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Research Article

Existence and Uniqueness of Solutions for a Discrete Fractional Mixed Type Sum-Difference Equation Boundary Value Problem

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By means of Schauder's fixed point theorem and contraction mapping principle, we establish the existence and uniqueness of solutions to a boundary value problem for a discrete fractional mixed type sum-difference equation with the nonlinear term dependent on a fractional difference of lower order. Moreover, a suitable choice of a Banach space allows the solutions to be unbounded and two representative examples are presented to illustrate the effectiveness of the main results.

1. Introduction

For $a, b \in \mathbb{R}$, such that $b - a$ is a nonnegative integer, we denote $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $\mathbb{N}_a^b = \{a, a+1, \ldots, b\}$ throughout this paper. It is also worth noting that, in what follows, for any function u defined on $\mathbb{N}_a$, we appeal to the convention $\sum_{s=k_1}^{k_2} u(s) = 0$, when $k_1, k_2 \in \mathbb{N}_a$ with $k_1 > k_2$.

In this paper, we will consider the following discrete fractional mixed type sum-difference equation boundary value problem:

$$\begin{gathered}\Delta^{\alpha} u(t) + f\left(t + \alpha - 1, u(t + \alpha - 1), \Delta^{\alpha-1} u(t), (Tu)(t), (Su)(t)\right) = 0, \quad t \in \mathbb{N}_0, \\ u(\alpha - 2) = 0, \\ \Delta^{\alpha-1} u(\infty) = u_{\infty},\end{gathered} \tag{1}$$

where $\alpha \in (1, 2]$, Δ^{α} and $\Delta^{\alpha-1}$ denote the discrete Riemann-Liouville fractional differences of order α and $\alpha - 1$, respectively, $f : \mathbb{N}_{\alpha-1} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\Delta^{\alpha-1} u(\infty) = \lim_{t\to+\infty} \Delta^{\alpha-1} u(t) = u_{\infty} \in \mathbb{R}$, and

$$\begin{gathered}(Tu)(t) = \sum_{s=0}^{t} k(t, s)\, u(s + \alpha - 1), \\ (Su)(t) = \sum_{s=0}^{\infty} h(t, s)\, u(s + \alpha - 1),\end{gathered} \tag{2}$$

where $k : D \to \mathbb{R}$, $D = \{(t, s) \in \mathbb{N}_0 \times \mathbb{N}_0 : s \le t\}$, and $h : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$.

Continuous fractional calculus is a generalization of ordinary differentiation and integration on an arbitrary order that can be noninteger. This subject, as old as the problem of ordinary differential calculus, can go back to the times when Leibniz and Newton invented differential calculus. The theory of fractional differential equations has received a lot of attention and now constitutes a new important mathematical branch due to its extensive applications in various fields of science and engineer. For more details, see [1–13] and references therein.

It is well known that discrete analogues of differential equations can be very useful in applications [14], in particular for using computer to simulate the behavior of solutions for certain dynamic equations. However, compared to the long and rich history of continuous fractional calculus, discrete fractional calculus attracted mathematicians and scientists into its fairly new research area in a short period of time. In this time period, the theory of discrete calculus has been developed in many directions parallel to the theory in continuous fractional calculus such as initial value problems and boundary value problems for fractional difference equations, discrete Mittag-Leffler functions, and inequalities with discrete fractional operators; see [15–39] and the references therein. At the same time, in [27], Atıcı and Şengül have shown the usefulness of discrete Gompertz fractional difference equation for tumor growth model, which implies that discrete fractional difference calculus will provide a new excellent tool to model real world phenomena in the future.

