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A note on the existence of solutions for a class of quasilinear elliptic equations: an Orlicz-Sobolev space setting

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Full list of author information is available at the end of the article**Abstract**

In this note, we study the existence and multiplicity of solutions for the quasilinear elliptic problem as follows:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x, u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary. The existence and multiplicity of solutions are obtained by a version of the symmetric mountain pass theorem.

Keywords: Orlicz-Sobolev spaces; symmetric mountain pass theorem; quasilinear elliptic equations

1 Introduction

In this note, we discuss the existence and multiplicity of solutions of the following boundary value problem:

$$\begin{cases} -\operatorname{div}(a(|\nabla u|)\nabla u) = f(x, u), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. The function a is such that $p: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$p(t) = \begin{cases} a(|t|)t, & t \neq 0; \\ 0, & t = 0, \end{cases}$$

is an increasing homeomorphism from \mathbb{R} onto itself and the continuous function $f(x, t) \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$ satisfies $f(x, 0) = 0$, $x \in \overline{\Omega}$. Especially, when $a(t) = |t|^{p-2}$, the problem (1.1) is the well-known p -Laplacian equation. There is a large number of papers on the existence of solutions for the p -Laplacian equation. But the problem (1.1) possesses more complicated nonlinearities. For example, it is inhomogeneous and has an important physical background, *e.g.*,

- (a) nonlinear elasticity: $P(t) = (1 + t^2)^\gamma - 1$, $\gamma > \frac{1}{2}$;
- (b) plasticity: $P(t) = t^\alpha (\log(1 + t))^\beta$, $\alpha \geq 1$, $\beta > 0$;
- (c) generalized Newtonian fluids: $P(t) = \int_0^t s^{1-\alpha} (\sinh^{-1} s)^\beta ds$, $0 \leq \alpha \leq 1$, $\beta > 0$.

So, in the discussions, some special techniques are needed, and the problem (1.1) has been studied in an Orlicz-Sobolev space and received considerable attention in recent years; see, for instance, the papers [1–9]. In paper [9], Fang and Tan discussed the problem (1.1) under the conditions that $f(x, t)$ was odd in t . They got the first result that when $h^+ < p^-$, and $f(x, t) \geq Ct^{q-1}$ for $0 < t < \delta$, $q < p^-$, the problem (1.1) had a sequence of solutions by genus theory. The second result is that when $f(x, t)$ satisfies $0 < \alpha F(x, t) \leq tf(x, t)$, $\forall x \in \overline{\Omega}$, $t \neq 0$, $\alpha > p^+$ and $f(x, t) = o(p(|t|))$ as $|t| \rightarrow 0$, the problem (1.1) has infinitely many pairs of solutions which correspond to the positive critical values by the symmetric mountain pass theorem.

Motivated by their results, in this note, we discuss the problem (1.1) when $f(x, t)$ is still odd in t but it satisfies weaker conditions than [9]; and furthermore, we need not know the behaviors of $f(x, t)$ near the zero. If $h^+ > p^-$, we can get multiplicity of solutions by a version of the symmetric mountain pass theorem.

The paper is organized as follows. In Section 2, we present some preliminary knowledge on the Orlicz-Sobolev spaces and give the main result. In Section 3, we make the proof.

2 Preliminaries

Obviously, the problem (1.1) allows a nonhomogeneous function p in the differential operator defining the problem (1.1). To deal with this situation, we introduce an Orlicz-Sobolev space setting for the problem (1.1) as follows.

Let

$$P(t) = \int_0^t p(s) ds, \quad \tilde{P}(t) = \int_0^t p^{-1}(s) ds, \quad t \in R,$$

then P and \tilde{P} are complementary N -functions (see [10]), which define the Orlicz spaces $L^P := L^P(\Omega)$ and $L^{\tilde{P}} := L^{\tilde{P}}(\Omega)$ respectively.

Throughout this paper, we assume the following condition on P :

$$(p) \quad 1 < p^- := \inf_{t>0} \frac{tp(t)}{P(t)} \leq p^+ := \sup_{t>0} \frac{tp(t)}{P(t)} < +\infty.$$

Under the condition (p), the Orlicz space L^P coincides with the set (equivalence classes) of measurable functions $u : \Omega \rightarrow R$ such that

$$\int_{\Omega} P(|u|) dx < +\infty,$$

and is equipped with the (Luxemburg) norm, *i.e.*,

$$\|u\|_P := \inf \left\{ k > 0 : \int_{\Omega} P\left(\frac{|u|}{k}\right) dx < 1 \right\}.$$

