Zhai and Hao *Advances in Difference Equations* (2015) 2015:147
DOI 10.1186/s13662-015-0487-6

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Multi-point boundary value problems for a coupled system of nonlinear fractional differential equations

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Abstract

In this paper, we investigate the existence and uniqueness of solutions to the coupled system of nonlinear fractional differential equations

$$\begin{cases} -D_{0+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \\ -D_{0+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \end{cases}$$

where D_{0+}^{ν} is the standard Riemann-Liouville fractional derivative of order ν, $t \in (0,1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in \mathbf{N}$, and $\lambda_1, \lambda_2 > 0$, with the multi-point boundary value conditions: $y_1^{(i)}(0) = 0 = y_2^{(i)}(0)$, $0 \le i \le n-2$; $D_{0+}^{\beta} y_1(1) = \sum_{i=1}^{m-2} b_i D_{0+}^{\beta} y_1(\xi_i)$; $D_{0+}^{\beta} y_2(1) = \sum_{i=1}^{m-2} b_i D_{0+}^{\beta} y_2(\xi_i)$, where $n-2 < \beta < n-1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $b_i \ge 0$ $(i = 1, 2, \ldots, m-2)$ with $\rho_1 := \sum_{i=1}^{m-2} b_i \xi_i^{\nu_1 - \beta - 1} < 1$, and $\rho_2 := \sum_{i=1}^{m-2} b_i \xi_i^{\nu_2 - \beta - 1} < 1$. Our analysis relies on the Banach contraction principle and Krasnoselskii's fixed point theorem.

MSC: 26A33; 34B18; 34B27

Keywords: existence and uniqueness; solutions; Riemann-Liouville fractional derivative; multi-point boundary value problems; Banach contraction principle; Krasnoselskii's fixed point theorem

1 Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non-integer order. The first definition of fractional derivative was introduced at the end of the nineteenth century by Liouville and Riemann, but the concept of non-integer derivative and integral, as a generalization of the traditional integer order differential and integral calculus, was mentioned already in 1695 by Leibniz and L'Hospital. With the help of fractional calculus, we can describe natural phenomena and mathematical models more accurately. The fractional differential equations play an important role in various fields of engineering, physics, economics and biological sciences, *etc.* (see [1–4] for example). In consequence, the subject of fractional differential equations is gaining much importance and attention. For more details on basic theory of fractional differential equations, one can see the monographs of Diethelm [1], Kilbas *et al.* [2], Miller and Ross [3], Podlubny [4] and Tarasov [5], and the papers [6–13] and the references therein.

As is known to all, the initial and boundary value problems for nonlinear fractional differential equations arise from the study of models of control, porous media, electrochemistry, viscoelasticity, electromagnetics, *etc.* Recently, the existence and uniqueness of solutions of initial and boundary value problems for nonlinear fractional equations have been extensively studied (see [14–20]), and some are coupled systems of nonlinear fractional differential equations (see [8, 14, 17, 21, 22]).

In [15], Mophou studied the mild solutions to impulsive fractional differential equations

$$\begin{cases} D_t^\alpha x(t) = Ax(t) + f(t, x(t)), \quad t \in I = [0, T], t \neq t_k, \\ x(0) = x_0 \in X, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \quad k = 1, 2, \ldots, m, \end{cases}$$

where $0 < \alpha < 1$, the operator $A : D(A) \subset X \to X$ is a generator of C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X, D_t^α is the Caputo fractional derivative, $f : I \times X \to X$ is a given continuous function, $I_k : X \to X$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = x(t_k^+) - x(t_k^-)$. Some existence and uniqueness results for the equations were established by means of Krasnoselskii's fixed point theorem.

By using the same fixed point theorem, Goodrich [23] considered the existence of a positive solution to the following system of differential equations of fractional order:

$$\begin{cases} -D_{0^+}^{\nu_1} y_1(t) = \lambda_1 a_1(t) f(y_1(t), y_2(t)), \\ -D_{0^+}^{\nu_2} y_2(t) = \lambda_2 a_2(t) g(y_1(t), y_2(t)), \end{cases} \tag{1}$$

where $D_{0^+}^\nu$ is the standard Riemann-Liouville fractional derivative of order ν, $t \in (0, 1)$, $\nu_1, \nu_2 \in (n-1, n]$ for $n > 3$ and $n \in \mathbf{N}$, and $\lambda_1, \lambda_2 > 0$, with the following boundary value conditions:

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leq i \leq n-2,$$

$$\left[D_{0+}^\alpha y_1(t)\right]_{t=1} = 0 = \left[D_{0+}^\alpha y_2(t)\right]_{t=1}, \quad 1 \leq \alpha \leq n-2,$$

under the assumptions that a_1, a_2, f, g are nonnegative and continuous.

