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SOME TAUBERIAN THEOREMS FOR EULER AND BOREL SUMMABILITY

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ABSTRACT. The well-known summability methods of Euler and Borel are studied as mappings from ℓ^1 into ℓ^1. In this ℓ-ℓ setting, the following Tauberian results are proved: if x is a sequence that is mapped into ℓ^1 by the Euler-Knopp method E_r with $r > 0$ (or the Borel matrix method) and x satisfies $\Sigma_{n=0}^{\infty}|x_n - x_{n+1}|\sqrt{n} < \infty$, then x itself is in ℓ^1.

KEY WORDS AND PHRASES. *Tauberian condition, ℓ-ℓ method, Euler-Knopp means, Borel exponential method.*

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1. INTRODUCTION.

In [2, p. 121], G. H. Hardy described a Tauberian theorem as one which asserts that a particular summability method cannot sum a divergent series that oscillates too slowly. In this paper we shall state the results in sequence-to-sequence form, so a typical order-type Tauberian theorem for a method A would have the form, "if x is a sequence such that Ax is convergent and $\Delta x_k = x_k - x_{k+1} = o(d_k)$, then x itself is convergent." Our present task is not to give more theorems in the setting of ordinary convergence, but rather, we shall develop analogous results for methods that map ℓ^1 into ℓ^1. Such a transformation is called an ℓ-ℓ method, and we shall henceforth write ℓ for ℓ^1. In [5] Knopp and Lorentz proved that the matrix A determines an ℓ-ℓ method if and only if $\sup_k \Sigma_{n=0}^{\infty}|a_{nk}| < \infty$.

In order to prove Tauberian theorems in an ℓ-ℓ setting, it is necessary to formulate an ℓ-ℓ analogue of the above Tauberian condition $\Delta x_k = o(d_k)$. Since this condition means that $\Delta x/d$ is in c_0, an ℓ-ℓ analogue would be "$\Delta x/d$ is in ℓ," which we shall write in series form as $\Sigma_{k=0}^{\infty}|\Delta x_k|/d_k < \infty$.

2. EULER-KNOPP AND BOREL ℓ-ℓ METHODS.

The Euler-Knopp means [6, pp. 56-60] are given by the matrix

$$E_r[n,k] = \begin{cases} \binom{n}{k} r^k(1 - r)^{n-k}, & \text{if } k \le n, \\ 0, & \text{if } k > n. \end{cases}$$

In [1, Theorem 4] it is shown that E_r determines an ℓ-ℓ method if and only if $0 < r \le 1$. Moreover, for such r, $E_r^{-1}[\ell] \ne \ell$.

The customary form of Borel exponential summability is the sequence-to-function transformation ([2, p. 182], [6, p. 54]) given by

$$\text{if } \lim_{t\to\infty} \{e^{-t}\Sigma_{k=0}^{\infty} x_k t^k/k!\} = L, \text{ then x is Borel summable to L.}$$

In order to consider this method in an ℓ-ℓ setting, we must modify it into a sequence-to-sequence transformation. This can be achieved by letting t tend to ∞ through integer values and considering the resulting sequence Bx. Then B is the Borel matrix method [6, p. 56], which is given by the matrix

$$b_{nk} = e^{-n}n^k/k!.$$

By a direct application of the Knopp-Lorentz Theorem [5], one can show that B is an ℓ-ℓ matrix. We shall not use this direct approach, however, because the assertion will follow from our first theorem, which is an inclusion theorem between B and E_r.

THEOREM 1. If $r > 0$ and x is a sequence such that $E_r x$ is in ℓ, then Bx is in ℓ.

PROOF. We use the familiar technique of showing that BE_r^{-1} is an ℓ-ℓ matrix. Since $Bx = (BE_r^{-1})E_r x$, this will ensure that Bx is in ℓ whenever $E_r x$ is in ℓ. Since $E_r^{-1} = E_{1/r}$, we replace 1/r by s and show that BE_s is an ℓ-ℓ matrix for all positive s. The n,k-th term of BE_s is given by

$$\begin{aligned} BE_s[n,k] &= \Sigma_{j=k}^{\infty} \frac{e^{-n}n^j}{j!}\binom{j}{k}(1 - s)^{j-k}s^k \\ &= \frac{e^{-n}n^k s^k}{k!} \Sigma_{j=k}^{\infty} \frac{n^{j-k}}{(j-k)!}(1 - s)^{j-k} \\ &= \frac{(ns)^k e^{-ns}}{k!} . \end{aligned}$$