We will denote by $W^{1,P}(\Omega)$ the corresponding Orlicz-Sobolev space with the norm

$$\|u\|_{W^{1,P}(\Omega)} := \|u\|_P + \|\nabla u\|_P$$

and define $W_0^{1,P}(\Omega)$ as the closure of C_0^∞ in $W^{1,P}(\Omega)$. In this note, we will use the following equivalent norm on $W_0^{1,P}(\Omega)$:

$$\|u\| := \inf \left\{ k > 0 : \int_{\Omega} P\left(\frac{|\nabla u|}{k}\right) dx < 1 \right\}.$$

Now, we introduce the Orlicz-Sobolev conjugate P_* of P , which is given by

$$P_*^{-1}(t) := \int_0^t \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau,$$

where we suppose that

$$\lim_{t \rightarrow 0} \int_t^1 \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau < +\infty, \quad \lim_{t \rightarrow \infty} \int_1^t \frac{p^{-1}(\tau)}{\tau^{\frac{N+1}{N}}} d\tau = +\infty.$$

Let $p_*^- := \inf_{t>0} \frac{tP_*'(t)}{P_*(t)}$, $p_*^+ := \sup_{t>0} \frac{tP_*'(t)}{P_*(t)}$. Throughout this paper, we assume that $p^+ < p_*^-$. Now, we will make the following assumptions on $f(x, t)$.

(f_*) There exists an odd increasing homeomorphism h from R to R , and nonnegative constants c_1, c_2 such that

$$|f(x, t)| \leq c_1 + c_2 h(|t|), \quad \forall t \in R, \forall x \in \overline{\Omega},$$

and $\lim_{t \rightarrow +\infty} \frac{H(t)}{P_*(kt)} = 0, \forall k > 0$, where

$$H(t) := \int_0^t h(s) ds.$$

Let

$$\tilde{H}(t) := \int_0^t h^{-1}(s) ds,$$

then we can obtain complementary N -functions which define corresponding Orlicz spaces L^H and $L^{\tilde{H}}$.

Similar to the condition (p), we also assume the following condition on H :

$$(h) \quad 1 < h^- := \inf_{t>0} \frac{th(t)}{H(t)} \leq h^+ := \sup_{t>0} \frac{th(t)}{H(t)} < +\infty.$$

In order to prove our results, we now state some useful lemmas.

Lemma 2.1 [10] *Under the condition (p), the spaces $L^P(\Omega)$, $W_0^{1,P}(\Omega)$ and $W^{1,P}(\Omega)$ are separable and reflexive Banach spaces.*

Lemma 2.2 [10] *Under the condition (f_*), the embedding $W_0^{1,P}(\Omega) \hookrightarrow L^H(\Omega)$ is compact.*

Lemma 2.3 [2] *Let $\rho(u) = \int_{\Omega} P(u) dx$, we have*

$$(1) \text{ if } |u|_P < 1, \text{ then } |u|_P^{p^+} \leq \rho(u) \leq |u|_P^{p^-};$$

- (2) if $|u|_p > 1$, then $|u|_p^{p^-} \leq \rho(u) \leq |u|_p^{p^+}$;
- (3) if $0 < t < 1$, then $t^{p^+} P(u) \leq P(tu) \leq t^{p^-} P(u)$;
- (4) if $t > 1$, then $t^{p^-} P(u) \leq P(tu) \leq t^{p^+} P(u)$.

Lemma 2.4 [11–13] *Let $E = V + X$, where E is a real Banach space and V is finite dimensional. Suppose $I \in C^1(E, R)$ is an even functional satisfying $I(0) = 0$ and*

- (I₁) *there is a constant $\rho > 0$ such that $I|_{\partial B_\rho \cap X} \geq 0$;*
- (I₂) *there is a subspace W of E with $\dim V < \dim W < \infty$ and there is $M > 0$ such that $\max_{u \in W} I(u) < M$;*
- (I₃) *considering $M > 0$ given by (I₂), I satisfies $(PS)_c$ for $0 \leq c \leq M$.*

Then I possesses at least $\dim W - \dim V$ pairs of nontrivial critical points.

Using the version of the symmetric mountain pass theorem mentioned above, we can state our result as follows.

Theorem 2.1 *Assume that $f(x, t)$ is odd in t , satisfies (f_*) with $p^- < h^+ \leq p^+$ and the following assumptions:*

- (f₁) *there exist $\eta > p^+$ and $1 < \sigma < p^-$, and $a_1, a_2 > 0$, such that $\frac{1}{\eta} f(x, t)t - F(x, t) \geq -a_1 - a_2 |t|^\sigma$ for every $t \in R$, a.e. in Ω .*
- (f₂) *there is $\Omega_0 \subset \Omega$ with $|\Omega_0| > 0$ such that $\liminf_{|t| \rightarrow \infty} F(x, t)/|t|^{p^+} = \infty$ uniformly a.e. in Ω_0 .*

Then for any given $k \in N$, the problem (1.1) possesses at least k pairs of nontrivial solutions.

