# A METHOD OF SOLVING $y^{(k)} - f(x)y = 0$

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(Received June 4, 1991)

**ABSTRACT.** An alternative method is shown for solving the differential equation $y^{(k)} - f(x)y = 0$ by means of series. Also included is a result for a sequence of functions $\{S_n(x)\}_{n=1}^{\infty}$ which gives conditions under which $\lim_n\left(\frac{d^k}{dx^k}S_n(x)\right) = \frac{d^k}{dx^k}\left(\lim_n S_n(x)\right)$.

**KEY WORDS AND PHRASES.** Differential equations, sequence of functions.
**1991 AMS SUBJECT CLASSIFICATION CODE.** 40A30.

## 1. Introduction

Consider the differential equation $y'' - f(x)y = 0$ for $a < x < b$ where $f$ is a given function continuous on $a \le x \le b$. If $f$ is analytic then the method of power series may be used to solve for $y$. However, for more general $f$, heuristics suggest that one "iterate" to a solution by finding a sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ that satisfy $f_n''(x) = f_{n-1}(x)f(x)$. Then possibly $\sum_{n=1}^{\infty} f_n(x)$ is a solution. Under suitable hypothesis this is indeed the case. The results can be generalized to the differential equation $y^{(k)} - f(x)y = 0$ as shown in Theorem A. The proof depends on an interesting result, Theorem 1, which gives conditions that insure that the limit of the $k^{\text{th}}$ derivative is the $k^{\text{th}}$ derivative of the limit. Theorem 1 generalizes the usual result found in Advanced Calculus books for differentiating the limit of a sequence of functions. We also include two examples that illustrate the method of solution when $k = 2$.

## 2. Statement of Theorems

**Theorem A.** Suppose $f$ is continuous on $[a,b]$, $c \in [a,b]$, and $k$ is a natural number. Define the sequence of functions $\{f_n(x)\}_{n=0}^{\infty}$ by

$$f_0(x) = a_0^{(0)} + a_1^{(0)}x + \ldots + a_{k-1}^{(0)}x^{k-1} \neq 0,$$

$$f_n(x) = \int_c^x \int_c^{u_{k-1}} \cdots \int_c^{u_1} f_{n-1}(u) \cdot f(u)\,du\,du_1 \ldots du_{k-1} + \sum_{j=0}^{k-1} a_j^{(n)}x^j, \quad n = 1,2,\ldots .$$

where $a_0^{(n)}, a_1^{(n)}, \ldots, a_{k-1}^{(n)}$, are constants, $n = 0,1,2,\ldots$ (Note $f_n(x)$ is any $k^{\text{th}}$ antiderivative of $f_{n-1}(x)f(x)$.)

If the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on $[a,b]$ to some function $S(x)$ then $\sum_{n=0}^{\infty} f_n^{(j)}(x)$ converges uniformly to $S^{(j)}(x)$ for $a \le x \le b$, $j = 1,2,\ldots,k$, and $S^{(k)}(x) = S(x) \cdot f(x)$ on $[a,b]$.

**Remark:** All derivatives at the endpoints $a$ and $b$ are necessarily one sided.

As mentioned, the proof of Theorem A depends on the following interesting result for sequences of differentiable functions:

**Theorem 1.** Suppose (i) $\{S_n(x)\}_{n=1}^{\infty}$ is a sequence of real functions defined on an interval $[a,b]$ and $k$ is a natural number;

(ii) $S_n'(x), S_n''(x), \ldots, S_n^{(k)}(x)$ exist at each $x \in [a,b]$, $n = 1,2,\ldots$

(iii) $\{S_n^{(k)}(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$;

(iv) either there is a $c \in [a,b]$ such that each of $\{S_n(c)\}_{n=1}^{\infty}, \{S_n'(c)\}_{n=1}^{\infty}, \ldots, \{S^{(k-1)}(c)\}_{n=1}^{\infty}$ converge or there are distinct points $c_1, \ldots, c_k$ such that each of $\{S_n(c_1)\}_{n=1}^{\infty}, \{S_n(c_2)\}_{n=1}^{\infty}, \ldots, \{S_n(c_k)\}_{n=1}^{\infty}$ converge.

