

ON THE STRUCTURE OF SUPPORT POINT SETS

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ABSTRACT. Let X be a metrizable compact convex subset of a locally convex space. Using Choquet's Theorem, we determine the structure of the support point set of X when X has countably many extreme points. We also characterize the support points of certain families of analytic functions.

KEY WORDS AND PHRASES: Support point, Extreme point, Choquet's Theorem.

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1. INTRODUCTION.

Let X be a subset of a locally convex space E . A continuous linear functional J on X is said to be associated with $f \in X$ if $\operatorname{Re} J(f) = \max\{\operatorname{Re} J(g) : g \in X\}$ and $\operatorname{Re} J$ is non constant on X . In this case we call f a support point of X . The set of support points of X will be denoted by $\operatorname{Supp} X$. The set of extreme points of a convex subset F of E will be denoted by $\operatorname{Ext} F$.

Let $D = \{z : |z| < 1, z \in \mathbb{C}\}$ and equip the space A of functions analytic in D with the topology of uniform convergence on compact subsets of D . This topology is metrizable [1, p.1]. Every continuous linear functional J on A is induced by a sequence $\{b_n\}_{n=0}^{\infty}$ which satisfies $\limsup |b_n|^{1/n} < 1$ and $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ [1, p.36]. Recently, the support points of many subclasses of A have been studied. For more details see [1] and [2].

In Section 2, we consider a metrizable compact convex set X in a locally convex space. Using Choquet's theorem we determine the structure of $\operatorname{Supp} X$ when $\operatorname{Ext} X$ is countable (Theorem 2.1).

In Section 3, we consider the classes: $P(p) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1\}$, $1 \leq p < \infty$. In Theorem 3.4, we determine $\text{Supp } P(p)$. Indeed, it is shown that $\text{Supp } X$ is in 1-1 correspondence with a proper subset of $\text{Supp Ball}(\ell_p)$.

2. SUPPORT POINTS OF SETS WITH COUNTABLY MANY EXTREME POINTS.

Let E be a locally convex space, and suppose that X is a metrizable compact convex subset of E . A theorem by Choquet [3, p.19] says that if $x \in X$ then there exists a probability measure μ_x on X , supported by $\text{Ext } X$, such that $L(x) = \int_{\text{Ext } X} L d\mu_x$ for every L in E^* . In case $\text{Ext } X$ is countable (possibly finite), we have the following:

CHOUQUET'S THEOREM (Countable Case). Suppose $\text{Ext } X = \{f_n\}$ is countable. Then $X = \{\sum_{n=1}^{\infty} \lambda_n f_n : \lambda_n \geq 0 \text{ for each } n \text{ and } \sum_{n=1}^{\infty} \lambda_n = 1\}$.

PROOF. Let $f \in X$. By Choquet's Theorem, there exists a probability measure μ_f on X , supported by $\{f_n\}$, such that $L(f) = \int_{\{f_n\}} L d\mu_f$. Thus $L(f) = \sum_n \mu_f(f_n) L(f_n)$. Hence $L(f - \sum_n \mu_f(f_n) f_n) = 0$.

Since this is true for every L in E^* , we get $f = \sum_n \mu_f(f_n) f_n$, as required.

We proceed to the main result of this section.

THEOREM 2.1. Let X be a metrizable compact convex subset of a locally compact space E such that $\text{Ext } X = \{f_n\}$ is countable. For each positive integer n , set K_n equal to the closed convex hull of $\{f_i : i \neq n\}$. Then

- (1) $\text{Supp } X$ is contained in the union of those K_n which are proper subsets of X .
- (2) $K_n \subseteq \text{Supp } X$ if and only if $f_n \notin$ closed affine hull of $\{f_i : i \neq n\}$.

PROOF. To prove (1), let $f \in \text{Supp } X$. By Choquet's Theorem, we can write $f = \sum_i \lambda_i f_i$ with each $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Let ϕ be a continuous linear functional associated with f . Then $\text{Re } \phi(f) = \sum_i \lambda_i \text{Re } \phi(f_i) \leq \sum_i \lambda_i \text{Re } \phi(f) = \text{Re } \phi(f)$. Hence we must have $\text{Re } \phi(f_i) = \text{Re } \phi(f)$ whenever $\lambda_i > 0$. On the other hand, since $\text{Re } \phi$ is non-constant on X , we must have $\text{Re } \phi(f_i) \neq \text{Re } \phi(f)$ for some i . We conclude that $\lambda_i = 0$ for some i , as required.

