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On some nonlinear boundary value problems related to a Black-Scholes model with transaction costs

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Abstract

We deal with some generalizations on a Black-Scholes model arising in financial mathematics. As a novelty in this paper, we consider a variable volatility and abstract functional boundary conditions, which allow us to treat a very large class of problems involving Black-Scholes equation. Our main results involve the existence of extremal solutions in presence of lower and upper solutions. Some examples of applications are provided too.

Keywords: Black-Scholes equation; functional conditions; discontinuous ODE's

1 Introduction

In this paper we are concerned with the following nonlinear boundary value problem:

$$\begin{cases} x^3(V'')^2(x) + p(x)x^2 V''(x) + q(x)(xV'(x) - V(x)) = 0, \\ B_1(V(c), V) = 0, \qquad B_2(V(d), V) = 0, \end{cases} \tag{1}$$

where p, q are nonnegative bounded functions which could be discontinuous in $[c, d]$, $c, d > 0$, and $B_i : \mathbb{R} \times \mathcal{C}([c, d]) \longrightarrow \mathbb{R}$, $i = 1, 2$, are functions which satisfy some conditions that we will state later. We observe that under this framework, a large class of boundary conditions is included, namely:

(1) Dirichlet conditions: $B_1(V(c), V) = V(c) - V_c$, $B_2(V(d), V) = V(d) - V_d$;

(2) Initial-integral conditions: $B_1(V(c), V) = V(c) - \int_c^d k(x)V(x)\,dx$;

(3) Multipoint conditions: $B_2(V(d), V) = V(d) - \sum_{j=1}^n V(x_j)$,

with $V_c, V_d \in \mathbb{R}$.

This study follows and generalizes the results contained in [1] with respect to the problem

$$\begin{cases} x^3(V'')^2(x) + px^2 V''(x) + q(xV'(x) - V(x)) = 0, \\ V(c) = V_c, \qquad V(d) = V_d. \end{cases} \tag{2}$$

In [1], it is assumed that p, q are positive constants and $V_c < V_d$. The contributions of the present paper are the following. First we address this problem but drop the condition $V_c < V_d$, and replace the constants p, q by two functions $p(x)$, $q(x)$; second, we replace

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the Dirichlet conditions by functional boundary conditions, which allows us to consider a very large class of problems for the equation of (2).

These problems are related to financial option pricing since they address the existence of stationary solutions of a class of generalizations of the classical Black-Scholes model (BS), introduced in 1973 [2], with equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r\left(S\frac{\partial V}{\partial S} - V\right) = 0, \tag{3}$$

where V represents the value of a call or put option, depending on an underlying asset S and on time t, r is the interest short rate and σ is the volatility of the asset price. In the (BS) model, S is modeled as a geometric Brownian motion, and no costs are considered when financial transactions hold.

Suppose that transaction costs are included in the model under the assumption that they are a percentage of the transaction, given as in [3] by a linear function h of the number of shares traded, *i.e.*, $h(\eta) = a - b\eta$, where η is the number of shares traded and $a, b > 0$. The following nonlinear BS type equation is obtained, where Δt is the interval between transactions,

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} - a\sigma S^2 \sqrt{\frac{2}{\pi \Delta t}}|V_{SS}| + bS^3\sigma^2 V_{SS}^2 + r(SV_S - V) = 0.$$

Then, if a is small enough and $V_{SS} > 0$ (see [1, 3, 4]), the following nonlinear version of (3) is obtained:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\tilde{\sigma}^2 S^2 \frac{\partial^2 V}{\partial S^2} + b\sigma^2 S^3 \left(\frac{\partial^2 V}{\partial S^2}\right)^2 + r\left(S\frac{\partial V}{\partial S} - V\right) = 0, \tag{4}$$

where $\tilde{\sigma} = \sigma^2\left(1 - 2\frac{a}{\sigma}\sqrt{\frac{2}{\pi \Delta t}}\right) > 0$ and is commonly denoted by adjusted volatility. Now, if we consider the stationary version of (4), we obtain the above ordinary differential equation (2) where $p = \frac{\tilde{\sigma}^2}{2b\sigma^2}$ and $q = \frac{r}{b\sigma^2}$ are constants.

This paper is organized as follows. In Section 2, we introduce an auxiliary nonlinear boundary value problem and the notions of upper and lower solutions used later. In Section 3, we consider problem (1) with Dirichlet boundary conditions. So, we start from paper [1] and we generalize it by considering $p(x), q(x)$ instead of constants p, q, which corresponds to variable volatility in the (BS) model, and by dropping the condition $V_c < V_d$. In Section 4, we deal with problem (1) on its full version, that is, with functional boundary conditions. Namely, we provide a result on the existence of extremal solutions between lower and upper solutions by using a generalized iteration with Dirichlet problems. Some examples of application are provided, too.