Then

each of the sequences $\{S_n^{(j)}(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to differentiable functions, $j = 0,1,2,\ldots,k-1$, and

$$\frac{d^j}{dx^j}\left(\lim_{n\to\infty} S_n(x)\right) = \lim_{n\to\infty}\left(\frac{d^j}{dx^j} S_n(x)\right), \quad j = 1,2,\ldots,k .$$

## 3. Discussion and Proofs

In order to prove Theorem 1 we need some preliminary results. First is a standard result from Advanced Calculus.

**Theorem 0.** Suppose that $\{S_n(x)\}_{n=1}^{\infty}$ is a sequence of real functions differentiable on an interval $a \le x \le b$ and such that

(i) $\{S_n'(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$;

(ii) $\{S_n(c)\}_{n=1}^{\infty}$ converges for some $c \in [a,b]$.

Then $\{S_n(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to a function $S(x)$, and $\frac{d}{dx}\left(\lim_{n\to\infty} S_n(x)\right) = S'(x) = \lim_{n\to\infty}\left(\frac{d}{dx} S_n(x)\right)$, $a \le x \le b$.

For a justification of Theorem 0, see [1], pp. 451-2.

Also required is the following

**Lemma.** Suppose $k$ is a natural number, $\{p_n(x)\}_{n=1}^{\infty}$ is a sequence of polynomials each of degree $\le k$, and $c_1, \ldots, c_{k+1}$ are $k+1$ distinct numbers. If $\{p_n(c_j)\}_{n=1}^{\infty}$ converges for $j = 1, \ldots, k+1$ then $\{p_n(x)\}_{n=1}^{\infty}$ converges for each $x \in \mathbf{R}$ to a polynomial $h(x)$ where either $h(x) \equiv 0$ or degree of $h(x)$ is $\le k$, and convergence is uniform on each bounded closed interval in $\mathbf{R}$. Moreover, $\lim_n p_n^{(\nu)}(x) = h^{(\nu)}(x)$, $\nu = 1, \ldots, k, x \in \mathbf{R}$.

**Proof:** Let $Q(x) = (x-c_1)(x-c_2)\ldots(x-c_{k+1})$. Using the Lagrange Interpolation formula, we have for each $n$,

$$p_n(x) = \sum_{j=1}^{k+1} \frac{p_n(c_j)Q(x)}{Q'(c_j)(x-c_j)} . \tag{2.1}$$

Clearly for each $x$,

$$h(x) := \lim_{n\to\infty} p_n(x) - \sum_{j=1}^{k+1} \frac{\left[\lim_n p_n(c_j)\right] Q(x)}{Q'(c_j)(x-c_j)}$$

exists and is finite; moreover $h(x)$ is a polynomial of degree $\le k$. For each $x$ in some interval $[a,b]$,

$$|p_n(x)-h(x)| \le \sum_{j=1}^{k+1} \left| \frac{(p_n(c_j)-h(c_j))Q(x)}{Q'(c_j)(x-c_j)} \right| \le M \sum_{j=1}^{k+1} |p_n(c_j)-h(c_j)| \tag{2.2}$$

where $M > 0$ is such that

$$\max_{a\le x\le b} \left| \frac{Q(x)}{Q'(c_j)(x-c_j)} \right| \le M \quad \text{for} \quad j = 1,2,\dots,k+1 .$$

Uniform convergence follows from inequality (2.2). By first differentiating (2.1) and then passing to the limit with $n$ we obtain $\lim_n p_n^{(\nu)}(x) = h^{(\nu)}(x), x \in \mathbf{R}$. ∎

We proceed to the

**Proof of Theorem 1.** We use induction on $k$. The case $k = 1$ is given by Theorem 0. So assume the theorem holds for $k-1 \ge 1$ and let $\{S_n(x)\}_{n=1}^{\infty}$ satisfy (i)-(iv). If $\{S_n(c)\}_1^{\infty}$, $\{S_n'(c)\}_1^{\infty}, \dots, \{S_n^{(k-1)}(c)\}_1^{\infty}$, each converge then $\{S_n^{(k-1)}(x)\}_{n=1}^{\infty}$ converges uniformly by Theorem 0 and hence the conclusion follows from the induction hypothesis and Theorem 0. Next suppose $\{S_n(c_1)\}_1^{\infty}, S_n(c_2)\}_1^{\infty}, \dots, \{S_n(c_k)\}_1^{\infty}$ each converge and define