To prove (2), suppose that f_n does not belong to the closed affine hull H of $\{f_i : i \neq n\}$ and fix $g \in K_n$. Then $H - g$ is a closed real subspace of E not containing $f_n - g$. A version of the Hahn-Banach theorem [4, page 59] gives a functional J in E^* whose real part ϕ vanishes on $H - g$ while $\phi(f_n - g) = -1$. Set $\phi(f_{n+1}) = b$. Then $\phi(f_n) = b - 1$ while $\phi(f_i) = \phi(f_{n+1}) = b$ for every $i \neq n$. Thus, $\phi(g) = b$ for all $g \in K_n$. For any h in X , by Choquet's Theorem, we have $h = \sum_i \beta_i f_i$ with $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. Thus $\phi(h) = \beta_n(b-1) + \sum_{i \neq n} \beta_i b = b - \beta_n \leq b$. This shows that $g \in \text{Supp } X$.

Conversely, assume that $K_n \subseteq \text{Supp } X$. For ease of notation we take $n = 1$ and assume $\text{Ext } X = \{f_n\}_{n=1}^{\infty}$ is infinite. By assumption, $f = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} f_i$ is a support point of X . Let ϕ be an associated linear functional in E^* and set $S = \{g \in E : \text{Re } \phi(g) = \text{Re } \phi(f)\}$. Note that S is a closed affine subspace of E . Since $\text{Re } \phi(f) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f_i) \leq \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f) = \text{Re } \phi(f)$, we have $\text{Re } \phi(f_i) = \text{Re } \phi(f)$ for all $i \geq 2$. Thus the closed affine hull

of $\{f_i: i \neq 1\} \subseteq S$. On the other hand, in view of Choquet's Theorem, if $f_1 \in S$ then $\text{Re } \phi$ would be constant on X . Thus $f_1 \notin S$ and consequently, $f_1 \notin$ closed affine hull of $\{f_i: i \neq 1\}$.

EXAMPLES. (1) Let X be a triangle in \mathbb{R}^2 with vertices f_1, f_2 and f_3 . These vertices are the extreme points of X and the affine hull of any two of them is a line, not containing the third. The theorem guarantees that $\text{Supp } X = \bigcup_{n=1}^3 K_n$, which is indeed the boundary of X .

(2) Let X be a square in \mathbb{R}^2 with vertices f_1, f_2, f_3 and f_4 . The affine hull of any three of the f_i 's is all of \mathbb{R}^2 . In particular, each $f_i \in$ affine hull of $\{f_j: j \neq i\}$. The theorem guarantees that no K_n is contained in $\text{Supp } X$. In fact, $\text{Supp } X =$ the boundary of X has no interior.

(3) Let T be the family of all functions which are analytic and univalent in D , and take the form $f(z) = z \cdot \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. By [5], $\text{Ext } T = \{f_n\}_{n=1}^{\infty}$, where $f_1(z) = z$ and $f_n(z) = z - \frac{1}{n} z^n$ for $n > 1$. For $n > 1$, it is clear that f_n does not belong to the closed affine hull of the remaining $\{f_i\}$, so $\bigcup_{n=2}^{\infty} K_n \subseteq \text{Supp } X$ by the second part of the Theorem. Since f_1 is a limit point of the remaining f_i 's, $K_1 = X$ and $\text{Supp } X = \bigcup_{n=2}^{\infty} K_n$ by the first part of the Theorem.

COROLLARY 2.2. Let X be as in Theorem 2.1. Then $\text{Supp } X = \bigcup_{\alpha} \overline{\text{co}}(E_{\alpha})$, where each E_{α} is a subset of $\text{Ext } X$.

PROOF. Suppose $f \in \text{Supp } X$ and ϕ is an associated linear functional with f . Writing $f = \sum_i \lambda_i f_i$, we see that $\text{Re } \phi(f) = \text{Re } \phi(f_i)$ whenever $\lambda_i \neq 0$. Take $E_{\alpha} = \{f_i: \lambda_i \neq 0\}$. Then $f \in \overline{\text{co}}(E_{\alpha}) \subseteq \text{Supp } X$.

