Internat. J. Math. & Math. Sci.
VOL. 13 NO. 1 (1990) 25-30

ON THE STRUCTURE OF SUPPORT POINT SETS

E. AZOFF

Department of Mathematics
University of Georgia
Athens, Georgia 30602

R. YOUNIS

Department of Mathematics
Kuwait University
P.O. Box 5969
13060 Safat
Kuwait

(Received September 7, 1988)

ABSTRACT. Let X be a metrizable compact convex subset of a locally convex space. Using Choquet's Theorem, we determine the structure of the support point set of X when X has countably many extreme points. We also characterize the support points of certain families of analytic functions.

KEY WORDS AND PHRASES: Support point, Extreme point, Choquet's Theorem.

1980 SUBJECT CLASSIFICATION (1985 Revision): Primary: 46A55, 52A07, Secondary: 30C99.

1. INTRODUCTION.

Let X be a subset of a locally convex space E. A continuous linear functional J on X is said to be associated with $f \in X$ if $\text{Re } J(f) = \max\{\text{Re } J(g)\colon g \in X\}$ and $\text{Re } J$ is non constant on X. In this case we call f a support point of X. The set of support points of X will be denoted by $\text{Supp } X$. The set of extreme points of a convex subset F of E will be denoted by $\text{Ext } F$.

Let $D = \{z\colon |z| < 1 , z \in C\}$ and equip the space A of functions analytic in D with the topology of uniform convergence on compact subsets of D. This topology is metrizable [1, p.1]. Every continuous linear functional J on A is induced by a sequence $\{b_n\}_{n=0}^{\infty}$ which satisfies $\limsup |b_n|^{1/n} < 1$ and $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ [1, p.36]. Recently, the support points of many subclasses of A have been studied. For more details see [1] and [2].

In Section 2, we consider a metrizable compact convex set X in a locally convex space. Using Choquet's theorem we determine the structure of $\text{Supp } X$ when $\text{Ext } X$ is countable (Theorem 2.1).

In Section 3, we consider the classes: $P(p) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1\}$, $1 \leq p < \infty$. In Theorem 3.4, we determine Supp $P(p)$. Indeed, it is shown that Supp X is in 1-1 correspondence with a proper subset of Supp Ball(ℓ_p) .

2. SUPPORT POINTS OF SETS WITH COUNTABLY MANY EXTREME POINTS.

Let E be a locally convex space, and suppose that X is a metrizable compact convex subset of E . A theorem by Choquet [3, p.19] says that if $x \in X$ then there exists a probability measure μ_x on X , supported by Ext X , such that $L(x) = \int_{\text{Ext } X} L \, d\mu_x$ for every L in E^* . In case Ext X is countable (possibly finite), we have the following:

CHOQUET'S THEOREM (Countable Case). Suppose Ext $X = \{f_n\}$ is countable. Then $X = \{\sum_n \lambda_n f_n: \lambda_n \geq 0$ for each n and $\sum_n \lambda_n = 1\}$.

PROOF. Let $f \in X$. By Choquet's Theorem, there exists a probability measure μ_f on X , supported by $\{f_n\}$, such that $L(f) = \int_{\{f_n\}} L \, d\mu_f$. Thus $L(f) = \sum_n \mu_f(f_n) L(f_n)$. Hence $L(f - \sum_n \mu_f(f_n) f_n) = 0.$

Since this is true for every L in E^* , we get $f = \sum_n \mu_f(f_n) f_n$, as required.

We proceed to the main result of this section.

THEOREM 2.1. Let X be a metrizable compact convex subset of a locally compact space E such that Ext $X = \{f_n\}$ is countable. For each positive integer n , set K_n equal to the closed convex hull of $\{f_i: i \neq n\}$. Then

(1) Supp X is contained in the union of those K_n which are proper subsets of X .

(2) $K_n \subseteq$ Supp X if and only if $f_n \notin$ closed affine hull of $\{f_i: i \neq n\}$.

PROOF. To prove (1), let $f \in$ Supp X . By Choquet's Theorem, we can write $f = \sum_i \lambda_i f_i$ with each $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Let ϕ be a continuous linear functional associated with f . Then Re $\phi(f) = \sum \lambda_i$ Re $\phi(f_i)$ $\leq \sum_i \lambda_i$ Re $\phi(f)$ = Re $\phi(f)$. Hence we must have Re $\phi(f_i)$ = Re $\phi(f)$ whenever $\lambda_i > 0$. On the other hand, since Re ϕ is non-constant on X , we must have Re $\phi(f_i) \neq$ Re $\phi(f)$ for some i . We conclude that $\lambda_i = 0$ for some i , as required.

