T_i(\xi_k)$ is an upper bound of $T_i(x'')$. The arbitrariness of x'' induces that $\max T_i(\xi_k)$ is also an upper bound of $T_i[S \cap (\xi_k]]$ and the arbitrariness of $i \in N$ induces that $\max T(\xi_k)$ is an upper bound of $T[S \cap (\xi_k]]$. Similarly, taking any $y'' \in S \cap [\xi_k)$, by the condition (II), we get $P_i(z_i', x_{-i}^k) \preceq^U P_i(z_i', y_{-i}'') \preceq^U P_i(z_i'', y_{-i}'')$, which implies that $\min T_i(\xi_k) = z_i' \preceq_i z_i''$. Hence $\min T_i(\xi_k)$ is a lower bound of $T_i(y'')$. The arbitrariness of y'' guarantees that $\min T_i(\xi_k)$ is a lower bound of $T_i[S \cap [\xi_k)]$ and the arbitrariness of $i \in N$ reduces that $\min T(\xi_k)$ is a lower bound of $T[S \cap [\xi_k)]$ once more. Consequently, T satisfies (H1) of Theorem 2.3.

Note that $\{\max T(\xi_k) : \xi_k \in \mathcal{P}\} = \{\xi^0\}$, Obviously, it is a strongly inversely inductive subset of S. Since S_i is a strongly inductive poset, Lemma 3.3 induces that S is also a strongly inductive poset. In the light of Lemma 2.2, the set $\{\min T(\xi_k) : \xi_k \in \mathcal{P}\} \subseteq S$ is a strongly inductive subset of S. Therefore, T satisfies (H2) of Theorem 2.3. As a conclusion of Theorem 2.3, T has the greatest fixed point x^* and the smallest fixed point x_* in $\mathcal{P}$.

Suppose that $\tilde{x}$ is a fixed point of T, that is, $\tilde{x}_i \in T_i(\tilde{x})$, which yields, for every fixed $i \in N$, $P_i(\tilde{x}_i, \tilde{x}_{-i})$ is a maximal element of $P_i(S_i, \tilde{x}_{-i})$. It is equivalent to

$$P_i(\tilde{x}_i, \tilde{x}_{-i}) \not\prec^U P_i(t_i, \tilde{x}_{-i}), \quad \forall t_i \in S_i$$

for every $i \in N$, which indicates that $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$ is an extended Nash equilibrium. Moreover, from Lemma 3.5, we see that x^* is a maximal extended Nash equilibrium and x_* is a minimal extended Nash equilibrium of this game. This completes the proof of the theorem. $\square$

Corollary 3.7 *Under the condition* (II) *of Theorem* 3.6, *if every player's strategy set* $(S_i, \preceq_i)$ *is a strongly inductive poset for* $i \in N$ *and his payoff function* P_i *furthermore satisfies*

(III) *for any fixed* $i \in N$ *and* $x_{-i} \in S_{-i}$, $P_i(S_i, x_{-i})$ *is an inversely inductive poset in* $(U, \preceq^U)$,

then the game Γ *has an extended Nash equilibrium. Furthermore,* Γ *has minimal and maximal extended Nash equilibriums.*

Proof In order to check that all conditions of Theorem 3.6 are satisfied, it suffices to check that S_i is inversely inductive for each $i \in N$. Let $C = \{x_k\}_{k=1}^{\infty}$ be an arbitrary *c.c.* of S_i. Then, for fixed $x_{-i} \in S_{-i}$, by the condition (II) $P_i(C, x_{-i})$ is also a *c.c.* of $P_i(S_i, x_{-i})$. By means of (III), there exists $y \in P_i(S_i, x_{-i})$ such that $y \preceq^U P_i(x_k, x_{-i})$ for each $k = 1, 2, \ldots$. We can choose $x \in S_i$ such that $y = P_i(x, x_{-i})$. By means of the condition (II), we have $x \preceq_i x_k$ for $k = 1, 2, \ldots$, that is, x is a lower bound of C. Consequently, S_i is inversely inductive. Now Theorem 3.6 guarantees the desired results. □

Remark 3.8 In addition to (II), all the rest of the hypotheses in Corollary 3.7, respectively, correspond to the conditions (i) and (2) in [6], Theorem 3.4. However, we not only use the weaker inductive poset concept but also reduce some tedious hypotheses.

4 Example

The purpose of this section is to show the advantages of our results by the following example.

Example 4.1 (Military manufacture example) Suppose that a war is kindling between two countries $C1$ and $C2$. To strengthen the combat effectiveness, they both intend to invest funds to acquire more weapons. Suppose that there are two military factories $F1$ and $F2$ offering the weapons to the two countries $C1$ and $C2$, respectively. As there is shortage of funds, each country can only invest 80 million dollars into its military factory. Suppose that the two factories just produce two weapons of the same type, tankers and fighter planes. The manufacturing cost of a tanker is 20 million dollars and a fighter plane will cost 30 million dollars. Suppose that the incomes are determined by the number of the two weapons that the factory can make. An arbitrarily considered outcome is a set of the total combat effectiveness of a factory. Let U be the collection of all possible outcomes. Assume that the combat effectiveness of a tanker and a fighter plane is incomparable. It is obvious that $(U, \preceq^U)$ is a poset.