The theorem says these E_{α} are proper subsets of $\text{Ext } X$, i.e., they cannot be "too big". The next proposition implies that they can't all be singletons, i.e., "too small".

PROPOSITION 2.3. Let X be a compact convex subset of a locally convex space. If X has more than two extreme points, then $\text{Supp } X$ is uncountable.

PROOF. Without loss of generality we may assume that $0 \in X$. Let f_1 and f_2 be two independent elements of X , and let ϕ_1 and ϕ_2 be continuous and linear functionals such that $\phi_1(f_1) = \phi_2(f_2) = 1$ and $\phi_1(f_2) = \phi_2(f_1) = 0$. Define $\psi: X \rightarrow \mathbb{R}^2$ by $\psi(f) = (\phi_1(f), \phi_2(f))$. Then $\psi(X)$ is a compact convex subset of \mathbb{R}^2 with non empty interior. Since $\psi(X)$ has uncountably many boundary points, $\text{Supp}(\psi(X))$ is uncountable. Since ψ^{-1} takes support points to support points, we see that $\text{Supp } X$ is uncountable too.

EXAMPLE. Take $f_n = e^{\frac{2\pi i}{n}}$ for $n = 1, 2, \dots$ and $X = \overline{\text{co}}\{f_n\}$ in \mathbb{R}^2 . Then $\text{Supp } X = \bigcup_{n=1}^{\infty} \overline{\text{co}}\{f_n, f_{n+1}\}$. Here all the E_{α} 's have cardinality two even though $\text{Ext } X$ is infinite.

COROLLARY 2.4. Let X be as in Theorem 2.1. Then $\text{Ext } X = \text{Supp } X$ if and only if X has two extreme points.

3. SUPPORT POINTS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS.

For $1 \leq p < \infty$, define $P(p) = \{ \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1 \}$. It is easy to see that the classes $P(p)$ are compact convex subsets of A . These classes are closely related to $\text{Ball}(\ell_p)$ and we will find that $\text{Supp } P(p)$ is in one-to-one correspondence with a proper subset of $\text{Supp } \text{Ball}(\ell_p)$. As a corollary, we determine the support points of certain families of univalent functions. We use the notation a for the sequence $\{a_n\}_{n=1}^{\infty}$.

We begin with a simple observation.

PROPOSITION 3.1. Let X be the unit ball of a Banach space E . Then $\text{Supp } X = \{x \in X : \|x\| = 1\}$. If ϕ is associated with x , then $\phi(x) = \|\phi\|$.

PROOF. That every vector of norm one belongs to $\text{Supp } X$ is a consequence of the Hahn-Banach theorem. Suppose conversely that the real part of $\phi \in X^*$ achieves its maximum over X at x . Since X is closed under multiplication by scalars of absolute value at most one, we have $\text{Re } \phi(x) = \sup_{y \in X} \text{Re } \phi(y) = \|\phi\|$. Thus $\|\phi\| = \text{Re } \phi(x) \leq \|\phi\| \|x\|$ and so $\|x\| = 1$. Moreover $\text{Re } \phi(x) = \|\phi\|$ implies $\text{Re } \phi(x) \geq |\phi(x)|$, so $\phi(x)$ is in fact real.

EXAMPLE. The family $P(p)$ "looks like" the unit ball of ℓ_p , but we cannot immediately apply Proposition 3.1 to find its support points. For example, the sequence $\{a_n\}_{n=1}^{\infty} = \{\sqrt{\frac{6}{\pi}} \frac{1}{n}\}_{n=1}^{\infty}$ belongs to the unit sphere of ℓ_2 , but $\sum_{n=1}^{\infty} a_n z^n$ is not a support point of $P(2)$. The problem is that any non-constant linear functional $\{b_n\}_{n=1}^{\infty} \in \ell_2^*$ which assumes its maximum at $\{a_n\}_{n=1}^{\infty}$ must be a scalar multiple of $\{a_n\}_{n=1}^{\infty}$. So $\limsup \sqrt[n]{|b_n|} = 1$, which does not correspond to a continuous linear functional on A .

We find the support points of $P(p)$ by making the remarks in the preceding example more precise.

PROPOSITION 3.2. Suppose $T : E \rightarrow F$ is a linear, injective, and continuous map between topological vector spaces E and F , and let X be a subset of E . Then $Tx \in \text{Supp } TX$ if and only if $x \in \text{Supp } X$ and some linear functional associated with x belongs to $\text{range } T^*$.