Although, among all recently research topics, the branch of discrete finite fractional difference boundary value problems is currently undergoing active investigation [16, 31–38], significantly less is known about discrete infinite fractional difference boundary value problems with the nonlinear term dependent on a fractional difference operator. Here, we should point out that in [39], Lv and Feng, by simple analogy with the ordinary case, introduced some basic definitions of discrete fractional calculus for Banach-valued functions and initially studied a class of discrete infinite fractional mixed type sum-difference equation boundary value problems in abstract spaces by using contracting mapping principle. Furthermore, as far as we know, the theory of discrete fractional mixed type sum-difference equations boundary value problems is still a new research area. So in this paper, we continue to focus on this topic for real-valued functions and provide some sufficient conditions for the existence and uniqueness of solutions to problem (1). Particularly note that problem (1) is not like the problem in [39] and the biggest difference is the nonlinear term f in (1) explicitly dependent on the discrete fractional difference operator of lower order. Hence, these differences that cause the main difficulties that we have to deal with in this paper are those of constructing a special Banach space and establishing an appropriate compactness criterion in it.

The outline for the remainder of this paper is as follows. In Section 2, we recall some useful preliminaries for discrete fractional calculus and present the basic space and its compactness criterion for studying problem (1). In Section 3, by employing Schauder's fixed point theorem and contraction mapping principle, we establish the existence and uniqueness results of problem (1). In Section 4, two concrete examples are provided to illustrate the possible applications of the obtained analytical results.

2. Preliminaries

In this section, we firstly present here some necessary definitions and basic results about discrete fractional calculus.

Definition 1 (see [30]). For any t and ν, the falling factorial function is defined as

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, \tag{3}$$

provided that the right-hand side is well defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2 (see [40]). The νth discrete fractional sum of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad t \in \mathbb{N}_{a+\nu}. \tag{4}$$

Also, we define the trivial sum $\Delta_a^{-0} f(t) = f(t)$, $t \in \mathbb{N}_a$.

Definition 3 (see [30]). The νth discrete Riemann-Liouville fractional difference of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, is defined by

$$\Delta_a^{\nu} f(t) = \Delta^n \Delta_a^{-(n-\nu)} f(t), \quad t \in \mathbb{N}_{a+n-\nu}, \tag{5}$$

where n is the smallest integer greater than or equal to ν and Δ^n is the nth order forward difference operator. If $\nu = n \in \mathbb{N}_1$, then $\Delta_a^n f(t) = \Delta^n f(t)$.

Remark 4. From Definitions 2 and 3, it is easy to see that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$ and Δ_a^{ν} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n-\nu}$, where n is the smallest integer greater than or equal to ν. For ease of notation, we throughout this paper omit the subscript a in $\Delta_a^{\nu} f(t)$ and $\Delta_a^{-\nu} f(t)$ when it is not to lead to domains confusion and general ambiguity.

Lemma 5 (see [30]). *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\nu, \mu > 0$. Then*

$$\Delta_{a+\mu}^{-\nu} \Delta_a^{-\mu} f(t) = \Delta_a^{-\nu-\mu} f(t) = \Delta_{a+\nu}^{-\mu} \Delta_a^{-\nu} f(t), \quad t \in \mathbb{N}_{a+\mu+\nu}. \tag{6}$$

Lemma 6 (see [31]). *Let $n \in \mathbb{N}_1$ and $f : \mathbb{N}_{\nu-n} \to \mathbb{R}$ with $\nu \in (n-1, n]$. Then*

$$\Delta_0^{-\nu} \Delta_{\nu-n}^{\nu} f(t) = f(t) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \cdots + c_n t^{\underline{\nu-n}}, \tag{7}$$

for $t \in \mathbb{N}_{\nu-n}$, $c_i \in \mathbb{R}$, $i \in \mathbb{N}_1^n$.