2 Auxiliary problem and upper and lower solutions

If we look to the equation of (1)

$$x^3\left(V''\right)^2(x) + p(x)x^2 V''(x) + q(x)\left(xV'(x) - V(x)\right) = 0,$$

and, as in [3, 4] and [1], solve it algebraically in order to V'', we obtain

$$V''(x) = -\frac{p(x)x^2 \pm \sqrt{p^2(x)x^4 - 4x^3 q(x)(xV'(x) - V(x))}}{2x^3}.$$ (5)

Since we are interested in convex solutions of (1), they can exist only if

$$V''(x) = -\frac{p(x)x^2 + \sqrt{p^2(x)x^4 - 4x^3 q(x)(xV'(x) - V(x))}}{2x^3},$$ (6)

and

$$-\big(xV'(x) - V(x)\big) \geq 0.$$ (7)

These facts lead us to consider first the following related problem:

$$\begin{cases} V''(x) + H(x, V(x), V'(x)) = 0, \\ B_1(V(c), V) = 0, \qquad B_2(V(d), V) = 0, \end{cases}$$ (8)

where

$$H(x, y, z) = \frac{p(x)x^2 - \sqrt{p^2(x)x^4 + 4x^3 q(x)|xz - y|}}{2x^3}.$$ (9)

Then we will see how solutions of this problem are solutions of our original problem.

We will use the method of upper and lower solutions for this problem and we will begin by considering the classical notions of $\mathcal{C}^2$-lower and upper solutions. However, afterwards, we will use some weaker notions, namely we will need a notion that allows lower solutions to have 'angles'. In fact, we will consider a maximum of two classical $\mathcal{C}^2$-lower solutions which is not necessarily differentiable and can exhibit an 'angle'. So, denoting by $D_- f(x)$ and $D^+ f(x)$, respectively, the lower left-hand and the upper right-hand Dini-derivatives of the function f at x, we introduce the following definitions (see [5]).

Definition 2.1 We say that $\alpha \in AC([c, d])$ is a lower solution for problem (8) if

$$B_1\big(\alpha(c), \alpha\big) \leq 0, \qquad B_2\big(\alpha(d), \alpha\big) \leq 0,$$

and for each $x_0 \in (c, d)$ one of the following conditions holds:
(1) $D_- \alpha(x_0) < D^+ \alpha(x_0)$;
(2) There exists an open interval J_0 such that $x_0 \in J_0 \subset (c, d)$, $\alpha_{|J_0} \in W^{2,1}(J_0)$, and for almost all $x \in J_0$ we have

$$\alpha''(x) + H\big(x, \alpha(x), \alpha'(x)\big) \geq 0.$$

We say that $\beta \in AC([c, d])$ is an upper solution for problem (8) if

$$B_1\big(\beta(c), \beta\big) \geq 0, \qquad B_2\big(\beta(d), \beta\big) \geq 0,$$

and for each $x_0 \in (c, d)$ one of the following conditions holds:
(1) $D^- \beta(x_0) > D_+ \beta(x_0)$;

(2) There exists an open interval J_0 such that $x_0 \in J_0 \subset (c,d)$, $\beta_{|J_0} \in W^{2,1}(J_0)$, and for almost all $x \in J_0$ we have

$$\beta''(x) + H\big(x, \beta(x), \beta'(x)\big) \leq 0.$$

Notice that if α, β are classical C^2-lower and upper solutions for problem (8), then they are also lower and upper solutions in the sense referred above.

3 Nonlinear problem with Dirichlet conditions

Consider problem (1) with

$$B_1\big(V(c), V\big) = V(c) - V_c, \qquad B_2\big(V(d), V\big) = V(d) - V_d,$$

that is, with standard Dirichlet conditions. So, we have in this case

$$\begin{cases} x^3 (V'')^2(x) + p(x)x^2 V''(x) + q(x)(xV'(x) - V(x)) = 0, \\ V(c) = V_c, \qquad V(d) = V_d. \end{cases} \tag{10}$$

The auxiliary problem referred in the previous sections is now

$$\begin{cases} V''(x) + H(x, V(x), V'(x)) = 0, \\ V(c) = V_c, \qquad V(d) = V_d, \end{cases} \tag{11}$$

where the function H is given by (9).

From the study of problem (11), we will deduce later existence and localization results for problem (10). Next proposition establishes adequately the existence of classical C^2-upper and lower solutions for (11).