PROOF. Recall that $T^* : F^* \rightarrow E^*$ is defined by $T^*\psi = \psi \circ T$. Suppose $Tx \in \text{Supp } TX$ and choose $\psi \in F^*$ with $\text{Re } \psi(Tx) = \max_{y \in TX} \text{Re } \psi(Ty)$. Set $\phi = \psi \circ T$; then $\psi \in \text{range } T^*$, $\text{Re } \phi(x) = \max_{y \in X} \text{Re } \phi(y)$, and injectivity of T implies that $\text{Re } \phi$ is not constant on X .

Conversely, let $\phi \in \text{range } T^*$ such that $\text{Re } \phi(x) = \max_{y \in X} \text{Re } \phi(y)$. Write $\phi = \psi \circ T$, $\psi \in F^*$. Then $\text{Re } \psi(Tx) = \max_{y \in TX} \text{Re } \psi(y)$, and $\text{Re } \psi$ cannot be constant on TX since $\text{Re } \phi$ is not constant on X .

PROPOSITION 3.3. Let $a \in X = \text{Ball}(\ell_p)$, $(1 < p < \infty)$, with $\|a\|_p = 1$, and $b \in \ell_q$. Then:

(1) If b is associated with a , then there exists $\beta \neq 0$ with $\beta |b_n|^q = |a_n|^p$ for all n .

$$(2) \text{ If } b_n = \begin{cases} \frac{a_n}{|a_n|} |a_n|^{p-1} & \text{if } a_n \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

then b is associated with a .

PROOF. (1) From Proposition 3.1, we learn that $b(a) = \|b\|_q = \|b\|_q \|a\|_p$. Thus we have Hölder equality, so there exists $\beta \neq 0$ with $\beta |b_n|^q = |a_n|^p$ for all n .

(2) $b(a) = \sum a_n b_n = \sum |a_n| |a_n|^{p-1} = \sum |a_n|^p = 1$, while $\|b\|_q = \sum_{n=1}^{\infty} |a_n|^{(p-1)q} = 1$, so this result follows from Hölder's inequality.

The following is the main result of this section.

THEOREM 3.4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be in $P(p)$. Then f is a support point of $P(p)$ if and only if

(1) f is analytic in \bar{D} and $\sum_{n=1}^{\infty} |a_n|^p = 1$, for $1 < p < \infty$.

(2) $f(z) = \sum_{n=1}^N a_n z^n$, where N is some positive integer and $\sum_{n=1}^N |a_n| = 1$ for $p = 1$.

PROOF. Define $T: \ell_p \rightarrow A$ by $T(a) = \sum_{n=1}^{\infty} a_n z^n$. Clearly T maps $\text{Ball}(\ell_p)$ onto $P(p)$ and T is injective. Moreover for any $r < 1$ and $a \in \ell_p$, ($1 < p < \infty$), we have $\sup_{|z| \leq r} |T(a)(z)| \leq \sum_{n=1}^{\infty} |a_n| r^n \leq \|a\|_p \left(\frac{1}{1-r}\right)^{1/q}$, by Hölder's inequality, so T is continuous. Similarly for $p = 1$.

If $\phi \in A^*$ is given by $\phi\left(\sum_{n=1}^{\infty} a_n z^n\right) = \sum_{n=1}^{\infty} a_n b_n$, then $(T^*\phi)(a) = \phi(Ta) = \sum_{n=1}^{\infty} a_n b_n$ for every $a \in \ell_p$. So $T^*\phi$ is the sequence $\{b_n\}_{n=1}^{\infty}$ considered as a member of $(\ell_p)^* = \ell_q$. Thus $\{b_n\}_{n=1}^{\infty} \in (\ell_p)^*$ is in the range of T^* if and only if $\limsup n \sqrt[n]{|b_n|} < 1$.

(1) Suppose $f = Ta \in \text{Supp } P(p)$. By Proposition 3.2, $a \in \text{Supp Ball}(\ell_p)$. Thus by Proposition 3.1, we get $\sum_{n=1}^{\infty} |a_n|^p = 1$. If the functional associated with Ta is given by $\{b_n\}_{n=1}^{\infty}$, then $\limsup n \sqrt[n]{|b_n|} < 1$. By Proposition 3.3, there exists $\beta \neq 0$ such that $|a_n|^p = \beta |b_n|^q$ for all n . Thus $\limsup n \sqrt[n]{|a_n|} < 1$ and so f is analytic in \bar{D} .