Very recently, Sun *et al.* [14] considered the coupled system of multi-term nonlinear fractional differential equations

$$\begin{cases} D^\alpha u(t) = f(t, v(t), D^{\beta_1} v(t), \ldots, D^{\beta_N} v(t)), \quad D^{\alpha-i} u(0) = 0, \quad i = 1, 2, \ldots, n_1, \\ D^\sigma v(t) = g(t, u(t), D^{\rho_1} u(t), \ldots, D^{\rho_N} u(t)), \quad D^{\sigma-j} v(0) = 0, \quad j = 1, 2, \ldots, n_2, \end{cases}$$

where $t \in (0, 1]$, $\alpha > \beta_1 > \beta_2 > \cdots > \beta_N > 0$, $\sigma > \rho_1 > \rho_2 \cdots > \rho_N > 0$, $n_1 = [\alpha] + 1$, $n_2 = [\sigma] + 1$, $\beta_q, \rho_q < 1$ and $q \in \{1, 2, \ldots, N\}$. By using the Schauder fixed point theorem and the Banach contraction principle, some results of existence and uniqueness of solutions for the coupled system are obtained.

However, to our knowledge, there are few works that deal with multi-point boundary value problems for a coupled system of nonlinear fractional differential equations. The purpose of this article is to investigate the solutions for the coupled system of nonlinear fractional differential equations (1) with the multi-point boundary conditions:

$$y_1^{(i)}(0) = 0 = y_2^{(i)}(0), \quad 0 \leq i \leq n-2, \tag{2}$$

$$D_{0^+}^{\beta} y_1(1) = \sum_{i=1}^{m-2} b_i D_{0^+}^{\beta} y_1(\xi_i), \tag{3}$$

$$D_{0^+}^{\beta} y_2(1) = \sum_{i=1}^{m-2} b_i D_{0^+}^{\beta} y_2(\xi_i), \tag{4}$$

where $n-2 < \beta < n-1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $b_i \geq 0$ $(i = 1, 2, \ldots, m-2)$ with $\rho_1 := \sum_{i=1}^{m-2} b_i \xi_i^{\nu_1-\beta-1} < 1$, and $\rho_2 := \sum_{i=1}^{m-2} b_i \xi_i^{\nu_2-\beta-1} < 1$.

Motivated by the above-mentioned works and recent works on coupled systems of fractional differential equations, we consider the existence and uniqueness of solutions of coupled system (1)-(4) by means of the Banach contraction principle and Krasnoselskii's fixed point theorem. In our paper, we do not suppose that a_1, a_2, f, g are nonnegative.

With this context in mind, the outline of this paper is as follows. In Section 2 we recall certain results from the theory of continuous fractional calculus. In Section 3 we provide some conditions under which problem (1)-(4) will have a unique solution or at least one solution.

2 Preliminaries

For the convenience of the reader, we present here some definitions, lemmas and basic results that will be used in the proofs of our theorems.

Definition 2.1 (see [4]) Let $\nu > 0$ with $\nu \in \mathbf{R}$. Suppose that $y : [a, +\infty) \to \mathbf{R}$. Then the νth Riemann-Liouville fractional integral is defined to be

$$D_{a^+}^{-\nu} y(t) := \frac{1}{\Gamma(\nu)} \int_a^t y(s)(t-s)^{\nu-1}\, ds,$$

whenever the right-hand side is defined. Similarly, with $\nu > 0$ and $\nu \in \mathbf{R}$, we define the νth Riemann-Liouville fractional derivative to be

$$D_{a^+}^{\nu} y(t) := \frac{1}{\Gamma(n-\nu)} \frac{d^n}{dt^n} \int_a^t \frac{y(s)}{(t-s)^{\nu+1-n}}\, ds,$$

where $n \in \mathbf{N}$ is the unique positive integer satisfying $n-1 \leq \nu < n$ and $t > a$.

Lemma 2.2 (see [23]) *Let $\alpha \in \mathbf{R}$. Then $D^n D_{a^+}^{\alpha} y(t) = D_{a^+}^{n+\alpha} y(t)$, for each $n \in N_0$, where $y(t)$ is assumed to be sufficiently regular so that both sides of the equality are well defined. Moreover, if $\beta \in (-\infty, 0]$ and $\gamma \in [0, +\infty)$, then $D_{a^+}^{\gamma} D_{a^+}^{\beta} y(t) = D_{a^+}^{\gamma+\beta} y(t)$.*

Lemma 2.3 (see [23]) *The general solution to $D_{a^+}^{\nu} y(t) = 0$, where $n-1 < \nu \leq n$ and $\nu > 0$, is the function $y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n}$, where $c_i \in \mathbf{R}$ for each i.*

Lemma 2.4 *Let $h \in C^n([0,1])$ be given. Then the unique solution to problem $-D_{0^+}^{\nu} y(t) = h(t)$ together with the boundary conditions $y^{(i)}(0) = 0$ and $D_{0^+}^{\beta} y(1) = \sum_{i=1}^{m-2} b_i D_{0^+}^{\beta} y(\xi_i)$, where $n-2 < \beta < n-1$ and $0 \leq i \leq n-2$, is*

$$y(t) = \int_0^1 G(t,s) h(s)\, ds, \tag{5}$$

where

$$G(t,s) = \begin{cases} -\frac{(t-s)^{\nu-1}}{\Gamma(\nu)} + \frac{t^{\nu-1}}{\Gamma(\nu)(1-\rho)}[(1-s)^{\nu-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i - s)^{\nu-\beta-1}], \\ \quad s \le t, \xi_{i-1} < s \le \xi_i, i = 1,2,\dots m-1, \\ \frac{t^{\nu-1}}{\Gamma(\nu)(1-\rho)}[(1-s)^{\nu-\beta-1} - \sum_{i=1}^{m-2} b_i(\xi_i - s)^{\nu-\beta-1}], \\ \quad t \le s, \xi_{i-1} < s \le \xi_i, i = 1,2,\dots m-1, \end{cases} \tag{6}$$

is the Green function for this problem, where $\rho = \sum_{i=1}^{m-2} b_i \xi_i^{\nu-\beta-1} < 1$ *and* $\xi_0 = 0$, $\xi_{m-1} = 1$.