Proposition 3.1 *The following assertions hold*:
(1) *If $\frac{V_d}{d} \leq \frac{V_c}{c}$, then the function*

$$\alpha_1(x) = \frac{V_d}{d} x \tag{12}$$

   is a C^2-lower solution for problem (11).
   The converse is also valid.
(2) *Take $k > 0$ such that*

$$k \geq \sqrt{\frac{Q}{c^3}} \sqrt{\max_{x \in [c,d]} \left| \frac{k}{2}(x^2 - cd) + \frac{cV_d - dV_c}{d - c} \right|}, \tag{13}$$

   where

$$Q = \max_{x \in [c,d]} q(x).$$

   Then the function

$$\alpha_k(x) = \frac{k}{2} x^2 + \left(\frac{V_d - V_c}{d - c} - \frac{k}{2}(d + c) \right) x + \frac{k}{2} cd - \frac{cV_d - dV_c}{d - c} \tag{14}$$

   is a C^2-lower solution for problem (11).

(3) *The function*

$$\beta(x) = \frac{V_d - V_c}{d - c}x + \frac{dV_c - cV_d}{d - c}, \quad x \in [c, d] \tag{15}$$

is a C^2-upper solution for problem (11).

Proof (1) If $\frac{V_d}{d} \leq \frac{V_c}{c}$, the thesis follows since

$$\alpha_1(c) = \frac{V_d}{d}c \leq V_c, \qquad \alpha_1(d) = V_d,$$

and, in (c, d),

$$\alpha_1''(x) + H\big(x, \alpha_1(x), \alpha_1'(x)\big) = 0.$$

Conversely, if α_1 is a lower solution for problem (11), then $\alpha_1(c) = \frac{V_d}{d}c \leq V_c$, which implies $\frac{V_d}{d} \leq \frac{V_c}{c}$.

(2) Observe that $\alpha_k(c) = V_c$, $\alpha_k(d) = V_d$. On the other hand, as

$$H\big(x, \alpha_k(x), \alpha_k'(x)\big) \geq \frac{p(x)x^2 - \sqrt{p^2(x)x^4} - \sqrt{4x^3 q(x)|x\alpha_k'(x) - \alpha_k(x)|}}{2x^3}$$

$$\geq -\sqrt{\frac{q(x)}{x^3}}\sqrt{|x\alpha_2'(x) - \alpha_k(x)|},$$

$$\geq -\sqrt{\frac{Q}{c^3}}\sqrt{\max_{x\in[c,d]}\left|\frac{k}{2}(x^2 - cd) + \frac{V_d c - V_c d}{d - c}\right|},$$

then

$$\alpha_k''(x) + H\big(x, \alpha_k(x), \alpha_k'(x)\big) \geq k - \sqrt{\frac{Q}{c^3}}\sqrt{\max_{x\in[c,d]}\left|\frac{k}{2}(x^2 - cd) - V_c + \frac{V_d - V_c}{d - c}c\right|}.$$

Since $k > 0$ satisfies hypothesis (13), we derive

$$\alpha_k''(x) + H\big(x, \alpha_k(x), \alpha_k'(x)\big) \geq 0.$$

Then the assertion holds.

(3) The thesis follows easily from the fact that

$$\beta''(x) + H\big(x, \beta(x), \beta'(x)\big) = H\big(x, \beta(x), \beta'(x)\big) \leq 0$$

and

$$\beta(c) = V_c, \qquad \beta(d) = V_d. \qquad \qquad \square$$

Remark 3.1 (1) Notice that there is no ambiguity in considering k big enough such that condition (13) holds. In fact, it is easy to see that the maximum in (13) depends on k and

is attained in the following way:

$$\max_{x\in[c,d]}\left|\frac{k}{2}(x^2-cd)+\frac{V_dc-V_cd}{d-c}\right|$$

$$=\max\left\{\left|\frac{k}{2}(d^2-cd)+\frac{V_dc-V_cd}{d-c}\right|,\left|\frac{k}{2}(c^2-cd)+\frac{V_dc-V_cd}{d-c}\right|\right\}.$$

So, it is clear that in the proof of the previous proposition, we could choose k satisfying (13) since

$$\lim_{k\to\infty}\frac{k}{\sqrt{k}}=+\infty.$$

(2) Observe also that the lower solution α_k can be written as $\alpha_k(x)=\beta(x)+\theta(x)$, where

$$\theta(x)=\frac{k}{2}(x^2-(d+c)x+cd).$$

In fact,

$$\alpha_k(x)=\frac{k}{2}x^2+\left(\frac{V_d-V_c}{d-c}-\frac{k}{2}(d+c)\right)x+\frac{k}{2}cd-\frac{V_dc-V_cd}{d-c}$$

$$=\left(\frac{V_d-V_c}{d-c}\right)x+\frac{dV_c-cV_d}{d-c}+\frac{k}{2}(x^2-(d+c)x+cd).$$

The function θ is quadratic, vanishes at $x=c$ and $x=d$, and is negative in $]c,d[$.

