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Beamforming matrix quantization with variable feedback rate

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Abstract

We propose a new technique to quantize and feedback the parameters when a beamforming matrix is compressed with the Givens Rotation (GR). We suggest to feedback the parameters with variable feedback rate, and use efficient source coding and codebook to quantize the GR parameters. The variable feedback rate means that the number of bits used to represent the quantized beamforming matrix is based on the value of the matrix itself. And due to the non-uniform distribution of the GR parameters, source coding and code book can be designed to quantize those parameters in a more effective manner. Compared with the fixed feedback rate scheme, the proposed method delivers a better performance without incurring additional feedback bandwidth.

Introduction

Multiple transmit and receive antennas system has been adopted in several communication standards in order to achieve a higher throughput. The open-loop multiple-input multiple-output (MIMO) technique has already been shown to achieve a high performance gain. With the availability of either the full or partial channel state information (CSI) at the transmitter, we can achieve further performance gain or receiver complexity reduction. Such closed-loop schemes have been considered in many communication standards for application of beamforming or multi-user precoding.

However, CSI estimation for the downlink channel at the base station is not possible in Frequency Division Duplex systems. It is also not straightforward to implement CSI estimation in Time Division Duplex (TDD) systems due to the mismatch in the radio front end. Hence in general, the CSI will be estimated at the mobile clients and be sent back to the base station. For example, in the 802.11n wireless LAN system, when the system is operated in TDD mode, the channel can either be estimated by the transmitter through calibration or the channel is fed back by the receiver [1]. This unfortunately requires a high and undesirable feedback bandwidth.

Another popular way to reduce the amount of CSI feedback is through differential encoding [2,3]. However, such technique suffers accumulated error propagation. Therefore, the mobile client often computes the beamforming matrix, which is usually a unitary matrix and "compress" such a matrix before feeding back to the base station. The "compression" can significantly reduce the feedback bandwidth requirement. And 802.11 ac wireless LAN system [4] adopts this methodology to feedback the beamforming matrix for single- and multi-user MIMO. Although we only use unitary beamforming matrix as an example in this article, the techniques that are discussed in this article apply to the feedback of channel matrix as well. For example, we may perform singular value decomposition (SVD) on the channel matrix, and feedback the eigenvalues and both the left and right eigenvectors matrices, which are both unitary.

There have been several proposals in the literature to compress the beamforming vector. One is codebook based such as the vector quantization (VQ) scheme proposed in [5-10], and another is by using the Givens Rotation (GR) [1,4,11-13]. Compared with the GR-based scheme, the VQ approach requires a higher storage, as a set of codebooks is needed for a particular antenna setting. It has a higher complexity than the GR approach, especially when the number of codewords in the codebook increases. Due to these reasons, the GR approach has been adopted in the 802.11n and 802.11 ac standards [1,4].

In this article, we investigate an effective approach to quantize and feedback the GR parameters that compress the beamforming matrix. The proposed scheme is capable of achieving a better performance, in the absence of

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extra bandwidth, than existing techniques that quantize and feedback the GR parameters.

Signal model
MIMO model

Consider a point-to-point MIMO channel with N_T transmit antennas and N_R receive antennas, the $N_T \times 1$ transmitted signal is denoted by $\mathbf{x}$ and the $N_R \times N_T$ channel denoted by $\mathbf{H}$. The $N_R \times 1$ received signal $\mathbf{y}$ can be expressed as

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n} \tag{1}$$

To demonstrate the idea of beamforming, we use the eigen-subspace beamforming as an example. By using SVD, a MIMO channel $\mathbf{H}$ can be decomposed into

$$\mathbf{H} = \mathbf{UDV}^{\mathrm{H}} \tag{2}$$

where $\mathbf{U}$ of size $N_R \times R$ and $\mathbf{V}$ of size $N_T \times R$ are both unitary matrices, and $\mathbf{D}$ is an $R \times R$ diagonal matrix consisting of the singular values of $\mathbf{H}$ as its diagonal elements, and R is the rank of $\mathbf{H}$. In order to perform eigen-subspace beamforming, $\mathbf{V}$ needs to be fed back to the base station. An effort to reduce the amount of information in $\mathbf{V}$ was reported in [11-13] where a matrix $\boldsymbol{\Sigma}$ was multiplied with $\mathbf{V}$ to form $\bar{\mathbf{V}}$, such that the last row of $\bar{\mathbf{V}}$ consists of only real numbers. Hence, we may re-express (2) as:

$$\mathbf{H} = \mathbf{UD\Sigma}\,\bar{\mathbf{V}}^{\mathrm{H}} = \mathbf{U}\,\bar{\mathbf{D}}\,\bar{\mathbf{V}}^{\mathrm{H}}\text{where } \bar{\mathbf{D}} = \mathbf{D\Sigma} \tag{3}$$

where

$$\boldsymbol{\Sigma} = \mathrm{diag}\left[\exp\left\{j * \arg\left(\vec{\mathbf{v}}_{N_T}^{\mathrm{H}}\right)\right\}\right] \tag{4}$$

and $\vec{\mathbf{v}}_{N_T}^{\mathrm{H}}$ represents the last column of $\bar{\mathbf{V}}^{\mathrm{H}}$.