Proof We know that the general solution to our problem is

$$y(t) = c_1 t^{\nu-1} + c_2 t^{\nu-2} + \cdots + c_n t^{\nu-n} - D_{0^+}^{-\nu} h(t),$$

we immediately observe that the boundary value condition $y^{(i)}(0) = 0, 0 \le i \le n-2$, implies that $c_2 = c_3 = \cdots = c_n = 0$. On the other hand, $D_{0^+}^{\beta} y(1) = \sum_{i=1}^{m-2} b_i D_{0^+}^{\beta} y(\xi_i)$ implies that

$$\begin{aligned} &-\frac{1}{\Gamma(\nu-\beta)} \int_0^1 (1-s)^{\nu-\beta-1} h(s)\,ds + c_1 \frac{\Gamma(\nu)}{\Gamma(\nu-\beta)} \\ &\quad = \sum_{i=1}^{m-2} b_i \biggl(-\frac{1}{\Gamma(\nu-\beta)} \int_0^{\xi_i} (\xi_i - s)^{\nu-\beta-1} h(s)\,ds + c_1 \frac{\Gamma(\nu)}{\Gamma(\nu-\beta)} \xi_i^{\nu-\beta-1} \biggr). \end{aligned}$$

That is to say,

$$c_1 = \frac{1}{\Gamma(\nu)(1-\rho)} \biggl[\int_0^1 (1-s)^{\nu-\beta-1} h(s)\,ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu-\beta-1} h(s)\,ds \biggr].$$

Therefore, the unique solution is

$$\begin{aligned} y(t) &= -\int_0^t \frac{(t-s)^{\nu-1}}{\Gamma(\nu)} h(s)\,ds \\ &\quad + \frac{t^{\nu-1}}{\Gamma(\nu)(1-\rho)} \biggl[\int_0^1 (1-s)^{\nu-\beta-1} h(s)\,ds - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu-\beta-1} h(s)\,ds \biggr] \\ &= \int_0^1 G(t,s) h(s)\,ds. \end{aligned}$$

The proof is complete. □

3 Main results

This section deals with the existence and uniqueness of solutions to problem (1)-(4).

Let E represent the Banach space of $C[0,1]$ when equipped with the usual supremum norm $\|\cdot\|$. Then put $X := E \times E$, where X is equipped with the norm

$$\bigl\|(y_1,y_2)\bigr\| := \|y_1\| + \|y_2\|$$

for $(y_1, y_2) \in X$. Observe that X is also a Banach space (see [24]). In addition, define two operators $T_1, T_2 : X \to E$ by

$$\big(T_1(y_1,y_2)\big)(t) := \lambda_1 \int_0^1 G_1(t,s)a_1(s)f\big(s,y_1(s),y_2(s)\big)\,ds$$

and

$$\big(T_2(y_1,y_2)\big)(t) := \lambda_2 \int_0^1 G_2(t,s)a_2(s)g\big(s,y_1(s),y_2(s)\big)\,ds,$$

where $G_1(t,s)$ is the Green function of Lemma 2.4 with ν replaced by ν_1 and ρ replaced by ρ_1, and likewise, $G_2(t,s)$ is the Green function of Lemma 2.4 with ν replaced by ν_2 and ρ replaced by ρ_2. Now, we define an operator $S : X \to X$ by

$$\begin{aligned}\big(S(y_1,y_2)\big)(t) &:= \Big(\big(T_1(y_1,y_2)\big)(t), \big(T_2(y_1,y_2)\big)(t)\Big)\\ &= \bigg(\lambda_1 \int_0^1 G_1(t,s)a_1(s)f\big(s,y_1(s),y_2(s)\big)\,ds,\\ &\qquad \lambda_2 \int_0^1 G_2(t,s)a_2(s)g\big(s,y_1(s),y_2(s)\big)\,ds\bigg). \end{aligned} \tag{7}$$

We claim that whenever $(y_1, y_2) \in X$ is a fixed point of the operator defined in (7), it follows that $y_1(t)$ and $y_2(t)$ solve problem (1)-(4). We shall look for fixed points of the operator S, seeing as these fixed points coincide with solutions of problem (1)-(4).

To establish the main results, we need the following assumptions:

(H_1) $f, g : [0,1] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ and $a_1, a_2 : [0,1] \to \mathbf{R}$;

(H_2) f, g, a_1, a_2 are continuous;

(H_3) there exist positive functions $L_1(t)$ and $L_2(t)$ such that

$$\begin{aligned}&\big|a_1(t)\big|\cdot\big|f\big(t,y_1(t),y_2(t)\big) - f\big(t,u_1(t),u_2(t)\big)\big| \le L_1(t)\big\|(y_1-u_1, y_2-u_2)\big\|,\\ &\big|a_2(t)\big|\cdot\big|g\big(t,y_1(t),y_2(t)\big) - g\big(t,u_1(t),u_2(t)\big)\big| \le L_2(t)\big\|(y_1-u_1, y_2-u_2)\big\|\end{aligned}$$

for all $t \in [0,1]$ and $(y_1,y_2), (u_1,u_2) \in X$.