Notation 3.1 Given two functions $\phi\le\psi$ in $[c,d]$, let us denote by $[\phi,\psi]$ the functional interval

$$[\phi,\psi]=\left\{V\in W^{2,1}([c,d]):\phi(x)\le V(x)\le\psi(x)\text{ for all }x\in[c,d]\right\}.$$

Next, we state an existence and localization result for problem (11).

Theorem 3.1 *Let α_1, α_k and β be the functions defined in the previous proposition.*
(a) *If $\frac{V_d}{d}\le\frac{V_c}{c}$, then problem (11) has extremal $W^{2,1}$-solutions, that is, the least and the greatest one, in the functional interval $[\alpha_1,\beta]$.*
(b) *If $k>0$ satisfies (13), then problem (11) has extremal $W^{2,1}$-solutions in the functional interval $[\alpha_k,\beta]$.*

Proof Consider in case (a)

$$E_1=\left\{(x,y,z)\in[c,d]\times\mathbb{R}^2:\alpha_1(x)\le y\le\beta(x)\right\},$$

and in case (b)

$$E_2=\left\{(x,y,z)\in[c,d]\times\mathbb{R}^2:\alpha_k(x)\le y\le\beta(x)\right\}.$$

It is clear that H is continuous in E_1 and E_2. Moreover, we have that

$$\left|H(x,y,z)\right| \leq \frac{p(x)}{x} + \sqrt{\frac{q(x)}{x^3}\beta(x)} + \frac{\sqrt{q(x)}}{x}\sqrt{|z|}$$

for all $(x,y,z) \in E_1$ or $(x,y,z) \in E_2$. Then, putting

$$\hat{A} = \max_{x\in[c,d]}\left(\frac{p(x)}{x} + \sqrt{\frac{q(x)}{x^3}\beta(x)}\right), \qquad \hat{B} = \max_{x\in[c,d]}\frac{\sqrt{q(x)}}{x}, \tag{16}$$

we derive that

$$\left|H(x,y,z)\right| \leq \hat{A} + \hat{B}\sqrt{|z|} \tag{17}$$

for $(x,y,z) \in E_1$ or $(x,y,z) \in E_2$, respectively. This inequality guarantees that the function H satisfies the (classical) Nagumo condition both in E_1 and E_2. Using the fact that α_1, α_k are C^2-lower solutions in cases (a) and (b), respectively, and β is a C^2-upper solution for problem (11) such that

$$\alpha_1 \leq \beta, \qquad \alpha_k \leq \beta,$$

the conclusion holds by application of a well-known result contained in [6]. □

Corollary 3.1 *Let $\frac{V_d}{d} \leq \frac{V_c}{c}$ and $k > 0$ satisfy* (13) *and consider the functions α_1, α_k and β defined, respectively, by* (12), (14), (15). *Define*

$$\alpha(x) = \max\left\{\alpha_1(x), \alpha_k(x)\right\}.$$

Then problem (11) *has the extremal $W^{2,1}$-solutions in the interval*

$$[\alpha, \beta] = \left\{V \in W^{2,1}\big([c,d]\big) : \alpha(x) \leq V(x) \leq \beta(x) \text{ for all } x \in [c,d]\right\}.$$

Proof We observe that

$$\alpha(x) = \max\left\{\alpha_1(x), \alpha_k(x)\right\}$$

is a lower solution for problem (11) in the sense defined in the previous section (not necessarily C^2) and, as before, β is a C^2-upper solution for problem (11). Similarly to the proof of Theorem 3.1, the Nagumo condition holds in

$$E = \left\{(x,y,z) \in [c,d] \times \mathbb{R}^2 : \alpha(x) \leq y \leq \beta(x)\right\}.$$

Then the result follows from an existence theorem contained in [6]. □

Proposition 3.2 *The following assertions hold*:
  (a) *Every solution V of problem* (11) *is convex.*

(b) *Consider α_1 defined by* (12). *Then every solution V of problem* (11) *such that $V \geq \alpha_1$ satisfies, for all in $x \in [c,d]$,*

$$xV'(x) - V(x) \leq 0.$$

Proof (a) Clearly, the convexity of solutions of (11) derives from the fact that

$$V''(x) = -H\big(x, V(x), V'(x)\big) \geq 0 \quad \text{for all } x \in \,]c,d[$$

and from the continuity in c and d.