Lemma 7 (see [15]). *Let $a \in \mathbb{R}$ and $\mu > 0$ be given. Then, for $\nu \in (n-1, n]$, $n \in \mathbb{N}_1$,*

$$\Delta_{a+\mu}^{\nu} (t-a)^{\underline{\mu}} = \mu^{\underline{\nu}} (t-a)^{\underline{\mu-\nu}}, \quad t \in \mathbb{N}_{a+\mu+n-\nu}. \tag{8}$$

Lemma 8 (see [28]). *Let $f : \mathbb{N}_a \to \mathbb{R}$, $p \in \mathbb{N}_1$ and $\nu > p$. Then*

$$\left(\Delta^p \Delta_a^{-\nu} f\right)(t) = \Delta_a^{-(\nu-p)} f(t). \tag{9}$$

Next, we define the space,

$$X = \left\{ u : \mathbb{N}_{\alpha-2} \longrightarrow \mathbb{R} \mid \sup_{t \in \mathbb{N}_{\alpha-2}} \frac{|u(t)|}{1 + t^{\underline{\alpha-1}}} < \infty, \sup_{t \in \mathbb{N}_0} \left| \Delta^{\alpha-1} u(t) \right| < \infty \right\} \tag{10}$$

equiped with the norm

$$\|u\|_X = \max \left\{ \sup_{t \in \mathbb{N}_{\alpha-2}} \frac{|u(t)|}{1 + t^{\underline{\alpha-1}}}, \sup_{t \in \mathbb{N}_0} \left| \Delta^{\alpha-1} u(t) \right| \right\}. \tag{11}$$

Furthermore, using the linear functional analysis theory, we can easily verify that $(X, \|\cdot\|_X)$ is a Banach space, and then we present the following compactness criterion in it.

Lemma 9. *Let $V \subseteq X$ be a bounded set. If for any given $\epsilon > 0$, there exists a positive integer $T = T(\epsilon)$ such that*

$$\left| \frac{u(t_2)}{1 + t_2^{\underline{\alpha-1}}} - \frac{u(t_1)}{1 + t_1^{\underline{\alpha-1}}} \right| < \epsilon, \tag{12}$$

$$\left| \Delta^{\alpha-1} u(s_2) - \Delta^{\alpha-1} u(s_1) \right| < \epsilon \tag{13}$$

whenever $t_1, t_2 \in \mathbb{N}_{\alpha+T}$, $s_1, s_2 \in \mathbb{N}_{T+1}$, and $u \in V$; then V is relatively compact in X.

Proof. Evidently, it is sufficient to prove that V is totally bounded. In what follows we divide this proof into two steps.

Step 1. Let us consider the case $t \in \mathbb{N}_{\alpha-2}^{\alpha+T}$.

Denote by $V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}$ the restriction of V on $\mathbb{N}_{\alpha-2}^{\alpha+T}$. Then $V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}$, equipped with the norm $\|u\|_\infty = \sup_{t \in \mathbb{N}_{\alpha-2}^{\alpha+T}} (|u(t)|/(1 + t^{\underline{\alpha-1}}))$, is a finite dimension Banach space. So we know that $V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}$ is relatively compact from the boundness of V; hence $V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}$ is totally bounded; namely, for any $\epsilon > 0$, there exist finitely many ball $B_\epsilon(u_i)$, $u_i \in V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}$, $i \in \mathbb{N}_1^n$, such that

$$V_{\mathbb{N}_{\alpha-2}^{\alpha+T}} \subset \bigcup_{i=1}^{n} B_\epsilon(u_i), \tag{14}$$

where $B_\epsilon(u_i) = \{u \in V_{\mathbb{N}_{\alpha-2}^{\alpha+T}} : \|u - u_i\|_\infty = \sup_{t \in \mathbb{N}_{\alpha-2}^{\alpha+T}} |u(t)/(1 + t^{\underline{\alpha-1}}) - u_i(t)/(1 + t^{\underline{\alpha-1}})| < \epsilon\}$.