Summing the k-th column of BE_s, we get

$$\begin{aligned} \Sigma_{n=0}^{\infty} |BE_s[n,k]| &= \frac{1}{k!} \Sigma_{n=0}^{\infty}(ns)^k e^{-ns} \\ &= 0\left(\frac{1}{k!}\int_0^{\infty}(ts)^k e^{-ts}dt\right) \\ &= 0(1/s). \end{aligned}$$

Hence, $\sup_k \sum_{n=0}^{\infty} |BE_s[n,k]| < \infty$, so BE_s is an ℓ-ℓ matrix.

Combining Theorem 1 with the knowledge that E_r is an ℓ-ℓ matrix, we get the following result as an immediate corollary.

THEOREM 2. The Borel matrix B determines an ℓ-ℓ method.

In addition to the inclusion relation given in Theorem 1, we can show that the ℓ-ℓ method B is strictly stronger than all E_r methods by the following example.

EXAMPLE. Suppose $r > 0$ and $x_k = (-s)^k$, where $s \geq -1 + 2/r$; then Bx is in ℓ but $E_r x$ is not in ℓ. For,

$$(Bx)_n = \sum_{k=0}^{\infty} e^{-n} \frac{n^k}{k!}(-s)^k = e^{-n}e^{-sn} = e^{-n(s+1)},$$

and

$$(E_r x)_n = \sum_{k=0}^{n} \binom{n}{k} (1 - r)^{n-k}(-rs)^k = (1 - r - rs)^n.$$

By solving $-1 < 1 - r' - rs < 1$, we see that $E_r x$ is in ℓ if and only if $-1 < s < -1 + 2/r$.

3. TAUBERIAN THEOREMS.

We are now ready to prove the principal results which show that B and E_r can not map a sequence from $\sim\ell$ into ℓ if the sequence oscillates too slowly.

THEOREM 3. If x is a sequence such that Bx is in ℓ and

$$(*) \qquad \sum_{r=0}^{\infty} |\Delta x_r|\sqrt{r} < \infty,$$

then x is in ℓ.

PROOF. It suffices to show that Bx - x is in ℓ; that is, $\sum_{n=0}^{\infty}|\sum_{k=0}^{\infty} b_{nk}x_k - x_n| < \infty$. Since $\sum_{k=0}^{\infty} b_{nk} = 1$ for each n, this sum can be written as $\sum_{n=0}^{\infty}|\sum_{k=0}^{\infty} b_{nk}(x_k - x_n)|$, and so it suffices to show that $A = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} b_{nk}|x_k - x_n| < \infty$.

We can write $A = C + D$, where

$$C = \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{\infty} b_{nk}|x_k - x_n|$$

and

$$D = \Sigma_{n=0}^{\infty}\Sigma_{k=n+1}^{\infty} b_{nk}|x_k - x_n|.$$

Then

$$C \le \Sigma_{n=0}^{\infty}\Sigma_{k=0}^{n-1} b_{nk}\Sigma_{r=k}^{n-1} |\Delta x_r|$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r} b_{nk}$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|C_r, \text{ say.}$$

Also,

$$D \le \Sigma_{n=0}^{\infty}\Sigma_{k=n+1}^{\infty} b_{nk}\Sigma_{r=n}^{k-1} |\Delta x_r|$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{\infty} b_{nk}$$

$$= \Sigma_{r=0}^{\infty} |\Delta x_r|D_r, \quad \text{say.}$$

By the Lemma following, $C_r = O(\sqrt{r})$ and $D_r = O(\sqrt{r})$, so

$$C + D \le H\ \Sigma_{r=0}^{\infty} |\Delta x_r|\sqrt{r} < \infty,$$

which proves the theorem.