Further, we set

$$\begin{aligned}&D_L^{-\nu} = \max\Big\{\sup_{t\in[0,1]}\big|D_{0^+}^{-\nu_1}L_1(t)\big|, \sup_{t\in[0,1]}\big|D_{0^+}^{-\nu_2}L_2(t)\big|\Big\},\\ &D^{-(\nu-\beta)}L(1) = \max\big\{\big|D_{0^+}^{-(\nu_1-\beta)}L_1(1)\big|, \big|D_{0^+}^{-(\nu_2-\beta)}L_2(1)\big|\big\},\\ &D^{-(\nu-\beta)}L(\xi_i) = \max\big\{\big|D_{0^+}^{-(\nu_1-\beta)}L_1(\xi_i)\big|, \big|D_{0^+}^{-(\nu_2-\beta)}L_2(\xi_i)\big|\big\}, \quad i = 1,2,\ldots,m-2.\end{aligned}$$

(H_4) The parameters λ_1, λ_2 satisfy $\lambda_1, \lambda_2 < \Lambda$, where

$$\begin{aligned}\Lambda = \min\Bigg\{&\frac{1}{4}\Bigg[D_L^{-\nu} + \frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)}\Bigg(\sum_{i=1}^{m-2} b_i D^{-(\nu-\beta)}L(\xi_i) + D^{-(\nu-\beta)}L(1)\Bigg)\Bigg]^{-1},\\ &\frac{1}{4}\Bigg[D_L^{-\nu} + \frac{\Gamma(\nu_2-\beta)}{\Gamma(\nu_2)(1-\rho_2)}\Bigg(\sum_{i=1}^{m-2} b_i D^{-(\nu-\beta)}L(\xi_i) + D^{-(\nu-\beta)}L(1)\Bigg)\Bigg]^{-1}\Bigg\}.\end{aligned}$$

(H_5) The parameters λ_1, λ_2 satisfy $\lambda_1, \lambda_2 < \Lambda$, where

$$\Lambda = \min\left\{\frac{1}{4}\left[\frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)}\left(\sum_{i=1}^{m-2} b_i D^{-(\nu-\beta)}L(\xi_i) + D^{-(\nu-\beta)}L(1)\right)\right]^{-1}, \right.$$
$$\left.\frac{1}{4}\left[\frac{\Gamma(\nu_2-\beta)}{\Gamma(\nu_2)(1-\rho_2)}\left(\sum_{i=1}^{m-2} b_i D^{-(\nu-\beta)}L(\xi_i) + D^{-(\nu-\beta)}L(1)\right)\right]^{-1}\right\}.$$

(H_6) There exists $\mu \in L^1([0,1],\mathbf{R}^+)$ such that

$$\left|a_1(t)f\left(t,y_1(t),y_2(t)\right)\right| \le \mu(t), \qquad \left|a_2(t)g\left(t,y_1(t),y_2(t)\right)\right| \le \mu(t),$$
$$\forall (t,y_1,y_2) \in [0,1]\times X.$$

We are ready to state the existence and uniqueness result.

Theorem 3.1 *Suppose that conditions* (H_1)-(H_4) *are satisfied. Then the boundary value problem* (1)-(4) *has a unique solution.*

Proof Let us set

$$M = \max\left\{\sup_{t\in[0,1]}\left|a_1(t)f(t,0,0)\right|, \sup_{t\in[0,1]}\left|a_2(t)g(t,0,0)\right|\right\}$$

and choose

$$r \ge \max\left\{4\lambda_1\left[\frac{M}{\Gamma(\nu_1+1)} + \frac{M}{\Gamma(\nu_1)(1-\rho_1)(\nu_1-\beta)}\left(\sum_{i=1}^{m-2} b_i\xi_i^{\nu_1-\beta}+1\right)\right],\right.$$
$$\left.4\lambda_2\left[\frac{M}{\Gamma(\nu_2+1)} + \frac{M}{\Gamma(\nu_2)(1-\rho_2)(\nu_2-\beta)}\left(\sum_{i=1}^{m-2} b_i\xi_i^{\nu_2-\beta}+1\right)\right]\right\}.$$