Similarly, denote $V_{\mathbb{N}_0^{T+1}}^{\alpha-1} = \{\Delta^{\alpha-1} u : \mathbb{N}_0^{T+1} \to \mathbb{R} \mid u \in V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}\}$. Then $V_{\mathbb{N}_0^{T+1}}^{\alpha-1}$ is also a Banach space with the norm $\|\Delta^{\alpha-1} u\|_\infty = \sup_{t \in \mathbb{N}_0^{T+1}} |\Delta^{\alpha-1} u(t)|$ and it can be covered by finitely many balls $B_\epsilon(\Delta^{\alpha-1} v_j)$; that is,

$$V_{\mathbb{N}_0^{T+1}}^{\alpha-1} \subset \bigcup_{j=1}^{m} B_\epsilon\left(\Delta^{\alpha-1} v_j\right), \quad v_j \in V_{\mathbb{N}_{\alpha-2}^{\alpha+T}}, \tag{15}$$

where $B_\epsilon(\Delta^{\alpha-1} v_j) = \{\Delta^{\alpha-1} u \in V_{\mathbb{N}_0^{T+1}}^{\alpha-1} : \|\Delta^{\alpha-1} u - \Delta^{\alpha-1} v_j\|_\infty = \sup_{t \in \mathbb{N}_0^{T+1}} |\Delta^{\alpha-1} u(t) - \Delta^{\alpha-1} v_j(t)| < \epsilon\}$.

Step 2. Define $V_{ij} = \{u \in V : u_{\mathbb{N}_{\alpha-2}^{\alpha+T}} \in B_\epsilon(u_i),\ \Delta^{\alpha-1} u_{\mathbb{N}_0^{T+1}} \in B_\epsilon(\Delta^{\alpha-1} v_j)\}$.

Let us consider the case $t \in \mathbb{N}_{\alpha-2}^{\alpha+T}$. It is obvious that $V_{\mathbb{N}_{\alpha-2}^{\alpha+T}} \subset \bigcup_{1 \le i \le n, 1 \le j \le m} V_{ij_{\mathbb{N}_{\alpha-2}^{\alpha+T}}}$. Now, let us take $u_{ij} \in V_{ij}$; then V can be covered by the balls $B_{4\epsilon}(u_{ij})$, $i \in \mathbb{N}_1^n$, $j \in \mathbb{N}_1^m$, where

$$B_{4\epsilon}\left(u_{ij}\right) = \left\{u \in V : \left\|u - u_{ij}\right\|_X < 4\epsilon\right\}. \tag{16}$$

In fact, for any $u \in V$, the argument in Step 1 implies that there exist i and j such that $u_{\mathbb{N}_{\alpha-2}^{\alpha+T}} \in B_\epsilon(u_i)$, $\Delta^{\alpha-1} u_{\mathbb{N}_0^{T+1}} \in B_\epsilon(\Delta^{\alpha-1} v_j)$. Hence, for $t \in \mathbb{N}_{\alpha-2}^{\alpha+T}$ and $s \in \mathbb{N}_0^{T+1}$, we have

$$\begin{aligned} &\left| \frac{u(t)}{1 + t^{\underline{\alpha-1}}} - \frac{u_{ij}(t)}{1 + t^{\underline{\alpha-1}}} \right| \\ &\quad \le \left| \frac{u(t)}{1 + t^{\underline{\alpha-1}}} - \frac{u_i(t)}{1 + t^{\underline{\alpha-1}}} \right| + \left| \frac{u_i(t)}{1 + t^{\underline{\alpha-1}}} - \frac{u_{ij}(t)}{1 + t^{\underline{\alpha-1}}} \right| \\ &\quad < 2\epsilon, \end{aligned} \tag{17}$$

$$\begin{aligned} &\left| \Delta^{\alpha-1} u(s) - \Delta^{\alpha-1} u_{ij}(s) \right| \\ &\quad \le \left| \Delta^{\alpha-1} u(s) - \Delta^{\alpha-1} v_j(s) \right| \\ &\qquad + \left| \Delta^{\alpha-1} v_j(s) - \Delta^{\alpha-1} u_{ij}(s) \right| < 2\epsilon. \end{aligned} \tag{18}$$

