

A SUFFICIENT CONDITION FOR STARLIKENESS OF ORDER α

PETRU T. MOCANU and GH. OROS

(Received 28 January 2001 and in revised form 12 June 2001)

ABSTRACT. We obtain a sufficient condition for starlikeness of order α , $|f'(z) - \lambda(f(z)/z) + \lambda - 1| < M = M_n(\lambda, \alpha)$, where $\lambda \in [0, 1]$, $\alpha \in [0, 1)$ and the function $f(z) = z + a_{n+1}z^{n+1} + \dots$ is analytic in the unit disc U .

2000 Mathematics Subject Classification. 30C45.

1. Introduction and preliminaries. Denote by U the unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\}. \quad (1.1)$$

Let $\mathcal{H}[U]$ be the space of holomorphic functions in U , and let

$$A_n = \{f \in \mathcal{H}[U], f(z) = z + a_{n+1}z^{n+1} + \dots, z \in U\} \quad (1.2)$$

with $A_1 = A$.

Let $\mathcal{H}[a, n]$ denote the class of analytic functions in the unit disc of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \quad z \in U. \quad (1.3)$$

Let

$$S^*(\alpha) = \left\{ f \in A, \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha, z \in U \right\}, \quad 0 \leq \alpha < 1, \quad (1.4)$$

be the class of starlike functions of order α in U .

If f and g are analytic in U , then we say that f is subordinate to g , written $f < g$ or $f(z) < g(z)$, if there is a function w analytic in U , with $w(0) = 0$, $|w(z)| < 1$, for any $z \in U$, such that $f(z) = g(w(z))$, for $z \in U$.

If g is univalent, then $f < g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

We use the following subordination result due to Hallenbeck and Ruscheweyh [1, page 71].

LEMMA 1.1. *Let h be a convex function with $h(0) = a$, and let $y \in \mathbb{C}^*$ be a complex number with $\operatorname{Re} y \geq 0$. If $p \in \mathcal{H}[a, n]$ and*

$$p(z) + \frac{1}{y} z p'(z) < h(z), \quad (1.5)$$

then

$$p(z) < q(z), \quad (1.6)$$

where

$$q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt, \quad q \prec h. \tag{1.7}$$

2. Main results

THEOREM 2.1. *Let $\lambda \in [0, 1]$, $\alpha \in [0, 1)$, and*

$$M = M_n(\lambda, \alpha) = \frac{(1 - \alpha)(n + 1 - \lambda)}{|\lambda - \alpha| + \sqrt{(1 - \lambda)^2 + (n + 1 - \lambda)^2}}. \tag{2.1}$$

If $f \in A_n$ satisfies the inequality

$$\left| f'(z) - \lambda \frac{f(z)}{z} + \lambda - 1 \right| < M_n(\lambda, \alpha), \tag{2.2}$$

with $M_n(\lambda, \alpha)$ given by (2.1), then $f \in S^*(\alpha)$.

PROOF. In the case $\lambda = 1$, the proof is given in [3]. We suppose that $\lambda \in [0, 1)$. If we consider $P(z) = f(z)/z$, then

$$f(z) = zP(z), \quad f'(z) = P(z) + zP'(z), \tag{2.3}$$

and (2.2) can be written in the following form:

$$\left| P(z) + \frac{zP'(z)}{1 - \lambda} - 1 \right| < \frac{M}{1 - \lambda} \tag{2.4}$$

which is equivalent to the differential subordination

$$P(z) + \frac{zP'(z)}{1 - \lambda} < 1 + \frac{M}{1 - \lambda} z \equiv h(z), \tag{2.5}$$

and by using Lemma 1.1, we obtain

$$P(z) \prec q(z) = \frac{\gamma}{nz^{\gamma/n}} \int_0^z h(t)t^{\gamma/n-1} dt = 1 + \frac{M}{1 - \lambda + n} z. \tag{2.6}$$

Subordination (2.6) is equivalent to

$$|P(z) - 1| < \frac{M}{1 - \lambda + n} \equiv R. \tag{2.7}$$

After a simple computation, from (2.7) it follows that

$$R < \frac{1 - \alpha}{|\lambda - \alpha|}. \tag{2.8}$$

If we put

$$\frac{zf'(z)}{f(z)} = (1 - \alpha)p(z) + \alpha, \tag{2.9}$$

then

$$f'(z) = P(z)[(1 - \alpha)p(z) + \alpha] \tag{2.10}$$

and (2.2) can be written as

$$|P(z)[(1-\alpha)p(z) + \alpha - \lambda] + \lambda - 1| < M = (1-\lambda+n)R. \tag{2.11}$$

We have to show that (2.11) implies $\operatorname{Re} p(z) > 0$ in U . Suppose that this is false. Since $p(0) = 1$, there exist $z_0 \in U$ and a real ρ , such that $p(z_0) = i\rho$.

Therefore, in order to show that (2.11) implies $\operatorname{Re} p(z) > 0$ in U , it is sufficient to obtain the contradiction from the inequality

$$|P(z_0)[(1-\alpha)p(z_0) + \alpha - \lambda] + \lambda - 1| \geq (1-\lambda+n)R. \tag{2.12}$$

If we let $P(z_0) = P = u + iv$, then

$$\begin{aligned} E &= |P[(1-\alpha)i\rho + \alpha - \lambda] + \lambda - 1|^2 \\ &= |P|^2[(1-\alpha)^2\rho^2 + (\alpha - \lambda)^2] - 2(1-\lambda)\operatorname{Re}\{P(1-\alpha)i\rho + \alpha - \lambda\} + (1-\lambda)^2 \\ &= (u^2 + v^2)(1-\alpha)^2\rho^2 + 2(1-\lambda)(1-\alpha)v\rho + |P(\alpha - \lambda) - (1-\lambda)|^2. \end{aligned} \tag{2.13}$$