Now, we show that $S(\Omega_r)\subset\Omega_r$, where $\Omega_r = \{(y_1,y_2)\in X : \|(y_1,y_2)\| \le r\}$, and S is a contraction. In fact, for all $(y_1,y_2)\in\Omega_r$, we obtain

$$\begin{aligned}
&\left\|T_1(y_1,y_2)\right\| \\
&\quad= \max_{t\in[0,1]}\left|-\int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)}\lambda_1 a_1(s)f\left(s,y_1(s),y_2(s)\right)ds\right. \\
&\qquad+ \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)}\left[\int_0^1 (1-s)^{\nu_1-\beta-1}\lambda_1 a_1(s)f\left(s,y_1(s),y_2(s)\right)ds\right. \\
&\qquad\left.\left.-\sum_{i=1}^{m-2} b_i\int_0^{\xi_i}(\xi_i-s)^{\nu_1-\beta-1}\lambda_1 a_1(s)f\left(s,y_1(s),y_2(s)\right)ds\right]\right| \\
&\quad\le \max_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)}\lambda_1\left|a_1(s)f\left(s,y_1(s),y_2(s)\right)\right|ds\right. \\
&\qquad+ \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)}\left[\int_0^1 (1-s)^{\nu_1-\beta-1}\lambda_1\left|a_1(s)f\left(s,y_1(s),y_2(s)\right)\right|ds\right.
\end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_1-\beta-1} \lambda_1 \bigl|a_1(s) f\bigl(s, y_1(s), y_2(s)\bigr)\bigr| \, ds \Biggr]\Biggr\} \\
&\leq \max_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 \bigl|a_1(s)\bigr| \bigl(\bigl| f\bigl(s, y_1(s), y_2(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\bigr) \, ds \\
&\quad + \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)} \Biggl[\int_0^1 (1-s)^{\nu_1-\beta-1} \lambda_1 \bigl|a_1(s)\bigr| \bigl(\bigl| f\bigl(s, y_1(s), y_2(s)\bigr) - f(s,0,0)\bigr| \\
&\quad + \bigl|f(s,0,0)\bigr|\bigr) \, ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_1-\beta-1} \lambda_1 \bigl|a_1(s)\bigr| \bigl(\bigl| f\bigl(s, y_1(s), y_2(s)\bigr) - f(s,0,0)\bigr| + \bigl|f(s,0,0)\bigr|\bigr) \, ds \Biggr]\Biggr\} \\
&\leq \max_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 \bigl(L_1(s) \bigl\|(y_1, y_2)\bigr\| + M\bigr) \, ds \\
&\quad + \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)} \Biggl[\int_0^1 (1-s)^{\nu_1-\beta-1} \lambda_1 \bigl(L_1(s) \bigl\|(y_1, y_2)\bigr\| + M\bigr) \, ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_1-\beta-1} \lambda_1 \bigl(L_1(s) \bigl\|(y_1, y_2)\bigr\| + M\bigr) \, ds \Biggr]\Biggr\} \\
&\leq \max_{t\in[0,1]} \Biggl\{ \lambda_1 \biggl(D_{0^+}^{-\nu_1} L_1(t) r + \frac{M t^{\nu_1}}{\Gamma(\nu_1+1)} \biggr) + \lambda_1 \frac{\Gamma(\nu_1-\beta) t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)} \\
&\quad \times \Biggl[\sum_{i=1}^{m-2} b_i \biggl(D_{0^+}^{-(\nu_1-\beta)} L_1(\xi_i) r + \frac{M \xi_i^{\nu_1-\beta}}{\Gamma(\nu_1-\beta+1)} \biggr) + D_{0^+}^{-(\nu_1-\beta)} L_1(1) r + \frac{M}{\Gamma(\nu_1-\beta+1)} \Biggr]\Biggr\} \\
&\leq \lambda_1 \biggl(D_L^{-\nu} r + \frac{M}{\Gamma(\nu_1+1)} \biggr) + \lambda_1 \frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)} \\
&\quad \times \Biggl[\sum_{i=1}^{m-2} b_i \biggl(D^{-(\nu-\beta)} L(\xi_i) r + \frac{M \xi_i^{\nu_1-\beta}}{\Gamma(\nu_1-\beta+1)} \biggr) + D^{-(\nu-\beta)} L(1) r + \frac{M}{\Gamma(\nu_1-\beta+1)} \Biggr] \\
&\leq \lambda_1 \Biggl[D_L^{-\nu} + \frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)} \Biggl(\sum_{i=1}^{m-2} b_i D^{-(\nu-\beta)} L(\xi_i) + D^{-(\nu-\beta)} L(1) \Biggr) \Biggr] r \\
&\quad + \lambda_1 M \Biggl[\frac{1}{\Gamma(\nu_1+1)} + \frac{1}{\Gamma(\nu_1)(1-\rho_1)(\nu_1-\beta)} \Biggl(\sum_{i=1}^{m-2} b_i \xi_i^{\nu_1-\beta} + 1 \Biggr) \Biggr] \\
&\leq \frac{1}{4} r + \frac{1}{4} r = \frac{r}{2},
\end{aligned}
$$

that is to say, $\|T_1(y_1, y_2)\| \leq \frac{r}{2}$.