$$\mathbf{y} = \mathbf{Hx} + \mathbf{n} = \left(\mathbf{U}\,\bar{\mathbf{D}}\,\bar{\mathbf{V}}^{\mathrm{H}}\right)\mathbf{x} + \mathbf{n} \tag{5}$$

To transmit data in the first K eigen modes (where $K \leq R$), the beamforming matrix $\mathbf{W}$ is simply the first K column vectors of $\bar{\mathbf{V}}$:

$$\mathbf{W} = \bar{\mathbf{V}}_{(1:K)} \tag{6}$$

The transmitted signal is related to the $K \times 1$ data signal $\mathbf{u}$ by

$$\mathbf{x} = \mathbf{Wu} \tag{7}$$

In order to retrieve $\mathbf{u}$, the mobile client multiplies the received signal with $\mathbf{U}^{\mathrm{H}}$,

$$\hat{\mathbf{u}} = \mathbf{U}^{\mathrm{H}}\mathbf{y} = \bar{\mathbf{D}}_{(1:K)}\mathbf{u} + \tilde{\mathbf{n}} \tag{8}$$

where $\tilde{\mathbf{n}}$ has the same statistics as $\mathbf{n}$ (as $\mathbf{U}$ is a unitary matrix). Since $\bar{\mathbf{D}}$ is a diagonal matrix, eigen-

beamforming leads to simple decoding, as the MIMO channel can be treated as a number of parallel subchannels.

In practice, due to the limited bandwidth in the feedback channel, $\mathbf{W}$ has to be quantized, and the base station receives the quantized version of $\mathbf{W}$, denoted by $\tilde{\mathbf{W}}$. We assume that the channels are estimated accurately, and there is no error or delay in the feedback channel. With these assumptions we consider only the impact of quantization error due to limited feedback bandwidth. Hence, $\tilde{\mathbf{W}}$, instead of $\mathbf{W}$, will be used as the beamforming matrix. In this article, we propose an effective method to quantize $\mathbf{W}$ and it will be shown that we can achieve a better performance than that of existing methods using the same average number of feedback bits.

GR model

Before we illustrate how the new proposed approach can easily be applied to the GR, we give a brief review on the GR. A unitary matrix, such as $\mathbf{W}$ in our case, can be represented as follows:

$$\mathbf{W} = \prod_{i=1}^{\min(N_T-1,K)}\left[\mathbf{D}_i\left(1_{i-1}\quad e^{j\phi_{1,i}}\quad \cdots \quad e^{j\phi_{N-1,i}}\right)\prod_{l=i+1}^{N_T}\mathbf{G}_{li}^{\mathrm{T}}(\psi_{li})\right] \times \mathbf{I}_{N_T \times K} \tag{9}$$

where $\mathbf{D}_i$ is a diagonal matrix and $\mathbf{G}$ is defined as

$$\mathbf{G}_{li}(\psi_{li}) = \begin{bmatrix} \mathbf{I}_{i-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \cos(\psi_{li}) & \mathbf{0} & \sin(\psi_{li}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{l-i-1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\sin(\psi_{li}) & \mathbf{0} & \cos(\psi_{li}) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{N_T-1} \end{bmatrix} \tag{10}$$

Take a 3×2 unitary matrix $\mathbf{W}$ as example, it can be described as

$$\mathbf{W} = \begin{bmatrix} e^{j\phi_{11}} & 0 & 0 \\ 0 & e^{j\phi_{21}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \mathbf{G}_{21}^{\mathrm{T}}(\psi_{21})\mathbf{G}_{31}^{\mathrm{T}}(\psi_{31})$$
$$\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{j\phi_{22}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \mathbf{G}_{32}^{\mathrm{T}}(\psi_{32}) \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{11}$$

Hence, the 3×2 unitary matrices $\mathbf{W}$ can fully be described by just six parameters: ϕ_{11}, ϕ_{21}, ψ_{21}, ψ_{31}, ϕ_{22}, and ψ_{32}. A 3×1 unit-norm vector only needs four parameters, namely ϕ_{11}, ϕ_{21}, ψ_{21}, and ψ_{31}. Whereas for 2×1 and 2×2 cases, two parameters, ϕ_{11} and ψ_{21}, will be sufficient. The full details can be founded in [1,4].

There are four combinations of bits assigned to the GR parameters in the IEEE 802.11n draft. They can be summarized as follows in the format of (b_ψ, b_ϕ), namely (1,3), (2,4), (3,5), and (4,6), where b_ψ represents the number of bits assigned to ψ, and b_ϕ represents the number of bits assigned to ϕ. Parameter ψ has a range

from 0 to $\pi/2$ whereas ϕ spans over a range from 0 to 2π [1,4].