To prove (2), suppose that f_n does not belong to the closed affine hull H of $\{f_i: i \neq n\}$ and fix $g \in K_n$. Then H - g is a closed real subspace of E not continuing f_n - g . A version of the Hahn-Banach theorem [4, page 59] gives a functional J in E^* whose real part ϕ vanishes on H - g while $\phi(f_n - g) = -1$. Set $\phi(f_{n+1}) = b$. Then $\phi(f_n) = b - 1$ while $\phi(f_i) = \phi(f_{n+1}) = b$ for every $i \neq n$. Thus, $\phi(g) = b$ for all $g \in K_n$. For any h in X , by Choquet's Theorem, we have $h = \sum_i \beta_i f_i$ with $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. Thus $\phi(h) = \beta_n(b-1) + \sum_{i \neq n} \beta_i b$ $= b - \beta_n \leq b$. This shows that $g \in$ Supp X .

Conversely, assume that $K_n \subseteq$ Supp X . For ease of notation we take n = 1 and assume Ext $X = \{f_n\}_{n=1}^{\infty}$ is infinite. By assumption, $f = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} f_i$ is a support point of X . Let ϕ be an associated linear functional in E^* and set $S = \{g \in E: \text{Re } \phi(g) = \text{Re } \phi(f)\}$. Note that S is a closed affine subspace of E . Since Re $\phi(f)$ = $\sum_{i=2}^{\infty} \frac{1}{2^{i-1}}$ Re $\phi(f_i) \leq \sum_{i=2}^{\infty} \frac{1}{2^{i-1}}$ Re $\phi(f)$ = Re $\phi(f)$, we have Re $\phi(f_i)$ = Re $\phi(f)$ for all $i \geq 2$. Thus the closed affine hull of $\{f_i: i \neq 1\} \subseteq S$. On the other hand, in view of Choquet's Theorem, if $f_1 \in S$ then $\mathrm{Re}\,\phi$ would be constant on X. Thus $f_1 \notin S$ and consequently, $f_1 \notin$ closed affine hull of $\{f_i: i \neq 1\}$.

EXAMPLES. (1) Let X be a triangle in R^2 with vertices f_1, f_2 and f_3. These vertices are the extreme points of X and the affine hull of any two of them is a line, not containing the third. The theorem guarantees that $\mathrm{Supp}\, X = \bigcup_{n=1}^{3} K_n$, which is indeed the boundary of X.

(2) Let X be a square in R^2 with vertices f_1, f_2, f_3 and f_4. The affine hull of any three of the f_i's is all of R^2. In particular, each $f_i \in$ affine hull of $\{f_j: j \neq i\}$. The theorem guarantees that no K_n is contained in Supp X. In fact, Supp X = the boundary of X has no interior.

(3) Let T be the family of all functions which are analytic and univalent in D, and take the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. By [5], $\mathrm{Ext}\, T = \{f_n\}_{n=1}^{\infty}$, where $f_1(z) = z$ and $f_n(z) = z - \frac{1}{n} z^n$ for $n > 1$. For $n > 1$, it is clear that f_n does not belong to the closed affine hull of the remaining $\{f_i\}$, so $\bigcup_{n=2}^{\infty} K_n \subseteq \mathrm{Supp}\, X$ by the second part of the Theorem. Since f_1 is a limit point of the remaining f_i's, $K_1 = X$ and $\mathrm{Supp}\, X = \bigcup_{n=2}^{\infty} K_n$ by the first part of the Theorem.

COROLLARY 2.2. Let X be as in Theorem 2.1. Then $\mathrm{Supp}\, X = \bigcup_{\alpha} \overline{\mathrm{co}}\,(E_\alpha)$, where each E_α is a subset of Ext X.