(b) Let V be a solution of problem (11) such that $V \geq \alpha_1$ in $[c,d]$. We claim that $\frac{V_d}{d} \geq V'(d)$. In fact, $\alpha_1 = \frac{V_d}{d}x \leq V(x)$ implies that $\frac{V_d}{d} \geq \frac{V(x)-V_d}{x-d}$ in $[c,d]$, and then, letting $x \to d$, we obtain $\frac{V_d}{d} \geq V'(d)$. This inequality together with the fact that the function $x \in [c,d] \longmapsto xV'(x) - V(x)$ is nondecreasing implies that $xV'(x) - V(x) \leq 0$ for all $x \in [c,d]$. $\qquad \square$

Next theorem establishes the relations between the convex solutions of (11) and of (10).

Theorem 3.2 *Consider problems* (11) *and* (10). *Then*

(a) *If $\frac{V_d}{d} \leq \frac{V_c}{c}$, every solution V of* (11) *provided by Theorem* 3.1 *is a convex solution of* (10).

(b) *Every convex solution V of problem* (10) *is a convex solution of problem* (11).

Proof As for (a), let $\frac{V_d}{d} \leq \frac{V_c}{c}$. Then α_1 is a lower solution of (11) and every solution V of (11) provided by Theorem 3.1 satisfies $V(x) \geq \alpha_1(x)$. Hence, by Proposition 3.2, V is convex and

$$\big|xV'(x) - V(x)\big| = -\big(xV'(x) - V(x)\big).$$

So

$$V''(x) = -H\big(x, V(x), V'(x)\big) = \frac{-p(x)x^2 + \sqrt{p(x)x^4 - 4q(x)x^3(xV'(x) - V(x))}}{2x^3},$$

which shows clearly that V is a convex function that solves (10).

As for (b), let V be a convex solution of (10). Then

$$x^3\big(V''\big)^2(x) + p(x)x^2V''(x) + q(x)\big(xV'(x) - V(x)\big) = 0,$$

and as $V'' \geq 0$, $p(x) \geq 0$ and $0 < c \leq x \leq d$, then

$$q(x)\big(xV'(x) - V(x)\big) \leq 0.$$

Then

$$x^6\big(V''\big)^2(x) + p(x)x^5V''(x) = x^3q(x)\big|xV'(x) - V(x)\big|,$$

that is, by adding $\frac{1}{4}p^2(x)x^4$ to both members,

$$\left(x^3V''(x) + \frac{1}{2}p(x)x^2\right)^2 = \frac{1}{4}p^2(x)x^4 + x^3q(x)\big|xV'(x) - V(x)\big|.$$

Applying the square root, we obtain

$$x^3 V''(x) + \frac{1}{2}p(x)x^2 = \sqrt{\frac{1}{4}p^2(x)x^4 + x^3 q(x)|xV'(x) - V(x)|},$$

and then

$$V''(x) = \frac{-p(x)x^2 + \sqrt{p^2(x)x^4 + 4x^3 q(x)|xV'(x) - V(x)|}}{2x^3} = -H(x, V(x), V'(x)).$$

So, V solves (10). □

From Theorem 3.1, Theorem 3.2 and Corollary 3.1, it is clear that the following existence and localization result holds.

Theorem 3.3 *Consider problem* (1) *with standard Dirichlet conditions, that is,*

$$\begin{cases} x^3(V'')^2(x) + p(x)x^2 V''(x) + q(x)(xV'(x) - V(x)) = 0, \\ V(c) = V_c, \qquad V(d) = V_d. \end{cases}$$

(1) *If* $\frac{V_d}{d} \leq \frac{V_c}{c}$, *then this problem has the extremal convex* $W^{2,1}$*-solutions in the functional interval* $[\alpha_1, \beta]$, *where* α_1 *and* β *are provided, respectively, by* (12) *and* (15);
(2) *If* $\frac{V_d}{d} \leq \frac{V_c}{c}$ *and* $k > 0$ *satisfies* (13), *then this problem has the extremal convex* $W^{2,1}$*-solutions in the functional interval* $[\alpha, \beta]$, *where*

$$\alpha(x) = \max\{\alpha_1(x), \alpha_k(x)\}$$

and α_1, α_k, β *are provided, respectively, by* (12), (14), (15).

Remark 3.2 Under the hypotheses of the above theorem, observe that if $\frac{V_d}{d} = \frac{V_c}{c}$ then α_1 is a solution of problem (10). On the other hand, in the periodic case, $V_c = V_d$, the constant function $V \equiv V_c$ is a solution of (10).