3 Main results and proofs

In this section, we assume that $N \geq 1$ and $E = W_0^{1,p}(\Omega)$, $u \in E$ is called a weak solution of the problem (1.1) if

$$\int_{\Omega} a(|\nabla u|) \nabla u \nabla \phi \, dx = \int_{\Omega} f(x, u) \phi \, dx, \quad \forall \phi \in E.$$

Set

$$I(u) = \int_{\Omega} P(|\nabla u|) \, dx - \int_{\Omega} F(x, u) \, dx, \quad \forall u \in E$$

and we know that the critical points of I are just the weak solutions of the problem (1.1).

For E is a separable and reflexive Banach space, then there exist (see [9]) $\{e_n\}_{n=1}^{\infty} \subset E$ and $\{e_n^*\}_{n=1}^{\infty} \subset E^*$ such that

$$e_n^*(e_m) = \delta_{n,m} = \begin{cases} 1, & \text{if } n = m; \\ 0, & \text{if } n \neq m. \end{cases} \quad \text{and} \quad e_n^*(v) = \alpha_n \quad \text{for } v = \sum_{i=1}^{\infty} \alpha_i e_i \in E.$$

Now, we set $V_j = \{u \in W_0^{1,p}(\Omega) : e_i^*(u) = 0, i > j\}$, $X_j = \{u \in W_0^{1,p}(\Omega) : e_i^*(u) = 0, i \leq j\}$, so

$$W_0^{1,p}(\Omega) = V_j \oplus X_j. \tag{3.1}$$

Lemma 3.1 *Given $\delta > 0$, there is $j \in N$ such that for all $u \in X_j$, $|u|_H \leq \delta \|u\|$.*

Proof We prove the lemma by contradiction. Suppose that there exist $\delta > 0$ and $u_j \in X_j$ for every $j \in N$ such that $|u_j|_H \geq \delta \|u_j\|$. Taking $v_j = \frac{u_j}{|u_j|_H}$, we have $|v_j|_H = 1$ for every $j \in N$ and $\|v_j\| \leq \frac{1}{\delta}$. Hence, $\{v_j\} \subset W_0^{1,p}(\Omega)$ is a bounded sequence, and we may suppose, without loss of generality, that $v_j \rightharpoonup v$ in $W_0^{1,p}(\Omega)$. Furthermore, $e_n^*(v) = 0$ for every $n \in N$ since $e_n^*(v_j) = 0$ for all $j \geq n$. This shows that $v = 0$. On the other hand, by the compactness of embedding $W_0^{1,p}(\Omega) \hookrightarrow L^H(\Omega)$, we conclude that $|v|_H = 1$. This proves the lemma. \square

Lemma 3.2 *Suppose f satisfies (f_*) , then there exist $j \in N$ and $\rho, \alpha > 0$ such that*

$$I|_{\partial B_\rho \cap X_j} \geq \alpha.$$

Proof Now suppose that $\|u\| > 1$. From (f_*) , we know that

$$\begin{aligned} I(u) &= \int_\Omega P(|\nabla u|) \, dx - \int_\Omega F(x, u) \, dx \\ &\geq \|u\|^{p^-} - C_1 |u|_H^{h^+} - C_2. \end{aligned}$$

Consequently, considering $\delta > 0$ to be chosen posteriorly by Lemma 3.1, we have for all $u \in X_j$ and j sufficiently large,

$$I(u) \geq \|u\|^{p^-} (1 - C_1 \delta^{h^+} \|u\|^{h^+ - p^-}) - C_2.$$

Now, taking $\|u\| = \rho(\delta) = (\frac{1}{2C_1 \delta^{h^+}})^{\frac{1}{h^+ - p^-}}$ and noting that $\rho(\delta) \rightarrow +\infty$, if $\delta \rightarrow 0$, we can choose $\delta > 0$ such that $\frac{1}{2} \rho^{p^-} > C_2$, $\rho > 1$, and $I(u) > 0$ for every $u \in X_j$, $\|u\| = \rho$, the proof is complete. \square

Lemma 3.3 *Suppose f satisfies (f_2) . Then given $m \in N$, there exist a subspace W of $W_0^{1,p}(\Omega)$ and a constant $M_m > 0$ such that $\dim W = m$ and $\max_{u \in W} I(u) < M_m$.*