By using (2.7) and the well-known triangle inequality, one obtains

$$\begin{aligned} |P(\alpha - \lambda) - (1-\lambda)| &= |P(\alpha - \lambda) + \alpha - \lambda - \alpha + \lambda - 1 + \lambda| \\ &= |(\alpha - \lambda)(P - 1) - (1-\alpha)| \\ &\geq 1 - \alpha - |\lambda - \alpha|R \end{aligned} \tag{2.14}$$

and we deduce

$$E \geq (u^2 + v^2)(1-\alpha)^2\rho^2 + 2(1-\lambda)(1-\alpha)v\rho + [(1-\alpha) - (\lambda - \alpha)R]^2. \tag{2.15}$$

If we let

$$\begin{aligned} F(\rho) &= E - M^2 \\ &\geq (u^2 + v^2)(1-\alpha)^2\rho^2 + 2(1-\lambda)(1-\alpha)v\rho \\ &\quad + [(1-\alpha) - |\lambda - \alpha|R]^2 - (1-\lambda+n)^2R^2, \end{aligned} \tag{2.16}$$

then (2.12) holds if $F(\rho) \geq 0$, for any real number ρ .

Because $(u^2 + v^2)(1-\alpha)^2 > 0$, the inequality $F(\rho) \geq 0$ holds if the discriminant Δ is negative, that is,

$$\Delta = (1-\alpha)^2\{(1-\lambda)^2v^2 - (u^2 + v^2)[(1-\alpha - |\lambda - \alpha|R])^2 - (1-\lambda+n)^2R^2\} \leq 0. \tag{2.17}$$

The last inequality is equivalent to

$$\begin{aligned} &v^2[(1-\lambda)^2 - (1-\alpha - |\lambda - \alpha|R])^2 + (1-\lambda+n)^2R^2] \\ &\leq u^2[(1-\alpha - |\lambda - \alpha|R])^2 - (1-\lambda+n)^2R^2]. \end{aligned} \tag{2.18}$$

After an easy computation, by using (2.7) we obtain the inequality

$$\frac{v^2}{u^2} \leq \frac{R^2}{1-R^2} \leq \frac{(1-\alpha - |\lambda - \alpha|R)^2 - (1-\lambda+n)^2R^2}{(1-\lambda)^2 - (1-\alpha - |\lambda - \alpha|R)^2 + (1-\lambda+n)^2R^2}, \tag{2.19}$$

which is equivalent to $\Delta \leq 0$. Therefore $F(\rho) \leq 0$, a contradiction of (2.11). It follows

that $\operatorname{Re} p(z) > 0$, and

$$\operatorname{Re} \frac{zf'(z)}{f(z)} = \operatorname{Re}(1 - \alpha)p(z) + \alpha = (1 - \alpha) \operatorname{Re} p(z) + \alpha \geq \alpha \quad (2.20)$$

hence $f \in S^*(\alpha)$. □

If $\lambda = 0$ then

$$M_n(0, \alpha) = \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}} \quad (2.21)$$

and we obtain the following corollary.

COROLLARY 2.2. *If $f \in A_n$ and*

$$|f'(z) - 1| < \frac{(1 - \alpha)(n + 1)}{\alpha + \sqrt{(n + 1)^2 + 1}}, \quad (2.22)$$

then $f \in S^*(\alpha)$.

For $\alpha = 0$ this result was obtained in [2].

If $\lambda = 1$,

$$M_n(1, \alpha) = \frac{n(1 - \alpha)}{n + 1 - \alpha}, \quad (2.23)$$

and we obtain the following corollary.

COROLLARY 2.3 (see [3]). *If $f \in A_n$ and*

$$\left| f'(z) - \frac{f(z)}{z} \right| < \frac{n(1 - \alpha)}{n + 1 - \alpha}, \quad (2.24)$$

then $f \in S^*(\alpha)$.

If $\lambda = \alpha$,

$$M_n(\alpha, \alpha) = \frac{(1 - \alpha)(n + 1 - \alpha)}{\sqrt{(1 - \alpha)^2 + (1 - \alpha + n)^2}}. \quad (2.25)$$

COROLLARY 2.4. *If $f \in A_n$ and*

$$\left| f'(z) - \alpha \frac{f(z)}{z} + \alpha - 1 \right| < \frac{(1 - \alpha)(n + 1 - \alpha)}{\sqrt{(1 - \alpha)^2 + (1 - \alpha + n)^2}}, \quad (2.26)$$

then $f \in S^*(\alpha)$.

REFERENCES

- [1] S. S. Miller and P. T. Mocanu, *Differential Subordinations: Theory and Applications*, Monographs and Textbooks in Pure and Applied Mathematics, vol. 225, Marcel Dekker, New York, 2000. [MR 2001e:30036](#). [Zbl 0954.34003](#).
- [2] P. T. Mocanu, *Some simple criteria for starlikeness and convexity*, *Libertas Math.* **13** (1993), 27–40. [MR 94k:30027](#). [Zbl 0793.30008](#).
- [3] G. Oros, *On a condition for starlikeness*, *The Second International Conference on Basic Sciences and Advanced Technology (Assiut, Egypt, November 5–8), 2000*, pp. 89–94.

PETRU T. MOCANU: DEPARTMENT OF MATHEMATICS, BABES-BOLYAI UNIVERSITY, 3400 CLUJ-NAPOCA, ROMANIA

E-mail address: pmocanu@math.ubbcluj.ro

GH. OROS: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ORADEA, 3700 ORADEA, ROMANIA



Hindawi

Submit your manuscripts at
<http://www.hindawi.com>