$$G_{n,1}(x) = \int_{c_1}^{x} S_n^{(k)}(u)du = S_n^{(k-1)}(x) - S_n^{(k-1)}(c_1)$$

$$G_{n,2}(x) = \int_{c_2}^{x} G_{n,1}(u)du = S_n^{(k-2)}(x) - S_n^{(k-2)}(c_2) - S_n^{(k-1)}(c_1) \cdot (x-c_2)$$

$$\vdots$$

$$G_{n,k}(x) = \int_{c_k}^{x} G_{n,k-1}(u)du = S_n(x) - p_n(x)$$

where $p_n(x) := S_n(x) - \int_{c_k}^{x} \dots \int_{c_2}^{u_3} \int_{c_1}^{u_2} S_n^{(k)}(u_1)du_1du_2\dots du_k = S_n(x) - G_{n,k}(x)$ is a uniquely determined polynomial of degree $\le k-1$, each $n$. Repeated use of Theorem 0 shows $\{G_{n,j}(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$, for $j = 1,2,\dots,k$, and in particular $\{G_{n,k}(x)\}_{n=1}^{\infty}$ converges uniformly. Since $\{S_n(c_j)\}_{n=1}^{\infty}$ converges then $p_n(c_j)\}_{n=1}^{\infty}$ converges, for $j = 1,\dots,k$. By the lemma the sequence of polynomials $\{p_n(x)\}_{n=1}^{\infty}$ converges uniformly to a polynomial $h(x)$ where either $h(x) \equiv 0$ or degree of $h(x)$ is $\le k-1$, and $\{p_n^{(k-1)}(x)\}_{n=1}^{\infty}$ converges to $h^{(k-1)}(x)$. Because $p_n^{(k-1)}(x) = S_n^{(k-1)}(c_1)$, $n = 1,2,\dots$ then $\{S_n^{(k-1)}(c_1)\}_{n=1}^{\infty}$ converges. It follows that $\{S_n^{(k-1)}(x)\}_{n=1}^{\infty}$ converges uniformly. Also $\lim_{n\to\infty} S_n^{(k)}(x) = \frac{d}{dx}\left(\lim_{n\to\infty} S_n^{(k-1)}(x)\right)$. Now the induction hypothesis can be used and the conclusion obtained. ∎

We are now able to give the

**Proof of Theorem A.** Apply Theorem 1 to the sequence of real functions

$$S_n(x) = f_0(x) + f_1(x) + \dots + f_n(x), \quad n = 0,1,2,\dots.$$

Note that each $S_n(x)$ has $k$ derivatives and

$$S_n^{(k)}(x) = (f_0(x) + f_1(x) + \dots + f_{n-1}(x)) \cdot f(x)$$
$$= S_{n-1}(x) \cdot f(x) \quad \text{for} \quad n \geq 1.$$

By hypothesis $\{S_n(x)\}_{n=1}^{\infty}$ converges uniformly on $[a,b]$ to $S(x)$. Hence $\{S_n^{(k)}(x)\}_{n=1}^{\infty}$ converges uniformly to $S(x) \cdot f(x)$. By Theorem 1, we obtain that the sequence of functions $\{S_n^{(j)}(x)\}_{n=1}^{\infty}$ converges uniformly and $\frac{d^j}{dx^j}\left(\lim_n S_n(x)\right) = \lim_n\left(\frac{d^j}{dx^j} S_n(x)\right)$ for $j = 1,2,\dots,k$. Thus for $a \leq x \leq b$,

$$S^{(k)}(x) = \frac{d^k}{dx^k}\left(\lim_n S_n(x)\right) = \lim_n\left(\frac{d^k}{dx^k} S_n(x)\right)$$
$$= \lim_n (S_{n-1}(x) \cdot f(x)) = S(x)f(x).$$

This proves Theorem A. ■

## 4. Examples and Remarks

We give some applications of Theorem A.