Then, for $(y_1, y_2), (u_1, u_2) \in X$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\bigl\| T_1(y_1, y_2) - T_1(u_1, u_2) \bigr\| \\
&\quad \leq \max_{t\in[0,1]} \Biggl\{ \int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 \bigl|a_1(s)\bigr| \cdot \bigl| f\bigl(s, y_1(s), y_2(s)\bigr) - f\bigl(s, u_1(s), u_2(s)\bigr)\bigr| \, ds \\
&\qquad + \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)} \Biggl[\int_0^1 (1-s)^{\nu_1-\beta-1} \lambda_1 \bigl|a_1(s)\bigr|
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \bigl|f\bigl(s,y_1(s),y_2(s)\bigr)-f\bigl(s,u_1(s),u_2(s)\bigr)\bigr|\,ds \\
&\quad + \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_1-\beta-1}\lambda_1\bigl|a_1(s)\bigr|\cdot\bigl|f\bigl(s,y_1(s),y_2(s)\bigr)-f\bigl(s,u_1(s),u_2(s)\bigr)\bigr|\,ds\Biggr]\Biggr\} \\
&\le \lambda_1 D_L^{-\nu}\bigl\|(y_1-u_1,y_2-u_2)\bigr\| + \lambda_1\frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)} \\
&\quad \times \Biggl[\sum_{i=1}^{m-2} b_i D^{-(\nu_1-\beta)}L(\xi_i) + D^{-(\nu_1-\beta)}L(1)\Biggr]\bigl\|(y_1-u_1,y_2-u_2)\bigr\| \\
&\le \frac{1}{4}\bigl\|(y_1-u_1,y_2-u_2)\bigr\|.
\end{aligned}
$$

That is to say, $\|T_1(y_1,y_2)-T_1(u_1,u_2)\| \le \frac{1}{4}\|(y_1-u_1,y_2-u_2)\| = \frac{1}{4}\|(y_1,y_2)-(u_1,u_2)\|$.

Hence, we find that $T_1:\Omega_r \to B_{\frac{r}{2}}$ and T_1 is a contraction, where $B_{\frac{r}{2}} = \{y\in B: \|y\| \le \frac{r}{2}\}$. Similarly, we have $T_2:\Omega_r \to B_{\frac{r}{2}}$ and T_2 is a contraction. Consequently, for any $(y_1,y_2)\in\Omega_r$,

$$
\bigl\|S(y_1,y_2)\bigr\| = \bigl\|\bigl(T_1(y_1,y_2),T_2(y_1,y_2)\bigr)\bigr\| \le \frac{r}{2}+\frac{r}{2} \le r,
$$

i.e., $S(\Omega_r)\subset\Omega_r$. And, for $(y_1,y_2),(u_1,u_2)\in\Omega_r$ and for each $t\in[0,1]$,

$$
\bigl\|S(y_1,y_2)-S(u_1,u_2)\bigr\| \le \frac{1}{2}\bigl\|(y_1-u_1,y_2-u_2)\bigr\| = \frac{1}{2}\bigl\|(y_1,y_2)-(u_1,u_2)\bigr\|.
$$

So, $S:\Omega_r\to\Omega_r$ and S is a contraction. Thus, the conclusion of the theorem follows from the contraction mapping principle. □

Our next result is based on the following well-known fixed point theorem due to Krasnoselskii.

Lemma 3.2 (Krasnoselskii [25]) *Let K be a closed convex and nonempty subset of a Banach space E. Let T, S be the operators such that*:
(i) *$Tx + Sy \in K$ whenever $x,y\in K$*;
(ii) *T is compact and continuous*;
(iii) *S is a contraction mapping.*
Then there exists $z\in K$ such that $z = Tz + Sz$.

Now we are ready to state and prove the following existence result.

Theorem 3.3 *Suppose that conditions* (H_1)-(H_3), (H_5), (H_6) *are satisfied. Then there exists at least one solution of the boundary value problem* (1)-(4).

Proof Let us fix

$$
\begin{aligned}
r \ge \max\Biggl\{&\frac{1}{2}\lambda_1\|\mu\|_{L^1}\Biggl[\frac{1}{\Gamma(\nu_1+1)}+\frac{1}{\Gamma(\nu_1)(1-\rho_1)(\nu_1-\beta)}\Biggl(\sum_{i=1}^{m-2} b_i\xi_i^{\nu_1-\beta}+1\Biggr)\Biggr], \\
&\frac{1}{2}\lambda_2\|\mu\|_{L^1}\Biggl[\frac{1}{\Gamma(\nu_2+1)}+\frac{1}{\Gamma(\nu_2)(1-\rho_2)(\nu_2-\beta)}\Biggl(\sum_{i=1}^{m-2} b_i\xi_i^{\nu_2-\beta}+1\Biggr)\Biggr]\Biggr\},
\end{aligned}
$$

and consider $\Omega_r = \{(y_1, y_2) \in X : \|(y_1, y_2)\| \leq r\}$. We define the operators Q_1 and Q_2 on Ω_r as

$$T_1(y_1, y_2) := Q_1(y_1, y_2) + Q_2(y_1, y_2),$$

where

$$\begin{aligned}
\bigl(Q_1(y_1, y_2)\bigr)(t) &= -\int_0^t \frac{(t-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 a_1(s) f\bigl(s, y_1(s), y_2(s)\bigr)\, ds, \\
\bigl(Q_2(y_1, y_2)\bigr)(t) &= \frac{t^{\nu_1-1}}{\Gamma(\nu_1)(1-\rho_1)} \Biggl[\int_0^1 (1-s)^{\nu_1-\beta-1} \lambda_1 a_1(s) f\bigl(s, y_1(s), y_2(s)\bigr)\, ds \\
&\quad - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_1-\beta-1} \lambda_1 a_1(s) f\bigl(s, y_1(s), y_2(s)\bigr)\, ds\Biggr].
\end{aligned}$$