Using the above bit assignment, ψ and ϕ can be quantized according to (12) (where $\widetilde{\psi}$ and $\widetilde{\phi}$ represent the quantized version of ψ and ϕ, respectively).

$$\widetilde{\psi} = \frac{k\pi}{2^{b_\psi+1}} + \frac{\pi}{2^{b_\psi+2}} \text{ where } k = 0, 1, \ldots, 2^{b_\psi} - 1 \quad (12)$$

$$\widetilde{\phi} = \frac{k\pi}{2^{b_\phi-1}} + \frac{\pi}{2^{b_\phi}} \text{ where } k = 0, 1, \ldots, 2^{b_\phi} - 1 \quad (13)$$

The beamforming matrix $\widetilde{\mathbf{W}}$ can be recovered at the base station by using (14):

$$\widetilde{\mathbf{W}} = \prod_{i=1}^{\min(N_T-1,K)} \left[\mathbf{D}_i \left(1_{i-1} \quad e^{j\widetilde{\phi}_{1,i}} \quad \ldots \quad e^{j\widetilde{\phi}_{N_T-1,i}} \right) \prod_{l=i+1}^{N_T} \mathbf{G}_{li}^{\mathrm{T}}(\widetilde{\psi}_{li}) \right] \times \mathbf{I}_{N_T \times K} \quad (14)$$

Methods

The three basic ideas of the proposed scheme are as follows:

A. *Dynamic bit assignment*:
   The bits assigned to the GR parameter ϕ can be made dependent on the value of ψ. When the resolution is "sparse" (which can be predetermined based on the value of ψ), we use more bits for the quantization of ϕ; when the resolution is "crowded", we use fewer bits for the quantization of ϕ. In other words, the bit assignment to ϕ is adaptively adjusted based on the value of ψ.

B. *Efficient source coding*:
   Due to the non-uniform distribution of the GR parameters ψ, efficient source coding such as the Huffman code [14] can be used to efficiently encode the GR parameter ψ and hence reducing the number of feedback bits required.

C. *Codebook design*:
   Due to the same reason of non-uniform distribution, instead of quantizing the GR parameter ψ in a uniform manner, codebook can be designed so as to quantize the parameter in a more effective manner.

Depending on the receiver structure or the design criteria, we can apply each of these ideas separately or jointly. We will illustrate each of the above in more details.

Dynamic bit assignment

To illustrate the idea of dynamic bit assignment, it is best to make use of a simple example of 2×1 beamforming vector. Consider a 2×1 unit-norm vector $\mathbf{w}$ as shown in (15), due to the unit-norm property, it must

satisfy the constraints in (16). In addition, there is a matching between the *GR* view point and the *Geometry* view point, i.e., both r_1 and r_2 are related to ψ_{21} by $r_1 = \cos(\psi_{21})$ and $r_2 = \sin(\psi_{21})$.

$$\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$
$$= \begin{bmatrix} r_1 e^{j\phi_{11}} \\ r_2 \end{bmatrix} \quad \leftarrow \text{Geometry view point}$$
$$= \begin{bmatrix} \cos\psi_{21} e^{j\phi_{11}} \\ \sin\psi_{21} \end{bmatrix} \leftarrow \text{Givens Rotation view point}$$

$$\quad (15)$$

Since $\mathbf{w}$ is a unit-norm vector, it must satisfy:

$$\begin{aligned} w_1^2 + w_2^2 &= 1 \\ \Rightarrow r_1^2 + r_2^2 &= 1 \end{aligned} \quad (16)$$

Based on the geometry view point in (15) and the constraints in (16), if we want to reduce the quantization error between the $\widetilde{\mathbf{w}}$ (quantized $\mathbf{w}$) with $\mathbf{w}$, we obtain the following insights:

- When r_2 is large (ψ_{21} is large), r_1 will be small; ϕ_{11} can have a lower resolution.
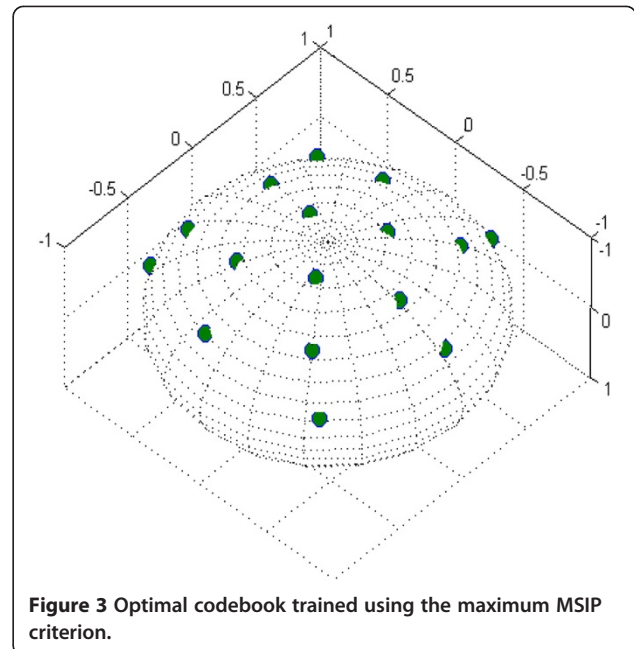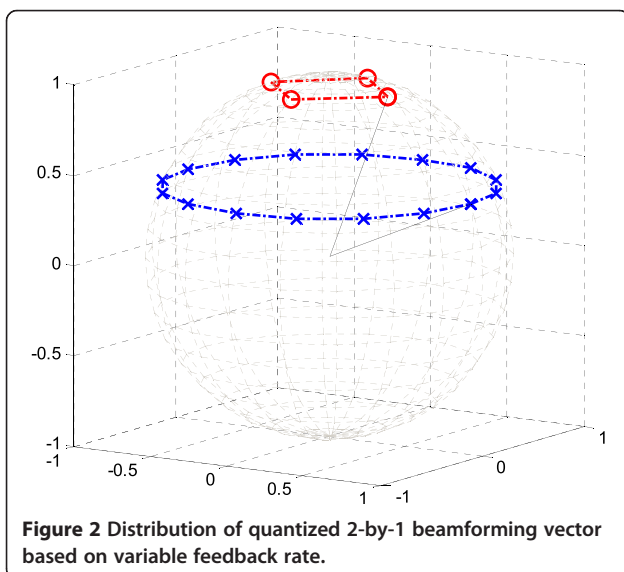- When r_2 is small (ψ_{21} is small), r_1 will be large; ϕ_{11} will need a higher resolution.

Hence, the number of bit assigned to ϕ_{11} should be a function of r_1 and r_2, which is in turn related to the value of ψ_{21} when GR is in use. This observation can be further illustrated in Figures 1 and 2. As shown in Figure 1, the radii of the two circles represent two possible values of r_1. This is equivalent to one bit assignment to ψ_{21}. When r_2 is small, r_1 will be large, as shown by the blue circle marked with "×" at the bottom of Figure 1. When r_2 is large, r_1 will be small, noting the red circle marked with "O" at the top of Figure 1. It can be seen that in this case, if we assign the same number of bits (e. g., 3 bits) to ϕ_{11}, it corresponds to eight points on each circle. The points on the upper circle are closer to each other, while the points on the lower circle are further apart. In this case, the total number of bits to represent $\mathbf{w}$ is $1 + 3 = 4$ bits. And this is the standard way of quantizing $\mathbf{w}$, e.g., in the 802.11n standard.

To achieve a lower quantization error, we can assign different number of bits to ϕ_{11} according to the value of r_1. For example, as shown in Figure 2, we assign two bits to ϕ_{11} when r_1 is small (i.e., ψ_{21} is large, the upper circle), and we assign 4 bits to ϕ_{11} when r_1 is large (i.e., ψ_{21} is small, the lower circle). It can be seen that the distance between the points are more evenly distributed in this case. Depending on the assignment of ψ_{21}, we can make the probability of having two cases (upper or lower circle) equal. Hence, in this scenario, the total number of bits

Figure 1 Distribution of quantized 2-by-1 beamforming vector based on fixed feedback rate.
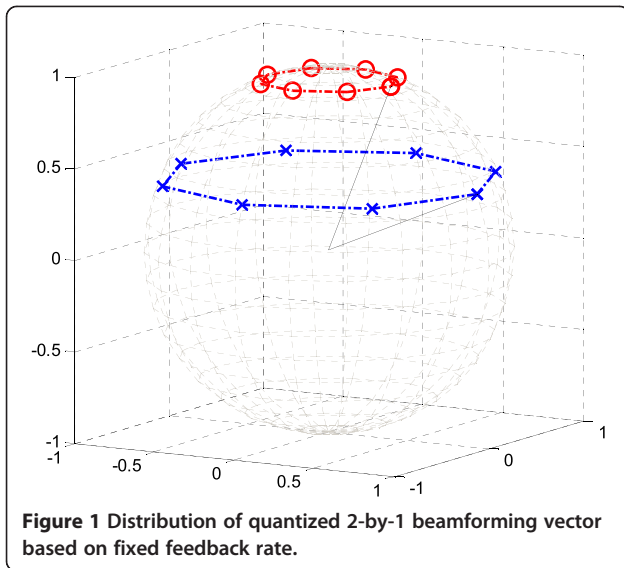
representing **w** is $1 + 2 = 3$ or $1 + 4 = 5$, which is 4 bits on average (if the probability ψ_{21} to be large and small, hence the probability of upper and lower circle are the same).

An optimal codebook obtained by VQ methodology as described in [5] is shown in Figure 3. The "optimum" codebook design criterion is to maximize the mean squared inner product (MSIP) of the beamforming vector and the codebook vectors. A modified form of the generalized Lloyd algorithm was used to train the codebooks used in the comparison. During the iterative training process, the nearest neighborhood condition was satisfied by identifying the partition cell R_i corresponding to the ith codebook vector that generates the highest MSIP with the training vector $\boldsymbol{v}$. At the end of

Figure 2 Distribution of quantized 2-by-1 beamforming vector based on variable feedback rate.