Conversely, suppose that $f = T(a)$ is analytic in \bar{D} with $\sum_{n=1}^{\infty} |a_n|^p = 1$. Then $a \in \text{Supp Ball}(\ell_p)$ by Proposition 3.1, and one can choose the functional associated with a as in the formula of Proposition 3.3. Since the radius of convergence of the power series of f is greater than one, $\limsup n \sqrt[n]{|a_n|} < 1$ so $\limsup n \sqrt[n]{|b_n|} < 1$ and thus $b \in \text{range } T^*$. Thus $f \in \text{Supp } P(p)$ by Proposition 3.2.

(2) Suppose $f = Ta \in \text{Supp } P(1)$ and b is a functional associated with a . Then $\|a\|_1 = 1$ and $b(a) = \|b\|_{\infty}$ by Propositions 3.2 and 3.1. Thus equality must hold at all points of the chain $b(a) \leq \sum_{n=1}^{\infty} |a_n| |b_n| \leq \sum_{n=1}^{\infty} |a_n| \|b\|_{\infty} \leq \|b\|_{\infty}$. In particular $|b_n| = \|b\|_{\infty}$ whenever $a_n \neq 0$. Since $\limsup n \sqrt[n]{|b_n|} < 1$, this means $a_n = 0$ for all but finitely many n , as required.

Conversely, suppose $Ta = f(z) = \sum_{n=1}^N a_n z^n$ and $\sum_{n=1}^N |a_n| = 1$. Then $a \in \text{Supp Ball}(\ell_1)$.

Define $b_n \equiv \begin{cases} \frac{a_n}{|a_n|} & \text{if } a_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$

Then $\limsup n \sqrt[n]{|b_n|} < 1$ and $\{b_n\}_{n=1}^{\infty} \in (\ell_1)^*$ is associated with a . By Proposition 3.2, f is a support point of $P(1)$, as required.

Let $Q(p) = \{f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A : \sum_{n=2}^{\infty} n |a_n|^p \leq 1\}$, $1 \leq p < \infty$. The class $Q(1)$ has been studied in [6].

We remark that each element of $Q(1)$ is univalent.

COROLLARY 3.5. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a support point of $Q(p)$ if and only if

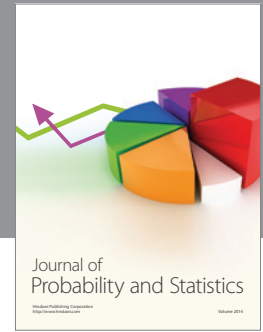
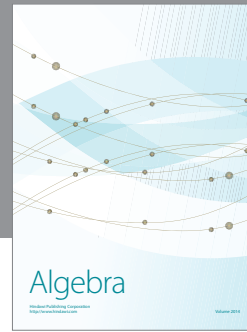
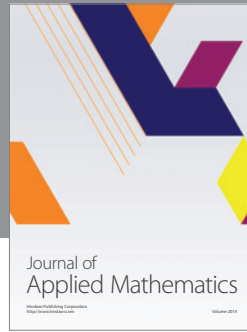
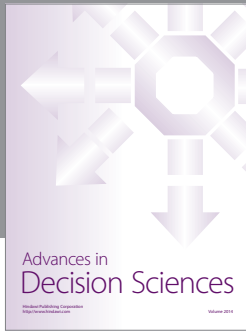
- (1) f is analytic in \bar{D} and $\sum_{n=2}^{\infty} n|a_n|^p = 1$, if $1 < p < \infty$
 (2) $f(z) = z + \sum_{n=2}^N a_n z^n$ and $\sum_{n=2}^N |a_n| = 1$, for some positive integer $N \geq 2$, if $p = 1$.

PROOF. One way to see this, is to replace ℓ_p by $\ell_p(\mu)$, where $\mu(n) = a_n$, $n = 2, 3, \dots$, in the proof of Theorem 3.4.

REMARK. One can define $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : \sup |a_n| \leq 1\}$. One can show, using an argument similar to the proof of Theorem 3.4, that $\text{Supp } P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : |a_n| = 1 \text{ for some } n \geq 1\}$.

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