PROOF. Suppose $f \in \mathrm{Supp}\, X$ and ϕ is an associated linear functional with f. Writing $f = \sum_i \lambda_i f_i$, we see that $\mathrm{Re}\,\phi(f) = \mathrm{Re}\,\phi(f_i)$ whenever $\lambda_i \neq 0$. Take $E_\alpha = \{f_i |\, \lambda_i \neq 0\}$. Then $f \in \overline{\mathrm{co}}\,(E_\alpha) \subseteq \mathrm{Supp}\, X$.

The theorem says these E_α are proper subsets of Ext X, i.e., they cannot be "too big". The next proposition implies that they can't all be singletons, i.e., "too small".

PROPOSITION 2.3. Let X be a compact convex subset of a locally convex space. If X has more than two extreme points, then Supp X is uncountable.

PROOF. Without loss of generality we may assume that $0 \in X$. Let f_1 and f_2 be two independent elements of X, and let ϕ_1 and ϕ_2 be continuous and linear functionals such that $\phi_1(f_1) = \phi_2(f_2) = 1$ and $\phi_1(f_2) = \phi_2(f_1) = 0$. Define $\psi: X \to \mathrm{R}^2$ by $\psi(f) = (\phi_1(f), \phi_2(f))$. Then $\psi(X)$ is a compact convex subset of R^2 with non empty interior. Since $\psi(X)$ has uncountably many boundary points, $\mathrm{Supp}(\psi(X))$ is uncountable. Since ψ^{-1} takes support points to support points, we see that Supp X is uncountable too.

EXAMPLE. Take $f_n = e^{\frac{2\pi i}{n}}$ for $n = 1, 2, \ldots$ and $X = \overline{\mathrm{co}}\,\{f_n\}$ in $\mathbf{R}^2$. Then $\mathrm{Supp}\, X = \bigcup_{n=1}^{\infty} \mathrm{co}\,\{f_n, f_{n+1}\}$. Here all the E_α's have cardinality two even though Ext X is infinite.

COROLLARY 2.4. Let X be as in Theorem 2.1. Then Ext X = Supp X if and only if X has two extreme points.

3. SUPPORT POINTS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS.

For $1 \leq p < \infty$, define $P(p) = \{\sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1\}$. It is easy to see that the classes $P(p)$ are compact convex subsets of A. These classes are closely related to $\mathrm{Ball}(\ell_p)$ and we will find that Supp $P(p)$ is in one-to-one correspondence with a proper subset of Supp $\mathrm{Ball}(\ell_p)$. As a corollary, we determine the support points of certain families of univalent functions. We use the notation a for the sequence $\{a_n\}_{n=1}^{\infty}$.

We begin with a simple observation.

PROPOSITION 3.1. Let X be the unit ball of a Banach space E. Then Supp $X = \{x \in X : \|x\| = 1\}$. If ϕ is associated with x, then $\phi(x) = \|\phi\|$.

PROOF. That every vector of norm one belongs to Supp X is a consequence of the Hahn-Banach theorem. Suppose conversely that the real part of $\phi \in X^*$ achieves its maximum over X at x. Since X is closed under multiplication by scalars of absolute value at most one, we have $\mathrm{Re}\,\phi(x) = \sup_{y \in X} \mathrm{Re}\,\phi(y) = \|\phi\|$. Thus $\|\phi\| = \mathrm{Re}\,\phi(x) \leq \|\phi\|\,\|x\|$ and so $\|x\| = 1$. Moreover $\mathrm{Re}\,\phi(x) = \|\phi\|$ implies $\mathrm{Re}\,\phi(x) \geq |\phi(x)|$, so $\phi(x)$ is in fact real.

EXAMPLE. The family $P(p)$ "looks like" the unit ball of ℓ_p, but we cannot immediately apply Proposition 3.1 to find its support points. For example, the sequence $\{a_n\}_{n=1}^{\infty} = \{\sqrt{\frac{6}{\pi}}\,\frac{1}{n}\}_{n=1}^{\infty}$ belongs to the unit sphere of ℓ_2, but $\sum_{n=1}^{\infty} a_n z^n$ is not a support point of $P(2)$. The problem is that any non-constant linear functional $\{b_n\}_{n=1}^{\infty} \in \ell_2^*$ which assumes its maximum at $\{a_n\}_{n=1}^{\infty}$ must be a scalar multiple of $\{a_n\}_{n=1}^{\infty}$. So $\limsup \sqrt[n]{|b_n|} = 1$, which does not correspond to a continuous linear functional on A.