Figure 3 Optimal codebook trained using the maximum MSIP criterion.

an iteration, the principal eigenvector of $E[\boldsymbol{v}\boldsymbol{v}^{\mathrm{T}}|\boldsymbol{v}\in R_i]$ was computed and became the ith codebook vector. From Figure 3, it can be seen that when using a variable feedback rate the codebook that appears in Figure 2 is closer to the optimal codebook than the one with a fixed feedback rate as shown in Figure 1.

Efficient source coding

In Figure 4a–f, we show the distribution of the GR parameters for a unitary beamforming matrix of dimension 3 × 2 when H is Rayleigh i.i.d. fading channel. Through our studies of 5,000 channel matrices, we observe that the parameter ϕ exhibits a uniform distribution from 0° to 360° (i.e., 0 to 2π), while the parameter ψ has a non-uniform distribution from the range of 0° to 90° (i.e., 0 to $\pi/2$). We also observe an asymmetric distribution for ψ_{31} and a symmetric distribution for ψ_{21} and ψ_{32}. For channel matrices with other dimensions, we have similar observation.

The distributions of the quantized version of ψ when using four levels of granularity are shown in Figure 4g–i. So, we should use less bits to source code those values with higher occurring probability, and more bits to source code those values with lower occurring probability. One possibility is the use of Huffman source coding [14].

Codebook design

Due to the non-uniform and asymmetric distribution of some of the parameters, instead of quantizing the GR parameters uniformly, we can design a codebook so as to reduce the quantization error. For example, instead of quantizing ψ_{31} uniformly with 2 bits using the value
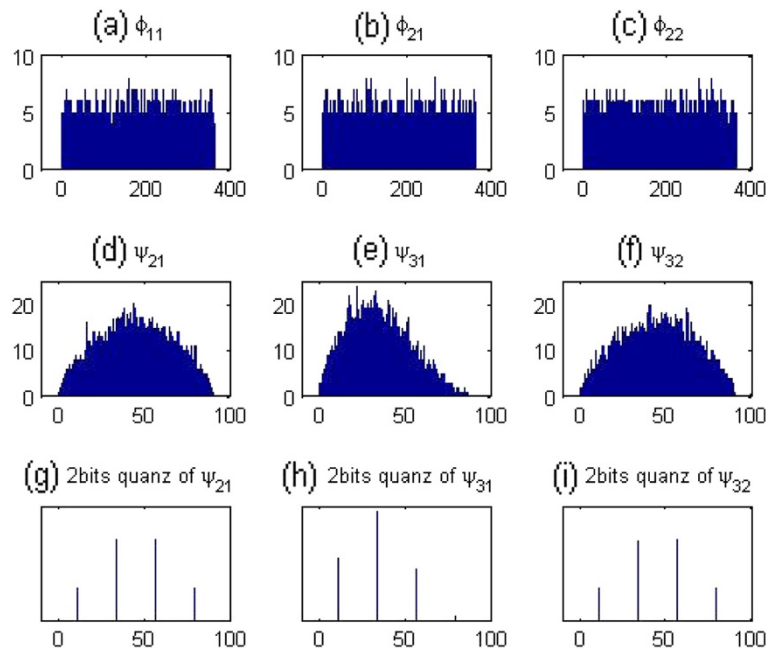
Figure 4 Distribution of the GR parameter for a 3 × 2 beamforming matrix.

[11.25, 33.75, 56.25, 78.75], we can use a codebook [8, 25, 41, 62] that is also 2 bits. As shown in Figure 4, ψ_{31} has a distribution that concentrate to the left-hand side (i.e., higher chances for smaller value), hence our codebook of [8, 25, 41, 62] also tends to have a lower value than the uniform codebook of [11.25, 33.75, 56.25, 78.75].

The above three techniques can be combined and optimized by certain design criteria, which can be a function of receiver. We will perform two case studies in the following sections, one based on the techniques A and C, and another based on the techniques A and B.

Case studies

Depending on the training symbol placement and the receiver design, we consider two cases. In the first case, a simple receiver is not retrained with the beamforming matrix, hence the receiver does not take the mismatch of the quantized beamforming matrix into account, and it simply uses a parallel decoder. On the other hand, in the second case, the receiver is retrained with the updated beamforming matrix, hence the mismatch between the quantized beamforming matrix and the channel is taken into account. It uses a more complicated receiver, such as an MMSE receiver.

Receiver with simple parallel decoder

In this section, we consider the receiver as stated in (8), which is repeated below by taking into account the mismatch in the beamforming matrix with the channel

$$\hat{\mathbf{u}} = \bar{\mathbf{D}}^{-1}\mathbf{U}^H\mathbf{y} = \bar{\mathbf{V}}\widetilde{\mathbf{W}}\mathbf{u} + \widetilde{\mathbf{n}} \qquad (17)$$

Due to quantization, $\bar{\mathbf{V}}\widetilde{\mathbf{W}}$ is no longer an identity matrix, therefore such a simple receiver should be highly sensitive to the quantization error.