Proof Let $x_0 \in \Omega_0$ and $r_0 > 0$ be such that $\overline{B(x_0, r_0)} \subset \Omega$, and $0 < |\overline{B(x_0, r_0)} \cap \Omega_0| < \frac{|\Omega_0|}{2}$. First, we take $v_1 \in C_0^\infty(\Omega)$ with $\text{supp}(v_1) = \overline{B(x_0, r_0)}$. Considering $\Omega_1 = \Omega_0 \setminus [\overline{B(x_0, r_0)} \cap \Omega_0] \subset \widehat{\Omega}_0 = \overline{\Omega} \setminus \overline{B(x_0, r_0)}$, we have $|\Omega_1| \geq \frac{|\Omega_0|}{2} > 0$. Let $x_1 \in \Omega_1$ and $r_1 > 0$ be such that $\overline{B(x_1, r_1)} \subset \widehat{\Omega}_0$, and $0 < |\overline{B(x_1, r_1)} \cap \Omega_1| < \frac{|\Omega_1|}{2}$. Next, we take $v_2 \in C_0^\infty(\Omega)$ with $\text{supp}(v_2) = \overline{B(x_1, r_1)}$. After a finite number of steps, we get v_1, v_2, \dots, v_m such that $\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$, $i \neq j$, and $|\text{supp}(v_j) \cap \Omega_0| > 0$ for all $i, j \in \{1, 2, \dots, m\}$. Let $W = \text{span}\{v_1, v_2, \dots, v_m\}$, by construction, $\dim W = m$, and $\int_\Omega |v|^{p^+} \, dx > 0$ for every $v \in W \setminus \{0\}$.

Since $\max_{u \in W \setminus \{0\}} I(u) = \max_{t>0, v \in W \cap \partial B_1(0)} (\int_\Omega P(t|\nabla v|) \, dx - \int_\Omega F(x, tv) \, dx)$, if $t > 1$, then $I(tv) \leq t^{p^+} - \int_\Omega F(x, tv) \, dx = t^{p^+} (1 - \frac{1}{t^{p^+}} \int_\Omega F(x, tv) \, dx)$. Now, it suffices to verify that

$$\lim_{t \rightarrow \infty} \frac{1}{t^{p^+}} \int_\Omega F(x, tv) \, dx > 1.$$

From the condition (f_2) , given $L > 0$, there is $C > 0$ such that for every $s \in R$, a.e. x in Ω_0 ,

$$F(x, s) \geq L|s|^{p^+} - C.$$

Consequently, for $v \in \partial B_1(0) \cap W$ and $t > 1$,

$$\int_{\Omega} F(x, tv) \, dx \geq Lt^{p^+} \int_{\Omega_0} |v|^{p^+} \, dx - Ct^{h^+} \int_{\Omega \setminus \Omega_0} H(v) \, dx - C_2,$$

and

$$\lim_{t \rightarrow \infty} \frac{\int_{\Omega} f(x, tv) \, dx}{t^{p^+}} \geq L \int_{\Omega_0} |v|^{p^+} \, dx - C \int_{\Omega \setminus \Omega_0} H(v) \, dx \geq Lr - CR,$$

where $r = \min\{\int_{\Omega_0} |v|^{p^+} \, dx, v \in \partial B_1(0) \cap W\}$ and $R = \max\{\int_{\Omega \setminus \Omega_0} H(v) \, dx, v \in \partial B_1(0) \cap W\}$. Observing that W is finite dimensional and we have $R < +\infty$, $r > 0$, the inequality is obtained by taking $L > \frac{1}{r}(1 + CR)$; the proof is complete. \square

Lemma 3.4 *Suppose f satisfies (f_1) , then I satisfies the (PS) condition.*

Proof We suppose that $\|u_n\| > 1$,

$$\begin{aligned} & M + o(1)\|u_n\| \\ & \geq I(u_n) - \frac{1}{\eta} I'(u_n)u_n \\ & = \int_{\Omega} P(|\nabla u_n|) \, dx - \frac{1}{\eta} \int_{\Omega} p(|\nabla u_n|) \nabla u_n \, dx + \int_{\Omega} \left(\frac{1}{\eta} f(x, u_n)u_n - F(x, u_n) \right) \, dx \\ & \geq \left(1 - \frac{p^+}{\eta} \right) \|u_n\|^{p^-} - a_1|\Omega| - C\|u_n\|^{\sigma}. \end{aligned}$$

Noting that $1 < \sigma < p^-, \eta > p^+$, $\{u_n\}$ is bounded. By [9], Lemma 3.1, we know that I satisfies the (PS) condition. \square

Proof of Theorem 2.1 First, we recall that $W_0^{1,p}(\Omega) = V_j \oplus X_j$, where V_j and X_j are defined in (3.1). Invoking Lemma 3.2, we find $j \in N$, and I satisfies I_1 with $X = X_j$. Now, by Lemma 3.3, there is a subspace W of $W_0^{1,p}(\Omega)$ with $\dim W = k + j = k + \dim V_j$ and such that I satisfies (I_2) . Since $I(0) = 0$ and I is even, we may apply Lemma 2.4 to conclude that I possesses at least k pairs of nontrivial critical points. The proof is complete. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors read and approved the final manuscript.

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