From what has been described above, we easily get the feasible strategy set of Fi as $S_i = \{A_1 = (4, 0), A_2 = (3, 0), A_3 = (2, 1), A_4 = (1, 2), A_5 = (0, 2)\}$, where the first and second components of A_k with $k \in \{1, 2, \ldots, 5\}$ denote the number of tankers and the number of fighter planes produced by Fi, respectively. Let $P_i : S \to U$ be the payoff function of Fi, where, for any $x \in S$, $P_i(x)$ denotes the total combat effectiveness of the weapons produced by Fi. We are now in the position to find an extended Nash equilibrium of this NNG.

Proof For any $A_k, A_t \in S_i$ with $A_k = (a_1, a_2)$ and $A_t = (b_1, b_2)$, we define the partial order $\preceq_i$ on S_i as follows:

$$A_k \preceq_i A_t \quad \text{if and only if} \quad a_i \leq b_i, \quad \text{for any } k, t \in \{1, 2, \ldots, 5\} \text{ and } i \in \{1, 2\}.$$

Obviously, $(S_i, \preceq_i)$ is a poset. Let $S = S_1 \times S_2$. For any $x, y \in S$ with $x = (A_k, A_t)$ and $y = (A_m, A_n)$, $k, t, m, n \in \{1, 2, \ldots, 5\}$, the partial order $\preceq^S$ on S is induced by the partial order $\preceq_i$ as follows:

$$x \preceq^S y \quad \text{if and only if} \quad A_k \preceq_1 A_m \text{ and } A_t \preceq_2 A_n, \quad \text{for any } k, t, m, n \in \{1, 2, \ldots, 5\}.$$

Then $(S, \preceq^S)$ is a poset.

It is obvious that the strategy set $(S_i, \preceq_i)$ is a strongly inductive and inversely inductive poset. Therefore the condition (I) of Theorem 3.6 is satisfied. For any two strategies $x = (x_1, x_2), y = (y_1, y_2) \in S$ with $x \preceq^S y$, we have $x_i \preceq_i y_i$ for each $i \in \{1, 2\}$, which means that for each factory, both the number of tankers and the number of fighter planes produced in x are less than that in y. Hence, the total combat effectiveness of the weapons in x is weaker than that in y for each factory, that is, $P_i(x) \preceq^U P_i(y)$ for $i \in \{1, 2\}$. Then the sufficiency of (II) in Theorem 3.6 is satisfied. We now show that the game also meets the necessity of (II). For any given $P_i(x), P_i(y) \in U$ with $P_i(x) \preceq^U P_i(y)$, $i \in \{1, 2\}$, we are in the position to prove that $x \preceq^S y$. If x and y are incomparable, then the numbers of tankers and fighter planes in x will not simultaneously be less or more than that in y. This implies that neither $P_i(x) \preceq^U P_i(y)$ nor $P_i(y) \preceq^U P_i(x)$ since the combat effectiveness of a tanker and a fighter plane is incomparable. Therefore, x and y are comparable by hypothesis $P_i(x) \preceq^U P_i(y)$, that is, $x \preceq^S y$ or $y \preceq^S x$. By the sufficiency of (II), if $y \preceq^S x$, then $P_i(y) \preceq^U P_i(x)$, which is a contradiction. Consequently, $x \preceq^S y$. By virtue of Theorem 3.6, the game has maximal and minimal extended Nash equilibriums. In fact, it easily checks that the strategy $(A_1, A_1) \in S$ is not only a maximal extended Nash equilibrium but also a minimal extended Nash equilibrium of this game. □

Remark 4.2 It is easy to see that $(S_i, \preceq_i)$ in Example 4.1 has neither a sup-center nor an inf-center, which indicates that Theorem 3.4 of [6] cannot solve the problem above. Hence our theorem compares favorably with that of [6] in its particular way.

References
1. Facchinei, F, Kanzow, C: Generalized Nash equilibrium problems. Ann. Oper. Res. **175**, 177-211 (2010). doi:10.1007/s10479-009-0653-x
2. Carl, S, Heikkilä, S: Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory. Springer, New York (2010)
3. Li, JL: Several extensions of the Abian-Brown fixed theorem and their applications to extended and generalized Nash equilibria on chain-complete posets. J. Math. Anal. Appl. **409**, 1084-1092 (2014)
4. Li, JL: Applications of fixed point theory to generalized Nash equilibriums of nonmonetized noncooperative games on Banach lattices. Nonlinear Anal. Forum **18**, 1-11 (2013)
5. Li, JL, Park, S: Generalized Nash equilibria of nonmonetized noncooperative games on lattices. Br. J. Econ. Manag. Trade **4**(1), 97-110 (2014)
6. Xie, LS, Li, JL, Wen, CF: Applications of fixed point theory to extended Nash equilibriums of nonmonetized noncooperative games on posets. Fixed Point Theory Appl. **2013**, 235 (2013)
7. Xie, LS, Li, JL, Yang, WS: Order-clustered fixed point theorems on chain-complete preordered sets and their applications to extended and generalized Nash equilibria. Fixed Point Theory Appl. **2013**, 192 (2013)

8. Agarwal, R, Balaj, M, O'Regan, D: A unifying approach to variational relation problems. J. Optim. Theory Appl. **155**, 417-429 (2012). doi:10.1007/s10957-012-0090-x
9. Aliprantis, CD, Burkinshaw, O: Positive Operators. Springer, Berlin (2006)
10. Dunford, N, Schwartz, JT: Linear Operators. Part I: General Theory. Wiley-Interscience, New York (1988)
11. Hong, SH, Qiu, ZY: Fixed points of discontinuous multivalued operators in ordered spaces with applications. Fixed Point Theory Appl. **2010**, 745769 (2010). doi:10.1155/2010/745769