Example 3.1 Consider problem (10) in the interval $[c, d] = [2, 6]$, with

$$p(x) = 1 + x^3, \qquad q(x) = [x],$$

where $[\cdot]$ denotes integer part, and boundary conditions $V(2) = 9$, $V(6) = 1$. Notice that in this case it is $V_c > V_d$. From condition (13), $k > 0$ must satisfy

$$k \geq \sqrt{\frac{6}{8}}\sqrt{\max\{|-4k - 13|, |12k - 13|\}},$$

so simple computations show that $k = 8$ satisfies (13). Then the function

$$\alpha(x) = 4x^2 - 34x + 61$$

is a lower solution for this problem. At the same time,

$$\beta(x) = 13 - 2x$$

is an upper solution.

Then, by application of Theorem 3.3, problem (10) with $p(x) = 1 + x^3$, $q(x) = [x]$, and boundary conditions $V(2) = 9$, $V(6) = 1$, has the extremal solutions in the functional interval

$$\left[\max\left\{ \frac{1}{6}x, 4x^2 - 34x + 61 \right\}, 13 - 2x \right].$$

4 Problem with functional boundary conditions

In this section we deal with problem (8) on its full expression and, as said in the introduction, we will use a generalized monotone method. In the construction of this method we will use two technical lemmas. First of them is the following generalization of Bolzano's theorem.

Lemma 4.1 ([7], Lemma 2.3) *Let $a, b \in \mathbb{R}$, $a \le b$, and $h : \mathbb{R} \longrightarrow \mathbb{R}$ be a function satisfying $h(a) \le 0 \le h(b)$ and*

$$\liminf_{z \to x^-} h(z) \ge h(x) \ge \limsup_{z \to x^+} h(z) \quad \text{for all } x \in [a, b]. \tag{18}$$

Then there exist $c_1, c_2 \in [a, b]$ such that $h(c_1) = 0 = h(c_2)$ and if $h(c) = 0$ for some $c \in [a, b]$, then $c_1 \le c \le c_2$, that is, c_1 and c_2 are, respectively, the least zero and the greatest one of h in $[a, b]$.

The second auxiliary result we need deals with the existence of extremal fixed points for nondecreasing operators defined in the space of absolutely continuous functions.

Lemma 4.2 ([8], Proposition 1.4.4) *Let $I \subset \mathbb{R}$ be a nonempty closed interval and $[\alpha, \beta]$ be a nonempty functional interval in $AC(I)$. Assume that $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ is a nondecreasing mapping and that there exists $\psi \in L^1(I, [0, +\infty))$ such that*

$$\left| (G\gamma)'(x) \right| \le \psi(x) \quad \text{for all } \gamma \in [\alpha, \beta] \text{ and almost all } x \in I.$$

Then G has in $[\alpha, \beta]$ the greatest, V^*, and the least, V_*, fixed points. Moreover, they satisfy

$$V_* = \min\{V : GV \le V\}, \qquad V^* = \max\{V : V \le GV\}. \tag{19}$$

Now we establish a new result on the existence of extremal convex solutions for problem (8).

Theorem 4.1 *Assume that there exist $\alpha, \beta \in AC([c, d])$ which are, respectively, a lower and an upper solution for problem (8) satisfying $\alpha(x) \le \beta(x)$ for all $x \in [c, d]$. Put*

$$[\alpha, \beta] = \left\{ \gamma \in AC([c, d]) : \alpha(x) \le \gamma(x) \le \beta(x) \text{ for all } x \in [c, d] \right\},$$

$$E = \left\{ y \in \mathbb{R} : \min_{x \in [c, d]} \alpha(x) \le y \le \max_{x \in [c, d]} \beta(x) \right\},$$

and assume, moreover, that the following conditions hold:

(H$_1$) *For all $\gamma \in [\alpha, \beta]$ and all $y \in E$, we have*

$$\liminf_{z \to y^-} B_i(z, \gamma) \geq B_i(y, \gamma) \geq \limsup_{z \to y^+} B_i(z, \gamma) \quad (i = 1, 2);$$

(H$_2$) *For all $y \in E$, the functions $B_i(y, \cdot)$ are nonincreasing in $[\alpha, \beta]$ ($i = 1, 2$), that is, if $\gamma_1, \gamma_2 \in [\alpha, \beta]$ are such that $\gamma_1(x) \leq \gamma_2(x)$ for all $x \in [c, d]$, then $B_i(y, \gamma_1) \geq B_i(y, \gamma_2)$.*