We find the support points of $P(p)$ by making the remarks in the preceeding example more precise.

PROPOSITION 3.2. Suppose $T : E \to F$ is a linear, injective, and continuous map between topological vector spaces E and F, and let X be a subset of E. Then $Tx \in$ Supp TX if and only if $x \in$ Supp X and some linear functional associated with x belongs to range T^*.

PROOF. Recall that $T^* : F^* \to E^*$ is defined by $T^*\psi = \psi \circ T$. Suppose $Tx \in$ Supp TX and choose $\psi \in F^*$ with $\mathrm{Re}\,\psi(Tx) = \max_{y \in X} \mathrm{Re}\psi(Ty)$. Set $\phi = \psi \circ T$; then $\psi \in$ range T^*, $\mathrm{Re}\,\phi(x) = \max_{y \in X} \mathrm{Re}\phi(y)$, and injectivity of T implies that $\mathrm{Re}\,\phi$ is not constant on X.

Conversely, let $\phi \in$ range T^* such that $\mathrm{Re}\,\phi(x) = \max_{y \in X} \mathrm{Re}\,\phi(y)$. Write $\phi = \psi \circ T$, $\psi \in F^*$. Then $\mathrm{Re}\,\psi(Tx) = \max_{y \in TX} \psi(y)$, and $\mathrm{Re}\,\psi$ cannot be constant on TX since $\mathrm{Re}\,\phi$ is not constant on X.

PROPOSITION 3.3. Let $a \in X = \mathrm{Ball}(\ell_p)$, $(1 < p < \infty)$, with $\|a\|_p = 1$, and $b \in \ell_q$. Then:

(1) If b is associated with a, then there exists $\beta \neq 0$ with $\beta|b_n|^q = |a_n|^p$ for all n.

(2) If $b_n = \begin{cases} \frac{a_n}{|a_n|}\,|a_n|^{p-1} & \text{if } a_n \neq 0 \\ 0 & \text{otherwise} \end{cases}$

then b is associated with a.

PROOF. (1) From Proposition 3.1, we learn that $b(a) = ||b||_q = ||b||_q||a||_p$. Thus we have Hölder equality, so there exists $\beta \neq 0$ with $\beta\,|b_n|^q = |a_n|^p$ for all n .

(2) $b(a) = \sum a_n b_n = \sum |a_n|\,|a_n|^{p-1} = \sum |a_n|^p = 1$, while $||b||_q = \sum_{n=1}^{\infty} |a_n|^{(p-1)q} = 1$, so this result follows from Holder's inequality.

The following is the main result of this section.

THEOREM 3.4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be in $P(p)$. Then f is a support point of $P(p)$ if and only if

(1) f is analytic in $\bar{D}$ and $\sum_{n=1}^{\infty} |a_n|^p = 1$, for $1 < p < \infty$.

(2) $f(z) = \sum_{n=1}^{N} a_n z^n$, where N is some positive integer and $\sum_{n=1}^{N} |a_n| = 1$ for $p = 1$.

PROOF. Define $T : \ell_p \to A$ by $T(a) = \sum_{n=1}^{\infty} a_n z^n$. Clearly T maps $\mathrm{Ball}(\ell_p)$ onto $P(p)$ and T is injective. Moreover for any $r < 1$ and $a \in \ell_p$, $(1 < p < \infty)$, we have $\sup_{|z|\le r} |T(a)(z)| \le \sum_{n=1}^{\infty} |a_n| r^n \le ||a||_p \left(\frac{1}{1-r^q}\right)^{1/q}$, by Hölder's inequality, so T is continuous. Similarly for $p = 1$.

If $\phi \in A^*$ is given by $\phi(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} a_n b_n$, then $(T^*\phi)(a) = \phi(Ta) = \sum_{n=1}^{\infty} a_n b_n$ for every $a \in \ell_p$. So $T^*\phi$ is the sequence $\{b_n\}_{n=1}^{\infty}$ considered as a member of $(\ell_p)^* = \ell_q$. Thus $\{b_n\}_{n=1}^{\infty} \in (\ell_p)^*$ is in the range of T^* if and only if $\limsup \sqrt[n]{|b_n|} < 1$.