**Example 1:** Consider $y'' - (Ax^k)y = 0$, $a \leq x \leq b$, where $A$, $k$ are constants, $k \geq 0$, and $f(x) = Ax^k$ is continuous and bounded by $M$ on $[a,b]$. We may assume $c = 0 \in [a,b]$ and $|a| \leq |b|$. Let $f_0(x) = 1$ and for $n \geq 1$ let

$$f_1(x) = \frac{Ax^{k+2}}{(k+1)(k+2)}, \quad f_2(x) = \frac{A^2x^{2k+4}}{(k+1)(k+2)(2k+3)(2k+4)}, \dots$$

$$f_n(x) = \frac{A^n x^{nk+2n}}{(k+1)(k+2)(2k+3)(2k+4)\dots(nk+2n-1)(nk+2n)}, \dots$$

Thus $f_n''(x) = f_{n-1}(x) \cdot f(x)$ and $|f_n(x)| \leq \frac{M^n|x|^{2n}}{(2n)!} \leq \frac{M^n b^{2n}}{(2n)!}$ for $a \leq x \leq b, n = 1,2,\dots$. The series $\sum \frac{M^n b^{2n}}{(2n)!}$ converges by the ratio test so $\sum_0^{\infty} f_n(x)$ converges uniformly on $[a,b]$ to a function $S(x)$ by the Weierstrass $M$-test. Now let $g_0(x) = x$ and for $n \geq 1$, let

$$g_1(x) = \frac{Ax^{k+3}}{(k+2)(k+3)}, \quad g_2(x) = \frac{A^2x^{2k+5}}{(k+2)(k+3)(2k+4)(2k+5)}, \dots$$

$$g_n(x) = \frac{A^n x^{nk+2n+1}}{(k+2)(k+3)(2k+4)(2k+5)\dots(nk+2n)(nk+2n+1)}, \dots$$

As before $g_n''(x) = g_{n-1}(x) \cdot f(x)$ and $|g_n(x)| \leq \frac{M^n|b|^{2n+1}}{(2n+1)!}$, $a \leq x \leq b$, $n = 1,2,\dots$ so that $\sum_{n=0}^{\infty} g_n(x)$ converges uniformly on $[a,b]$ to a function $T(x)$. By Theorem 1, $S(x)$ and $T(x)$ are solutions to $y'' - Ax^k y = 0$. Since the Wronskian of $S(x)$ and $T(x)$ is $W(x) = S(x)T'(x) - T(x) \cdot S'(x)$ and $W(0) \neq 0$ then $S(x)$ and $T(x)$ are linearly independent on $[a,b]$, see e.g. [2], pp. 111-113. It follows that the general solution is $C_1S(x) + C_2T(x)$ for constants $C_1$, $C_2$. In particular if $k = 0$ and $A > 0$ then

$$S(x) = \sum_{n=0}^{\infty} \frac{(\sqrt{A}\,x)^{2n}}{(2n)!} = \cosh(\sqrt{A}\,x) \quad \text{and} \quad T(x) = \frac{1}{\sqrt{A}} \sum_{0}^{\infty} \frac{(\sqrt{A}\,x)^{2n+1}}{(2n+1)!} = \frac{\sinh(\sqrt{A}\,x)}{\sqrt{A}}$$

and if $k = 0$ and $A < 0$ then

$$S(x) = \sum_{0}^{\infty} \frac{(-1)^n (\sqrt{|A|}\,x)^{2n}}{(2n)!} = \cos(\sqrt{|A|}\,x) \quad \text{and} \quad T(x) = \frac{1}{\sqrt{|A|}} \sum_{0}^{\infty} \frac{(-1)^n (\sqrt{|A|}\,x)^{2n+1}}{(2n+1)!} = \frac{\sin(\sqrt{|A|}\,x)}{\sqrt{|A|}}$$

These solutions are the same as those obtained by elementary methods.

**Example 2:** Consider $y'' - Ae^{kx}y = 0$, $-a < x < a$, where $A$, $k$ are constants, $a > 0$, $k \neq 0$ and $f(x) = Ae^{kx}$.