For all $(y_1, y_2), (u_1, u_2) \in \Omega_r$, we find that

$$\begin{aligned}
&\bigl\|Q_1(y_1, y_2) + Q_2(u_1, u_2)\bigr\| \\
&\quad\leq \|\mu\|_{L^1} \lambda_1 \Biggl[\frac{1}{\Gamma(\nu_1+1)} + \frac{1}{\Gamma(\nu_1)(1-\rho_1)(\nu_1-\beta)} \Biggl(\sum_{i=1}^{m-2} b_i \xi_i^{\nu_1-\beta} + 1\Biggr)\Biggr] \\
&\quad\leq \frac{r}{2}.
\end{aligned}$$

Thus, $Q_1(y_1, y_2) + Q_2(u_1, u_2) \in \Omega_{\frac{r}{2}}$ for all $(y_1, y_2), (u_1, u_2) \in \Omega_r$. From assumption (H_3), we have

$$\begin{aligned}
\bigl\|Q_2(y_1, y_2) - Q_2(u_1, u_2)\bigr\| &\leq \lambda_1 \frac{\Gamma(\nu_1-\beta)}{\Gamma(\nu_1)(1-\rho_1)} \Biggl[\sum_{i=1}^{m-2} b_i D^{-(\nu_1-\beta)} L(\xi_i) + D^{-(\nu_1-\beta)} L(1)\Biggr] \\
&\quad\times \bigl\|(y_1 - u_1, y_2 - u_2)\bigr\| \\
&\leq \frac{1}{4} \bigl\|(y_1 - u_1, y_2 - u_2)\bigr\| \\
&= \frac{1}{4} \bigl\|(y_1, y_2) - (u_1, u_2)\bigr\|,
\end{aligned}$$

where $(y_1, y_2), (u_1, u_2) \in X$, $t \in [0, 1]$. So, Q_2 is a contraction mapping. We next consider the operator Q_1. Evidently, the continuity of f implies that the operator Q_1 is continuous. Also, Q_1 is uniformly bounded on Ω_r as

$$\bigl\|Q_1(y_1, y_2)\bigr\| \leq \lambda_1 \frac{\|\mu\|_{L^1}}{\Gamma(\nu_1+1)} \leq 2r.$$

Now, we show that $Q_1(y_1, y_2)(t)$ is equicontinuous. In fact, since a_1, f are bounded on the compact set $[0, 1]$ and $[0, 1] \times \Omega_r$, respectively, we can define

$$M_1 = \max\Bigl\{\sup_{(t, y_1, y_2) \in [0,1] \times \Omega_r} \bigl|a_1(t) f(t, y_1, y_2)\bigr|, \sup_{(t, y_1, y_2) \in [0,1] \times \Omega_r} \bigl|a_2(t) g(t, y_1, y_2)\bigr|\Bigr\},$$

and we have, for any $t_1, t_2 \in [0,1]$,

$$\begin{aligned}
&\left|Q_1(y_1,y_2)(t_2) - Q_1(y_1,y_2)(t_1)\right| \\
&\quad = \left| -\int_0^{t_2} \frac{(t_2-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 a_1(s) f\big(s, y_1(s), y_2(s)\big)\,ds \right. \\
&\qquad \left. + \int_0^{t_1} \frac{(t_1-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 a_1(s) f\big(s, y_1(s), y_2(s)\big)\,ds \right| \\
&\quad = \left| \int_0^{t_1} \frac{(t_2-s)^{\nu_1-1} - (t_1-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 a_1(s) f\big(s, y_1(s), y_2(s)\big)\,ds \right. \\
&\qquad \left. + \int_{t_1}^{t_2} \frac{(t_2-s)^{\nu_1-1}}{\Gamma(\nu_1)} \lambda_1 a_1(s) f\big(s, y_1(s), y_2(s)\big)\,ds \right| \\
&\quad \le \lambda_1 \frac{M_1}{\Gamma(\nu_1)} \left| \int_0^{t_1} \big((t_2-s)^{\nu_1-1} - (t_1-s)^{\nu_1-1}\big)\,ds + \int_{t_1}^{t_2} (t_2-s)^{\nu_1-1}\,ds \right| \\
&\quad = \lambda_1 \frac{M_1}{\Gamma(\nu_1+1)} \left| -\big((t_2-t_1)^{\nu_1} - t_2^{\nu_1}\big) - t_1^{\nu_1} + (t_2-t_1)^{\nu_1} \right| \\
&\quad = \lambda_1 \frac{M_1}{\Gamma(\nu_1+1)} \left| t_2^{\nu_1} - t_1^{\nu_1} \right|,
\end{aligned}$$

which is independent of (y_1, y_2). Therefore, Q_1 is equicontinuous on Ω_r. Hence, by the Arzela-Ascoli theorem, Q_1 is compact on Ω_r. Similarly, we set

$$T_2(y_1,y_2) := P_1(y_1,y_2) + P_2(y_1,y_2),$$

where

$$\begin{aligned}
\big(P_1(y_1,y_2)\big)(t) &= -\int_0^t \frac{(t-s)^{\nu_2-1}}{\Gamma(\nu_2)} \lambda_2 a_2(s) g\big(s, y_1(s), y_2(s)\big)\,ds, \\
\big(P_2(y_1,y_2)\big)(t) &= \frac{t^{\nu_2-1}}{\Gamma(\nu_2)(1-\rho_2)} \left[\int_0^1 (1-s)^{\nu_2-\beta-1} \lambda_2 a_2(s) g\big(s, y_1(s), y_2(s)\big)\,ds \right. \\
&\qquad \left. - \sum_{i=1}^{m-2} b_i \int_0^{\xi_i} (\xi_i - s)^{\nu_2-\beta-1} \lambda_2 a_2(s) g\big(s, y_1(s), y_2(s)\big)\,ds \right].
\end{aligned}$$