For arbitrary $t \in \mathbb{N}_{\alpha+T}$, (12) and (17) yield that

$$\begin{aligned} &\left| \frac{u(t)}{1 + t^{\underline{\alpha-1}}} - \frac{u_{ij}(t)}{1 + t^{\underline{\alpha-1}}} \right| \\ &\quad \le \left| \frac{u(t)}{1 + t^{\underline{\alpha-1}}} - \frac{u(\alpha+T)}{1 + (\alpha+T)^{\underline{\alpha-1}}} \right| \\ &\qquad + \left| \frac{u(\alpha+T)}{1 + (\alpha+T)^{\underline{\alpha-1}}} - \frac{u_{ij}(\alpha+T)}{1 + (\alpha+T)^{\underline{\alpha-1}}} \right| \\ &\qquad + \left| \frac{u_{ij}(\alpha+T)}{1 + (\alpha+T)^{\underline{\alpha-1}}} - \frac{u_{ij}(t)}{1 + t^{\underline{\alpha-1}}} \right| < \epsilon + 2\epsilon + \epsilon \\ &\quad = 4\epsilon, \end{aligned} \tag{19}$$

and for any $s \in \mathbb{N}_{T+1}$, (13) and (18) ensure that

$$\begin{aligned} &\left| \Delta^{\alpha-1} u(s) - \Delta^{\alpha-1} u_{ij}(s) \right| \\ &\quad \le \left| \Delta^{\alpha-1} u(s) - \Delta^{\alpha-1} u(T+1) \right| \\ &\qquad + \left| \Delta^{\alpha-1} u(T+1) - \Delta^{\alpha-1} u_{ij}(T+1) \right| \\ &\qquad + \left| \Delta^{\alpha-1} u_{ij}(T+1) - \Delta^{\alpha-1} u_{ij}(s) \right| < \epsilon + 2\epsilon + \epsilon \\ &\quad = 4\epsilon. \end{aligned} \tag{20}$$

Relations (17)–(20) show that $\|u - u_{ij}\|_X < 4\epsilon$. Therefore, V is totally bounded and this lemma is proved. □

3. Main Result

In this section, we will establish the existence and uniqueness of solutions for problem (1) by using Schauder's fixed point theorem and contraction mapping principle. For the sake of convenience and to abbreviate our presentation, for any function $u \in X$, we denote

$$\begin{aligned} g_u(t) = f\Big(t + \alpha &- 1, u(t + \alpha - 1), \Delta^{\alpha-1} u(t), (Tu)(t), (Su)(t)\Big), \\ & t \in \mathbb{N}_0 \end{aligned} \tag{21}$$

in the sequel discussion and list here the following conditions:

(C_1) $k^* = \sup_{t\in\mathbb{N}_0}\sum_{s=0}^{t}|k(t,s)| < \infty$ and $h^* = \sup_{t\in\mathbb{N}_0}\sum_{s=0}^{\infty}(|h(t,s)|[1+(s+\alpha-1)^{\underline{\alpha-1}}]/(1+(t+\alpha-1)^{\underline{\alpha-1}})) < \infty$.

(C_2) There exist functions $p_i : \mathbb{N}_{\alpha-1} \to [0,\infty)$, $i \in \mathbb{N}_1^5$, with

$$p^* = \sum_{t=\alpha-1}^{\infty}\left\{\left(1+t^{\underline{\alpha-1}}\right)\left[p_1(t)+p_3(t)k^*+p_4(t)h^*\right] + p_2(t)\right\} < \Gamma(\alpha) \tag{22}$$

and $p_5^* = \sum_{t=\alpha-1}^{\infty} p_5(t) < \infty$ such that

$$\left|f(t,u,v,w,\varpi)\right| \le p_1(t)|u| + p_2(t)|v| + p_3(t)|w| + p_4(t)|\varpi| + p_5(t), \tag{23}$$

for $(t,u,v,w,\varpi) \in \mathbb{N}_{\alpha-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