In these conditions, problem (8) *has the extremal convex solutions in $[\alpha, \beta]$.*

Proof We define a mapping $G : [\alpha, \beta] \longrightarrow [\alpha, \beta]$ as follows: for all $\gamma \in [\alpha, \beta]$, $G\gamma$ is the greatest convex solution in $[\alpha, \beta]$ for the Dirichlet problem

$$\begin{cases} V''(x) + H(x, V(x), V'(x)) = 0 & \text{for all } x \in [c, d], \\ V(c) = \gamma_c, \qquad V(d) = \gamma_d, \end{cases} \tag{20}$$

where γ_c, γ_d are the greatest solutions in, respectively, $[\alpha(c), \beta(c)]$ and $[\alpha(d), \beta(d)]$, for the following respective algebraic equations:

$$B_1(y, \gamma) = 0, \tag{21}$$

$$B_2(y, \gamma) = 0. \tag{22}$$

Step 1: *The mapping G is well defined.* As α and β are lower and upper solutions of (8) and by condition (H$_2$), we have for all $\gamma \in [\alpha, \beta]$:

$$B_1\big(\alpha(c), \gamma\big) \leq B_1\big(\alpha(c), \alpha\big) \leq 0 \leq B_1\big(\beta(c), \beta\big) \leq B_1\big(\beta(c), \gamma\big),$$

$$B_2\big(\alpha(d), \gamma\big) \leq B_2\big(\alpha(d), \alpha\big) \leq 0 \leq B_2\big(\beta(d), \beta\big) \leq B_2\big(\beta(d), \gamma\big),$$

and so condition (H$_1$) implies that the numbers γ_c and γ_d are well defined by application of Lemma 4.1.

On the other hand, as γ_c and γ_d are the greatest solutions of equations (21)-(22) in, respectively, $[\alpha(c), \beta(c)]$ and $[\alpha(d), \beta(d)]$, this implies that $\alpha(c) \leq \gamma_c$, $\alpha(d) \leq \gamma_d$, $\beta(c) \geq \gamma_c$ and $\beta(d) \geq \gamma_d$. So, α and β are, respectively, a lower and an upper solution for problem (20). This guarantees that (20) has the greatest convex solution in $[\alpha, \beta]$. (Notice that (16)-(17) provides a Nagumo-type bound for H between our α and β.)

Step 2: *G is a nondecreasing mapping.* Let $\gamma_1, \gamma_2 \in [\alpha, \beta]$ such that $\gamma_1(x) \leq \gamma_2(x)$ for all $x \in [c, d]$, and we will show that $G\gamma_1 \leq G\gamma_2$. First, notice that

$$B_1\big(G\gamma_1(c), \gamma_2\big) \leq B_1\big(G\gamma_1(c), \gamma_1\big) = 0 \tag{23}$$

and

$$B_1\big(\beta(c), \gamma_2\big) \geq B_1\big(\beta(c), \beta\big) \geq 0, \tag{24}$$

so reasoning as above we obtain that $G\gamma_1(c) \leq G\gamma_2(c)$. (Note that $G\gamma_1(c) > G\gamma_2(c)$ coupled with (23)-(24) would contradict the fact that $G\gamma_2(c)$ is the greatest solution of $B_1(\cdot, \gamma_2)$ in

$[\alpha(c), \beta(c)]$.) In a similar way we prove that $G\gamma_1(d) \le G\gamma_2(d)$. Now, assume that $G\gamma_1 \not\le G\gamma_2$ and consider the function

$$\hat{\alpha}(x) = \max\{G\gamma_1(x), G\gamma_2(x)\}.$$

Thus defined, $\hat{\alpha}$ is a lower solution for problem (20) with conditions $V(c) = \gamma_{2c}$, $V(d) = \gamma_{2d}$. Indeed, if $G\gamma_1 < G\gamma_2$ in an interval $(\hat{x}_1, \hat{x}_2)$, then

$$\hat{\alpha}''(x) = G\gamma_2''(x) = -H\big(x, G\gamma_2(x), (G\gamma_2)'(x)\big) = -H\big(x, \hat{\alpha}(x), \hat{\alpha}'(x)\big) \quad \text{for all } x \in (\hat{x}_1, \hat{x}_2).$$

In the same way, if $G\gamma_1 > G\gamma_2$ in an interval $(\tilde{x}_1, \tilde{x}_2)$, then

$$\hat{\alpha}''(x) = -H\big(x, \hat{\alpha}(x), \hat{\alpha}'(x)\big) \quad \text{for all } x \in (\tilde{x}_1, \tilde{x}_2).$$

On the other hand, if $G\gamma_1(x_0) = G\gamma_2(x_0)$, then the convexity of $G\gamma_1$ and $G\gamma_2$ implies that $D_-\hat{\alpha}(x_0) < D^+\hat{\alpha}(x_0)$.