LEMMA. If $b_{nk} = e^{-n}n^k/k!$ and r is a positive integer, then

(i) $$\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r} b_{nk} = O(\sqrt{r}),$$

and

(ii) $$\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{\infty} b_{nk} = O(\sqrt{r}).$$

PROOF. Let $p = [\sqrt{r}]$, and write the sum in (i) as

$$\Sigma_{n=r+1}^{\infty}\Sigma_{k=0}^{r-p} b_{nk} + \Sigma_{n=r+1}^{\infty}\Sigma_{k=r-p+1}^{r} b_{nk} = F_r + G_r, \text{ say.}$$

If $s < n$, then (cf. [2, p. 202])

$$\Sigma_{k=0}^{s} \frac{n^k}{k!} = \frac{n^s}{s!}(1 + \frac{s}{n} + \frac{s}{n}\frac{s-1}{n} + \cdots)$$

$$\leq \frac{n^s}{s!}(1 + \frac{s}{n} + (\frac{s}{n})^2 + \cdots)$$

$$= \frac{n^s}{s!}(\frac{n}{n-s}).$$

In F_r we have $s = r - p$ and

$$\max_{n\geq r+1} \frac{n}{n-r+p} = \frac{r+1}{p+1} \leq \sqrt{r} + 1,$$

so

$$F_r < (\sqrt{r} + 1) \frac{1}{(r-p)!} \Sigma_{n=r+1}^{\infty} e^{-n} n^{r-p} \leq \sqrt{r} + 1.$$

In G_r we have

$$\Sigma_{k=r-p+1}^{r} b_{nk} \leq \sqrt{r} \max_{k\leq r} b_{nk} = \sqrt{r}\, e^{-n} \frac{n^r}{r!},$$

so

$$G_r \leq \sqrt{r} \frac{1}{r!} \Sigma_{n=r+1}^{\infty} e^{-n} n^r \leq \sqrt{r}.$$

Hence, (i) is proved.

Next write the sum in (ii) as

$$\Sigma_{n=0}^{r}\Sigma_{k=r+1}^{r+p-1} b_{nk} + \Sigma_{n=0}^{r}\Sigma_{k=r+p}^{r} b_{nk} = H_r + I_r, \text{ say.}$$

(Assume that $H_r = 0$ if $p = 1$.) Then

$$H_r \leq (p - 1) \Sigma_{n=0}^{r} e^{-n} \max_{k>r} \frac{n^k}{k!}$$

$$\leq (\sqrt{r} - 1) \frac{1}{(r+1)!} \Sigma_{n=0}^{r} e^{-n} n^{r+1}$$

$$\leq \sqrt{r} - 1.$$

If $s \geq n$, then

$$\Sigma_{k=s}^{\infty} \frac{n^k}{k!} = \frac{n^s}{s!}(1 + \frac{n}{s+1} + \frac{n}{s+1}\frac{n}{s+2} + \cdots)$$

$$\leq \frac{n^s}{s!}(1 + \frac{n}{s+1} + (\frac{n}{s+1})^2 + \cdots)$$

$$= \frac{n^s}{s!}\left(\frac{s+1}{s+1-n}\right).$$

Taking $s = r + p$, we have

$$I_r \le \frac{1}{(r+p)!} \Sigma_{n=0}^{r} e^{-n} n^{r+p} \left(\frac{r+p+1}{r+p+1-n}\right)$$

$$\le \left(\frac{r+p+1}{p+1}\right) \frac{1}{(r+p)!} \Sigma_{n=0}^{r} e^{-n} n^{r+p}$$

$$\le \sqrt{r} + 1.$$

This completes the proof of the Lemma.

By combining Theorem 3 with Theorem 1, we get an ℓ-ℓ Tauberian theorem for the Euler-Knopp means.

THEOREM 4. If $r > 0$ and x is a sequence satisfying (*) such that $E_r x$ is in ℓ, then x is in ℓ.

Next we give an application of these Tauberian theorems.

EXAMPLE. The following sequence is not mapped into ℓ by B -- or, a fortiori, by E_r, with $r > 0$. Define x by

$$x_0 = \pi^2/6 \quad \text{and} \quad \Delta x_j = 1/(j+1)^2.$$

Then x satisfies (*), but x is not in ℓ because if $k \geq 1$,

$$x_k = x_0 - \Sigma_{j=0}^{k-1} \Delta x_j = \frac{\pi^2}{6} - \Sigma_{m=1}^{k} m^{-2}$$

$$= \Sigma_{m \geq k+1} m^{-2} \sim 1/k.$$

Hence, by Theorem 3, Bx is not in ℓ.

It is possible -- but much more tedious -- to construct a real number sequence x such that $\Delta x_j = \pm(j+1)^{-2}$ and x changes sign infinitely many times, yet x is not in ℓ. For such an x, Theorem 3 implies that Bx cannot be in ℓ.

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