In this section, we demonstrate in details how the proposed scheme works for a 3 × 1 beamforming vector **w** as shown in (18). In Table 1, we show the bit assignment

Table 1 Number of bits allocation for three transmit antennas beamforming vector

Average feedback bits	Bits allocated								
	Givens				Proposed				
	$b_{\psi_{31}}$	$b_{\psi_{21}}$	$b_{\varphi_{21}}$	$b_{\varphi_{11}}$	$b_{\psi_{31}}$	$b_{\psi_{21}}$	$b_{\varphi_{21}}$	$b_{\varphi_{11}}$	Remark
8	1	1	3	3	1	1	2, 3, 4	2, 3, 4	Table 2
12	2	2	4	4	2	2	3, 4, 5, 6	3, 4, 5, 6	Table 3

Table 2 Bit allocation for φ_{21} and φ_{11} when the average number of feedback bit is 8

Bit representative of		Bits allocated for φ_{21} and φ_{11}		Total number of feedback bits
ψ_{31}^{*}	ψ_{21}^{*}	$b_{\varphi_{21}}$	$b_{\varphi_{11}}$	
0	0	3	4	9
0	1	4	3	9
1	0	2	3	7
1	1	3	2	7

*Possible values for ψ_{31} and ψ_{21} are [17 50] degree.

Table 3 Bit allocation for φ_{21} and φ_{11} when the average number of feedback bit is 12

Bit representative of		Bits allocated for φ_{21} and φ_{11}		Total number of feedback bits
ψ_{31}^*	ψ_{21}^*	$b_{\varphi_{21}}$	$b_{\varphi_{11}}$	
00	00	3	6	13
00	01	4	5	13
00	10	5	4	13
00	11	6	3	13
01	00	3	5	12
01	01	4	4	12
01	10	4	4	12
01	11	5	3	12
10	00	3	5	12
10	01	4	4	12
10	10	4	4	12
10	11	5	3	12
11	00	3	4	11
11	01	3	4	11
11	10	4	3	11
11	11	4	3	11

*Possible values for ψ_{31} and ψ_{21} are [8 25 41 62] degree.

for the GR of the traditional and the proposed schemes. For example, in an average of 8 bits feedback configuration, the traditional scheme allocates 1 bit each to ψ_{21} and ψ_{31}, and 3 bits each to ϕ_{11} and ϕ_{21}. In the proposed scheme, 1 bit is assigned to ψ_{21} and ψ_{31}, but 2, 3, or 4 bits to ϕ_{11}, with the actual assignment (whether 2, 3, or 4 bits) depending on the value of ψ_{21} and ψ_{31}, as shown in Table 2. For an average feedback rate of 12 bits, the actual bit assignment is shown in Table 3.

The quantization and reconstruction of ϕ are based on the formula in (13). For the ψ parameter, since ψ_{31} and ψ_{21} are not uniformly distributed as shown in Figure 4, we design a codebook for those two parameters, the codebook used are shown as the footnote to Tables 2 and 3. Hence, in this case study, we have used the technique "*dynamic bit assignment*" and "*codebook design*" that have been mentioned in the previous section.

$$\mathbf{w} = \begin{bmatrix} r_1 e^{j\phi_{11}} \\ r_2 e^{j\phi_{21}} \\ r_3 \end{bmatrix} = \begin{bmatrix} \cos\psi_{21}\cos\psi_{31}e^{j\phi_{11}} \\ \sin\psi_{21}\cos\psi_{31}e^{j\phi_{21}} \\ \sin\psi_{31} \end{bmatrix} \quad (18)$$

Since the receiver is not retrained with the beamforming matrix, the quantization error will be critical in this case. We first compare the error in quantization by using mean square error (MSE) in (19) or mean angular distance (MAD) [6] in (20).

$$MSE = E|\mathbf{w} - \widetilde{\mathbf{w}}|^2 \quad (19)$$

$$MAD = E\sqrt{1 - |\mathbf{w} \cdot \widetilde{\mathbf{w}}|^2} \quad (20)$$

where • in (20) denotes the dot product operation.
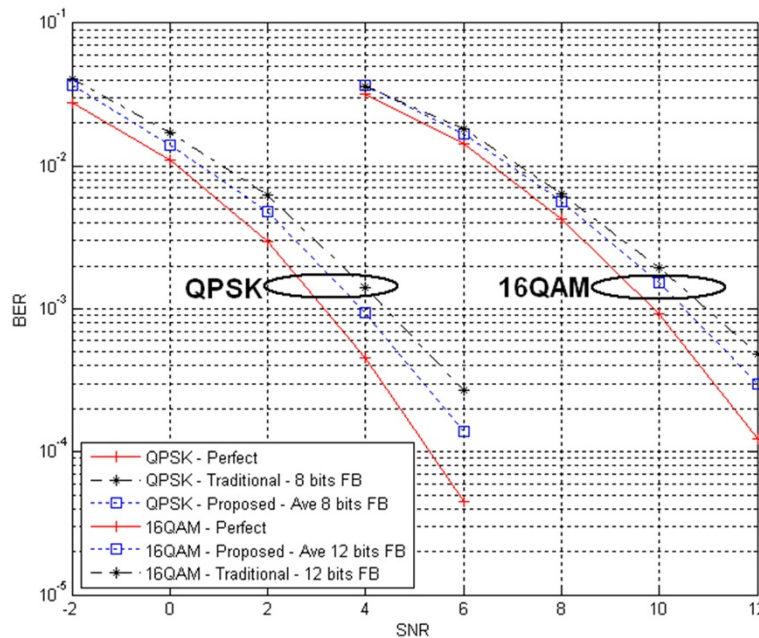
Figure 5 BER performance for 3 tx 3 rx, one stream based on SVD beamforming: QPSK with average 8 bits feedback and 16QAM with average 12 bits feedback.