And we can obtain a similar conclusion to the operator T_1. Now, we let

$$A(y_1,y_2) := \big(Q_1(y_1,y_2), P_1(y_1,y_2)\big), \qquad B(y_1,y_2) := \big(Q_2(y_1,y_2), P_2(y_1,y_2)\big).$$

Therefore, we have

$$\begin{aligned}
S(y_1,y_2) &= \big(T_1(y_1,y_2), T_2(y_1,y_2)\big) \\
&= \big(Q_1(y_1,y_2) + Q_2(y_1,y_2), P_1(y_1,y_2) + P_2(y_1,y_2)\big) \\
&= \big(Q_1(y_1,y_2), P_1(y_1,y_2)\big) + \big(Q_2(y_1,y_2), P_2(y_1,y_2)\big) \\
&= A(y_1,y_2) + B(y_1,y_2).
\end{aligned}$$

In view of the proof above, we get the following.

(I) For all $(y_1, y_2), (u_1, u_2) \in \Omega_r$, we find that

$$\|A(y_1, y_2) + B(u_1, u_2)\| = \|Q_1(y_1, y_2) + Q_2(u_1, u_2)\| + \|P_1(y_1, y_2) + P_2(u_1, u_2)\| \leq r.$$

Thus, $A(y_1, y_2) + B(u_1, u_2) \in \Omega_r$ for all $(y_1, y_2), (u_1, u_2) \in \Omega_r$.

(II) For $(y_1, y_2), (u_1, u_2) \in X$, $t \in [0, 1]$,

$$\begin{aligned} \|B(y_1, y_2) - B(u_1, u_2)\| &= \|Q_2(y_1, y_2) - Q_2(u_1, u_2)\| + \|P_2(y_1, y_2) - P_2(u_1, u_2)\| \\ &\leq \frac{1}{2}\|(y_1 - u_1, y_2 - u_2)\| = \frac{1}{2}\|(y_1, y_2) - (u_1, u_2)\|. \end{aligned}$$

So, B is a contraction mapping.

(III) The continuity of Q_1 and P_1 implies that the operator A is continuous. Also, A is uniformly bounded on Ω_r as

$$\|A(y_1, y_2)\| = \|Q_1(y_1, y_2)\| + \|P_1(y_1, y_2)\| \leq 4r.$$

Moreover, for any $t_1, t_2 \in [0, 1]$,

$$\begin{aligned} &\bigl|A(y_1, y_2)(t_2) - A(y_1, y_2)(t_1)\bigr| \\ &\quad = \bigl|\bigl(Q_1(y_1, y_2)(t_2) - Q_1(y_1, y_2)(t_1), P_1(y_1, y_2)(t_2) - P_1(y_1, y_2)(t_1)\bigr)\bigr| \\ &\quad \leq \bigl|Q_1(y_1, y_2)(t_2) - Q_1(y_1, y_2)(t_1)\bigr| + \bigl|P_1(y_1, y_2)(t_2) - P_1(y_1, y_2)(t_1)\bigr| \\ &\quad \leq \lambda_1 \frac{M_1}{\Gamma(\nu_1 + 1)}\bigl|t_2^{\nu_1} - t_1^{\nu_1}\bigr| + \lambda_2 \frac{M_1}{\Gamma(\nu_2 + 1)}\bigl|t_2^{\nu_2} - t_1^{\nu_2}\bigr|, \end{aligned}$$

which is independent of (y_1, y_2). Therefore, A is equicontinuous on Ω_r. Hence, by the Arzela-Ascoli theorem, A is compact on Ω_r.

Thus all the assumptions of Lemma 3.2 are satisfied, so $S(y_1, y_2) = A(y_1, y_2) + B(y_1, y_2)$ has at least one fixed point. Hence, we obtain that (1)-(4) has at least one solution. □

4 Conclusions

There are few works that deal with multi-point boundary value problems for a coupled system of nonlinear fractional differential equations. In this article, we study multi-point boundary value problems for a coupled system of nonlinear fractional differential equations (1)-(4). By using Green's function, the Banach contraction principle and Krasnoselskii's fixed point theorem, we establish some new existence, uniqueness theorems of solutions for multi-point boundary value problems for a coupled system of nonlinear fractional differential equations (1)-(4).

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

Acknowledgements

The research was supported by the Youth Science Foundation of China (11201272) and the Science Foundation of Shanxi Province (2010021002-1).

Received: 25 January 2015 Accepted: 27 April 2015 Published online: 08 May 2015

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