(C_2') $f^* = \sum_{t=\alpha-1}^{\infty}|f(t,0,0,0,0)| < \infty$, and there exist nonnegative numbers a_i, $i \in \mathbb{N}_1^4$, and a function $q : \mathbb{N}_{\alpha-1} \to [0,+\infty)$ with $q^* = \sum_{t=\alpha-1}^{\infty} q(t)[(1+t^{\underline{\alpha-1}})(a_1+a_3k^*+a_4h^*)+a_2] < \Gamma(\alpha)$ such that

$$\left|f(t,u,v,w,\varpi) - f(t,\overline{u},\overline{v},\overline{w},\overline{\varpi})\right| \le q(t) \cdot \left(a_1|u-\overline{u}| + a_2|v-\overline{v}| + a_3|w-\overline{w}| + a_4|\varpi-\overline{\varpi}|\right) \tag{24}$$

for $t \in \mathbb{N}_{\alpha-1}$, $u,v,w,\varpi,\overline{u},\overline{v},\overline{w},\overline{\varpi} \in \mathbb{R}$.

Lemma 10. *If (C_1) and (C_2) hold, then, for any $u \in X$,*

$$\sum_{t=0}^{\infty}\left|g_u(t)\right| \le p^*\|u\|_X + p_5^*. \tag{25}$$

Proof. For any $u \in X$, $t \in \mathbb{N}_0$, using (C_1), (C_2), and the monotonicity of $\iota^{\underline{\alpha-1}}$, $\iota \in \mathbb{N}_{\alpha-1}$ produces

$$\begin{aligned}
\left|g_u(t)\right| &= \left|f\left(t+\alpha-1, u(t+\alpha-1), \Delta^{\alpha-1}u(t), (Tu)(t), (Su)(t)\right)\right| \\
&\le p_1(t+\alpha-1)|u(t+\alpha-1)| + p_2(t+\alpha-1)\cdot\left|\Delta^{\alpha-1}u(t)\right| + p_3(t+\alpha-1)|(Tu)(t)| + p_4(t+\alpha-1)|(Su)(t)| + p_5(t+\alpha-1) \\
&\le \left[1+(t+\alpha-1)^{\underline{\alpha-1}}\right]\left\{p_1(t+\alpha-1) + p_3(t+\alpha-1) \cdot \sum_{s=0}^{t}\frac{|k(t,s)|\left[1+(s+\alpha-1)^{\underline{\alpha-1}}\right]}{1+(t+\alpha-1)^{\underline{\alpha-1}}} + p_4(t+\alpha-1) \cdot \sum_{s=0}^{\infty}\frac{|h(t,s)|\left[1+(s+\alpha-1)^{\underline{\alpha-1}}\right]}{1+(t+\alpha-1)^{\underline{\alpha-1}}}\right\} \\
&\quad\cdot \sup_{t\in\mathbb{N}_0}\frac{|u(t+\alpha-1)|}{1+(t+\alpha-1)^{\underline{\alpha-1}}} + p_2(t+\alpha-1)\left|\Delta^{\alpha-1}u(t)\right| + p_5(t+\alpha-1) \le \left\{\left[1+(t+\alpha-1)^{\underline{\alpha-1}}\right]\right. \\
&\quad\cdot\left[p_1(t+\alpha-1) + p_3(t+\alpha-1)k^* + p_4(t+\alpha-1)h^*\right] + \left.p_2(t+\alpha-1)\right\}\|u\|_X + p_5(t+\alpha-1).
\end{aligned} \tag{26}$$

Summating both sides of (26), we can get (25). The proof is completed. □

Lemma 11. *If (C_1) and (C_2) hold, then the unique solution of problem (1) is*

$$u(t) = \sum_{s=0}^{\infty}G(t,s)g_u(s) + \frac{u_\infty}{\Gamma(\alpha)}t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-2}, \tag{27}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)}\begin{cases} t^{\underline{\alpha-1}} - (t-s-1)^{\underline{\alpha-1}}, & s \in \mathbb{N}_0^{t-\alpha}, \\ t^{\underline{\alpha-1}}, & s \in \mathbb{N}_{t-\alpha+1}. \end{cases} \tag{28}$$