So, problem (20) with conditions $V(c) = \gamma_{2c}$, $V(d) = \gamma_{2d}$ has a solution in $[\hat{\alpha}, \beta]$, but this contradicts the fact that $G\gamma_2$ is the greatest solution for this problem in $[\alpha, \beta]$. Then we conclude that $G\gamma_1 \le G\gamma_2$, and so G is a nondecreasing mapping.

Step 3: *G has the extremal fixed points.* Let $\gamma \in [\alpha, \beta]$. For all $x \in [c, d]$, we have that

$$(G\gamma)'(x) = (G\gamma)'(c) - \int_c^x H\big(s, G\gamma(s), G\gamma'(s)\big)\,ds,$$

and so

$$\big|G\gamma'(x)\big| \le \max\left\{\frac{|\beta(d) - \alpha(c)|}{d - c}, \frac{|\beta(c) - \alpha(d)|}{d - c}\right\} + \int_c^x |\hat{A} + \hat{B}\sqrt{s}|\,ds,$$

where $\hat{A}$, $\hat{B}$ are as in (16). Then, by application of Lemma 4.2, we obtain that G has the extremal fixed points in $[\alpha, \beta]$, say V^*, V_*, which moreover satisfy (19).

Step 4: V^* *is the greatest convex solution of problem* (8) *in* $[\alpha, \beta]$. First, it is clear, as $GV^* = V^*$, that V^* is a solution of problem (8). Now, if V is another solution of (8), then we have that $V \le GV$ and so (19) implies that $V \le V^*$. So, V^* is the greatest convex solution of problem (8) in $[\alpha, \beta]$.

To obtain the least convex solution of (8) in $[\alpha, \beta]$, we only have to redefine the mapping G in the obvious way. □

Remark 4.1 Notice that condition (H$_1$) is satisfied, for example, if $B_1(\cdot, \gamma)$ is continuous or if it has only downwards discontinuities.

Remark 4.2 The same argument used in the proof of Theorem 3.3 provides that extremal convex solutions of problem (8) are also extremal convex solutions of problem (1) provided that $\frac{V(d)}{d} \le \frac{V(c)}{c}$.

Example 4.1 Consider problem (8) in an interval $[c, d]$, $c > 0$, with the following boundary conditions:

(B$_1$) 'The initial value of the solution is one half of its mean value on the whole interval $[c, d]$';

(B$_2$) 'The final value of the solution has integer part 4.'

Previous conditions can be written in the following form:

(B$_1$) $B_1(V(c), V) = V(c) - \frac{1}{2}\frac{1}{d-c}\int_c^d V(s)\, ds = 0$;

(B$_2$) $B_2(V(d), V) = -[V(d)] + 4 = 0$.

We will show that for $V(d) \in [4, 5)$ and $d \geq 3c$, $\alpha(x) = \frac{V(d)}{d}x$ and $\beta \equiv V(d)$ are, respectively, a lower and an upper solution for this problem. Indeed, we have $-[\alpha(d)] = -4$ and

$$\alpha(c) - \frac{1}{2}\frac{1}{d-c}\int_c^d \alpha(s)\, ds = \frac{V(d)}{d}\left(c - \frac{d+c}{4}\right) \leq 0$$

if $d \geq 3c$. On the other hand, the constant function $\beta \equiv V(d)$ is such that

$$B_1\big(\beta(c), \beta\big) = \frac{1}{2}V(d) \geq 0$$

and

$$B_2\big(\beta(d), \beta\big) = -\big[V(d)\big] + 4 = 0.$$

Finally, notice that for each $V(d) \in [4, 5)$, functions B_1 and B_2 satisfy conditions (H$_1$)-(H$_2$) between α and β. We can conclude, by application of Theorem 4.1, that if $d \geq 3c$ then problem (8) with boundary conditions (B$_1$)-(B$_2$) has the extremal solutions between α and β for each $V(d) \in [4, 5)$.

Remark 4.3 In papers [3] and [1] the authors obtained uniqueness of solutions for problem (2). Notice that uniqueness of solutions cannot be guaranteed when we include variable coefficients p and q and functional boundary conditions. For example, consider problem (1) with boundary conditions

$$\big[V(1)\big] = 2, \qquad \big[V(2)\big] = 4.$$

In this case, each function $V(x) = ax$ with $a \in [2, 2.5)$ is a solution of the problem.

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