Table 4 Huffman code for GR parameters

GR parameters ψ_{21} and ψ_{32}				
Quantized value of ψ_{21} or ψ_{32}	Probability (i)	Huffman code	Bits for ψ_{21} or ψ_{32} (ii)	Ave bits for ψ_{21} or ψ_{32} (i) × (ii)
11.25	0.14714	110	3	0.44142
33.75	0.35496	0	1	0.35496
56.25	0.35146	10	2	0.70292
78.75	0.14644	111	3	0.43932
				1.93862
GR parameters ψ_{31}				
Quantized value of ψ_{31}	Probability (i)	Huffman code	Bits for ψ_{31} (ii)	Ave bits for ψ_{31} (i) × (ii)
11.25	0.2722	10	2	0.5444
33.75	0.47748	0	1	0.47748
56.25	0.2299	110	3	0.6897
78.75	0.02042	111	3	0.06126
				1.77284

MSE readings for quantization of 3×1 beamforming vector based on the traditional fixed rate feedback approach versus that of the newly proposed scheme based on variable rate feedback are as follows: MSE of 0.11 versus 0.091 (for 8 bits feedback) and MSE of 0.03 versus 0.028 (for 12 bits feedback). MAD readings are as follows: MAD of 0.31 versus 0.282 (for 8 bits feedback) and MAD of 0.162 versus 0.156 (for 12 bits feedback). Hence, the proposed scheme always achieves a lower MSE and MAD than the traditional scheme for both cases of average 8 or 12 bits feedback.

The BER performance is shown in Figure 5 for QPSK (with average 8 bits feedback) and 16QAM (with average 12 bits feedback) modulated 3×3 MIMO, respectively. It can be seen that the proposed scheme outperforms the traditional approach. Notably such a performance gain is achieved without any additional feedback bandwidth.

Receiver with MMSE detector

In this section, we consider a different receiver from that in (8). It is assumed that the mismatch in the beamforming matrix is known, and we apply the MMSE detector as shown:

$$\hat{\mathfrak{u}} = \left(\mathbf{G}^{H}\mathbf{G} + \alpha\mathbf{I}\right)^{-1}\mathbf{G}^{H}\mathbf{y} \qquad \mathbf{G} = \mathbf{H}\,\tilde{\mathbf{W}} \tag{21}$$

In this case, we consider a three transmit antennas and two streams of data. Due to the non-uniform

distribution of the parameters ψ as shown in Figure 4, we can make use of Huffman coding [14] to encode the quantized value of ψ. Hence, we make use of the techniques "*dynamic bit assignment*" and "*efficient source coding*" discussed earlier in this case study.

As shown in Table 4, it is sufficient to represent the quantized version of ψ_{21} (or ψ_{32}) and ψ_{31} using 1.94 and 1.77 bits (instead of 2 bits for a granularity of four).

It is also found that when $\psi_{21} = 33.75$, 56.25, and $\psi_{31} = 11.25$, 33.75, the quantized version of the beamforming matrix will have higher chances of being poor in quality, hence we suggest that we use 3 bits to quantize ϕ_{11} and ϕ_{21} when the above-mentioned condition occurs. For the rest of the case, we simply use 2 bits to quantize ϕ_{11} and ϕ_{21}. As shown in Table 5, it can be seen that 5.06 bits on average were required for the quantization of ϕ_{11} and ϕ_{21} in our proposed scheme.