Let $f_0(x) = 1$, $$f_1(x) = \frac{A}{k^2}e^{kx}, \quad f_2(x) = \frac{A^2e^{2kx}}{(k^2)(2k)^2}, \ldots$$

$$f_n(x) = \frac{A^n e^{nkx}}{[(k)(2k)\ldots(nk)]^2}, \ldots$$

Then $f_n''(x) = f_{n-1}(x)f(x)$ and $|f_n(x)| \le \frac{(|A|e^{|ka|})^n}{[(k^n)(n!)]^2}$ for $|x| \le a$, $n = 1,2,\ldots$ The series $\sum_{n=1}^{\infty} \frac{(|A|e^{|ka|})^n}{[(k^n)(n!)]^2}$ converges by the Ratio Test so $\sum_{n=0}^{\infty} f_n(x) = 1 + \sum_{n=1}^{\infty} \frac{A^n e^{nkx}}{[(k^n)(n!)]^2}$ converges uniformly on $[-a,a]$ to a function $S(x)$. Now let $c = 0$, $g_0(x) = x$,

$$g_1(x) = \frac{Ae^{kx}}{k^2}\left(x - \frac{2}{k}\right), \quad g_2(x) = \frac{A^2e^{2kx}}{(k)^2(2k)^2}\left\{x - \frac{2}{k}\left(1 + \frac{1}{2}\right)\right\}, \ldots$$

$$g_n(x) = \frac{A^n e^{nkx}}{[(k^n)(n!)]^2}\left[x - \frac{2}{k}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\right], \ldots$$

Then $g_n''(x) = g_{n-1}(x) \cdot f(x)$ and $|g_n(x)| \le \frac{(|A|e^{|ka|})^n}{[(k^n)(n!)]^2}\left[|a| + \frac{2}{k}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right] := b_n$ for $|x| \le a$, $n = 1,2,\ldots$.

Since $\sum b_n$ converges by the Ratio Test then $\left(x + \sum_{n=1}^{\infty} g_n(x)\right)$ converges uniformly on $[-a,a]$ to some $T(x)$.

To see that $S(x)$ and $T(x)$ are linearly independent let $x = x(u) = k^{-1}ln\,u \Leftrightarrow u = e^{kx}$. Then

$$S(x(u)) = 1 + \sum_{n=1}^{\infty} \frac{A^n u^n}{(k^n \cdot n!)^2}$$

and

$$T(x(u)) = \frac{ln\,u}{k} + \sum_{n=1}^{\infty} \frac{A^n u^n}{(k^n \cdot n!)^2}\left[\frac{ln\,u}{k} - \frac{2}{k}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right]$$

$$= \left(\frac{ln\,u}{k}\right) \cdot S(x(u)) - \frac{2}{k}\sum_{n=1}^{\infty}\left\{\frac{A^n u^n}{(k^n \cdot n!)^2}\left(1 + \frac{1}{2} + \ldots + \frac{1}{n}\right)\right\}.$$

Note $S(x(u))$ has a Maclaurin Series but $T(x(u))$ does not; hence $S(x)$ and $T(x)$ are linearly independent. Thus $C_1S(x) + C_2T(x)$, $C_1$, $C_2$ constants, is the general solution to

$$y'' - Ae^{kx}y = 0, \quad -a \le x \le a.$$

We conclude with some remarks. Theorem 1 is a generalization of Theorem 0 and is of interest by itself. It is possible to improve Theorem 1 by generalizing hypothesis (iv) e.g., to include in (iv) a third alternative as follows: $c_1, \ldots, c_{k-1}$ are distinct points and each of $\{S_n(c_1)\}_1^\infty$, ..., $\{S_n(c_{k-1})\}_1^\infty$ and $\{S_n'(c_1)\}_1^\infty$ converge. It may be possible to generalize Theorem A to differential equations that include intermediate derivatives, e.g., $y'' + g(x)y' + f(x)y = 0$, $f(x)$ and $g(x)$ continuous on $[a,b]$; such a generalization would require an improvement of Theorem 1. As seen in the examples, linearly independent solutions to the differential equation are obtained by using linearly independent functions for $f_0(x)$. Finally, we note that difficulties in the application of Theorem A may occur when finding the $k^{th}$ antiderivative of $f_{n-1}(x)f(x)$, and thus this method of solution may be impractical for such cases.

*Note: The first author is deceased.

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