(1) Suppose $f = Ta \in \mathrm{Supp}\, P(p)$. By Proposition 3.2, $a \in \mathrm{Supp\ Ball}\,(\ell_p)$. Thus by Proposition 3.1, we get $\sum_{n=1}^{\infty} |a_n|^p = 1$. If the functional associated with Ta is given by $\{b_n\}_{n=1}^{\infty}$, then $\limsup \sqrt[n]{|b_n|} < 1$. By Proposition 3.3, there exists $\beta \neq 0$ such that $|a_n|^p = \beta |b_n|^q$ for all n . Thus $\limsup \sqrt[n]{|a_n|} < 1$ and so f is analytic in $\bar{D}$.

Conversely, suppose that $f = T(a)$ is analytic in $\bar{D}$ with $\sum_{n=1}^{\infty} |a_n|^p = 1$. Then $a \in \mathrm{Supp\ Ball}\,(\ell_p)$ by Proposition 3.1, and one can choose the functional associated with a as in the formula of Proposition 3.3. Since the radius of convergence of the power series of f is greater than one, $\limsup \sqrt[n]{|a_n|} < 1$ so $\limsup \sqrt[n]{|b_n|} < 1$ and thus $b \in \text{range } T^*$. Thus $f \in \mathrm{Supp}\, P(p)$ by Proposition 3.2.

(2) Suppose $f = Ta \in \mathrm{Supp}\, P(1)$ and b is a functional associated with a . Then $||a||_1 = 1$ and $b(a) = ||b||_\infty$ by Propositions 3.2 and 3.1. Thus equality must hold at all points of the chain $b(a) \le \sum_{n=1}^{\infty} |a_n||b_n| \le \sum_{n=1}^{\infty} |a_n|\, ||b||_\infty \le ||b||_\infty$. In particular $|b_n| = ||b||_\infty$ whenever $a_n \neq 0$. Since $\limsup \sqrt[n]{|b_n|} < 1$, this means $a_n = 0$ for all but finitely many n , as required.

Conversely, suppose $Ta = f(z) = \sum_{n=1}^{N} a_n z_n$ and $\sum_{n=1}^{N} |a_n| = 1$. Then $a \in \mathrm{Supp\ Ball}\,(\ell_1)$.

Define $b_n \equiv \begin{cases} \dfrac{a_n}{|a_n|} & \text{if } a_n \neq 0 , \\ 0 & \text{otherwise} . \end{cases}$

Then $\limsup \sqrt[n]{|b_n|} < 1$ and $\{b_n\}_{n=1}^{\infty} \in (\ell_1)^*$ is associated with a . By Proposition 3.2, f is a support point of $P(1)$, as required.

Let $Q(p) = \{f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A : \sum_{n=2}^{\infty} n|a_n|^p \le 1\}$, $1 \le p < \infty$. The class $Q(1)$ has been studied in [6]. We remark that each element of $Q(1)$ is univalent.

COROLLARY 3.5. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a support point of $Q(p)$ if and only if

(1) f is analytic in $\bar{D}$ and $\sum_{n=2}^{\infty} n|a_n|^p = 1$, if $1 < p < \infty$

(2) $f(z) = z + \sum_{n=2}^{N} a_n z^n$ and $\sum_{n=2}^{N} |a_n| = 1$, for some positive integer $N \geq 2$, if $p = 1$.

PROOF. One way to see this, is to replace ℓ_p by $\ell_p(\mu)$, where $\mu(n) = n$, $n = 2,3,\ldots$ in the proof of Theorem 3.4.

REMARK. One can define $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : \sup|a_n| \leq 1\}$. One can show, using an argument similar to the proof of Theorem 3.4, that Supp $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : |a_n| = 1$ for some $n \geq 1\}$.

REFERENCES

1. SCHOBER, G. Univalent Functions - Selected Topics, Springer-Verlag, 1975.

2. HALLENBECK, D.J. and MACGREGOR, T.H. Linear Problems and Convexity Techniques in Geometric Function Theory, Pitman, 1984.

3. PHELPS, R.R. Lectures on Choquet's Theorem, Van Nostrand, 1966.

4. RUDIN, W. Functional Analysis, McGraw-Hill, 1973.

5. SILVERMAN, H. Univalent functions with negative coefficients, *Proc. Amer. Math. Soc.* 51(1975), 109-116.

6. SILVERMAN, H. Order starlikeness for multipliers of univalent functions, *J. Math. Anal. Appl.* 103(1984) 48-57.