Combining everything, the average number of bits required to represent a 3×2 unitary beamforming matrix using our proposed scheme can be computed as the following:

$$\text{Ave bits required}: \ \text{E}\left[b_{\psi_{21}}\right] + \text{E}\left[b_{\psi_{31}}\right] + \text{E}\left[b_{\phi_{11}} + b_{\phi_{21}}\right] + \text{E}\left[b_{\psi_{32}}\right] + \text{E}\left[b_{\phi_{22}}\right]$$
$$= 1.93862 + 1.77284 + 5.0592 + 1.93862 + 2b$$
$$= 12.7093 \tag{22}$$

We compare five different quantization schemes in Table 6. It can be seen that our proposed scheme

Table 5 Bits assignment for GR parameters φ_{11} and φ_{21}

Bits for φ_{11} (i)	Bits for φ_{21} (ii)	Condition	Probability (iii)	Ave bits for φ_{11} and φ_{21} ((i) + (ii)) * (iii)
3	3	$\psi_{21} = 33.75$, 56.25 $\psi_{31} = 11.25$, 33.75	(0.35496 + 0.35146) * (0.2722 + 0.47748) = 0.5296	3.1776
2	2	Otherwise	1 − 0.5296 = 0.4704	1.8816
				5.0592

Table 6 Five schemes for comparisons

	Bit assignment	Ave number of feedback bits
Scheme A	Perfect feedback	∞
Scheme B	$b_\psi = 1$ and $b_\varphi = 3$	12
Scheme C	$b_\psi = 2$ and $b_\varphi = 2$	12
Scheme D	$b_\psi = 2$ and $b_\varphi = 3$	15
Scheme E (proposed)	Tables 4 and 5	12.71

(Scheme E with average 12.71 bits of feedback) achieve similar performance as Scheme D that requires 15 bits of feedback. Hence, we save about 2 bits of feedback, and this saving can be significant when there is a large number of subcarriers, especially in the future broadband system. Compare to Schemes B and C, we achieve a better performance with 0.71 additional bits in the average number of feedback bit.

The simulation results of the BER for the five schemes are shown in Figure 6. We assume three transmit antennas and three receive antennas with eigen beamforming and two data streams: one 64QAM and the other 16QAM, hence a 10 bits/s/Hz spectral efficiency is achieved. It can be seen that Schemes B and C gave the worst BER, while Scheme A delivered the best performance in terms of BER. Scheme D performed better than Schemes B and C by using three more average feedback bits. Our proposed scheme can capture most of the gain that can be provided by Scheme D, but we require only 0.71 additional feedback bits compared to the additional 3 bits required by Scheme D.

The newly proposed scheme can be considered as a hybrid of the traditional GR approach and VQ-based approach, i.e., we have a code book for the GR parameters ϕ and ψ. However, the new scheme has a lower storage requirement than those based on VQ codebooks.

The assignment of the number of bits and the codebook design in these case studies are just for illustration, there could be other assignment methods that lead to better performance, and different receiver or different system design may lead to different design criteria.

Conclusions

In this article, a simple quantization scheme has been presented for the unit-norm beamforming vector or unitary beamforming matrix based on variable-rate feedback. The basic idea is to provide for higher resolution in the dense area and lower resolution in the sparse area. The idea can directly be applied to the existing GR approach allocating variable bits to the ϕ parameter according to the value of ψ. Due to the non-uniform distribution of the GR parameter ψ, the performance can be further improved if we incorporate into the system efficient source coding and codebook design for GR parameters. Results show that the proposed scheme can achieve a lower MSE and lower MAD. The BER performance of the close-loop MIMO system based on the proposed quantization scheme also outperforms that of existing schemes.

The proposed idea is not restricted to the use of eigen-beamformer or GR which have been used as the
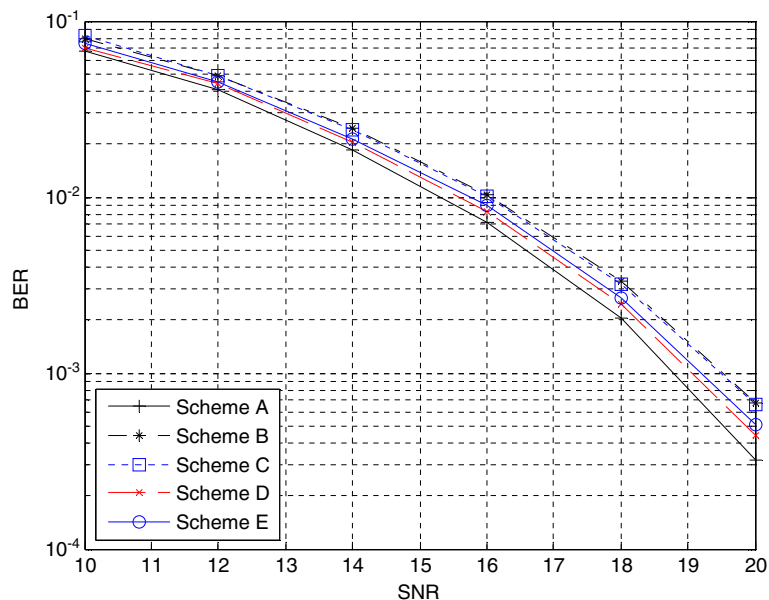
Figure 6 BER performance for 3tx 3rx two streams based on SVD beamforming with feedback bit assignment in Table 6. First stream with 64QAM and second stream with 16QAM, total spectral efficiency 10 bps/Hz.

baseline for comparison. Our proposed method gives a better accuracy when compressing a unit-norm vector or unitary matrix, and such accuracy plays an important role in many communications system including precoding for multi-user MIMO.

Competing interests
The authors declare that they have no competing interests.

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