Proof. If $u : \mathbb{N}_{\alpha-2} \to \mathbb{R}$ satisfies the equation of problem (1), then Lemma 6 implies that

$$u(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}g_u(s) + c_1t^{\underline{\alpha-1}} + c_2t^{\underline{\alpha-2}}, \tag{29}$$

for some $c_i \in \mathbb{R}$, $i \in \mathbb{N}_1^2$, $t \in \mathbb{N}_{\alpha-2}$. By $u(\alpha-2) = 0$, we get $c_2 = 0$.

Therefore,

$$u(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}g_u(s) + c_1t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-2}. \tag{30}$$

By virtue of Lemmas 5, 7, and 8, we have

$$\Delta^{\alpha-1}u(t) = -\sum_{s=0}^{t-1}g_u(s) + c_1\Gamma(\alpha), \quad t \in \mathbb{N}_0. \tag{31}$$

Using the condition $\Delta^{\alpha-1}u(\infty) = u_\infty$ in (31), we obtain

$$c_1 = \frac{1}{\Gamma(\alpha)}\left(\sum_{s=0}^{\infty}g_u(s) + u_\infty\right). \tag{32}$$

Now, substitution of c_1 into (30) gives

$$u(t) = -\frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}g_u(s) + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{\infty}t^{\underline{\alpha-1}}g_u(s) + \frac{u_\infty}{\Gamma(\alpha)}t^{\underline{\alpha-1}} = \sum_{s=0}^{\infty}G(t,s)\,g_u(s) + \frac{u_\infty}{\Gamma(\alpha)}t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-2}, \tag{33}$$

where $G(t,s)$ is defined by (28). The proof is completed. □

Remark 12. From the expression of $G(t,s)$, we can easily find that $G(t,s) \geq 0$ and $G(t,s)/(1+t^{\underline{\alpha-1}}) < 1/\Gamma(\alpha)$ for $(t,s) \in \mathbb{N}_{\alpha-2}\times\mathbb{N}_0$.

For any $u \in X$, define an operator $\mathcal{F}$ by

$$(\mathcal{F}u)(t) = \sum_{s=0}^{\infty}G(t,s)\,g_u(s) + \frac{u_\infty}{\Gamma(\alpha)}t^{\underline{\alpha-1}}, \quad t \in \mathbb{N}_{\alpha-2} \tag{34}$$

and due to Lemma 10 and Remark 12, we have

$$\frac{|(\mathcal{F}u)(t)|}{1+t^{\underline{\alpha-1}}} \leq \sum_{s=0}^{\infty}\frac{G(t,s)}{1+t^{\underline{\alpha-1}}}|g_u(s)| + \frac{|u_\infty|\,t^{\underline{\alpha-1}}}{\Gamma(\alpha)(1+t^{\underline{\alpha-1}})} \leq \frac{1}{\Gamma(\alpha)}\{p^*\|u\|_X + p_5^* + |u_\infty|\}, \quad t \in \mathbb{N}_{\alpha-2}. \tag{35}$$

On the other hand, by virtue of Lemmas 5, 7, 8, and 10, we get

$$\left(\Delta^{\alpha-1}\mathcal{F}u\right)(t) = \sum_{s=t}^{\infty}g_u(s) + u_\infty, \tag{36}$$

$$\left|\left(\Delta^{\alpha-1}\mathcal{F}u\right)(t)\right| \leq p^*\|u\|_X + p_5^* + |u_\infty| \tag{37}$$

which hold for $t \in \mathbb{N}_0$. So (35) and (37) imply that $\mathcal{F}: X \to X$ is well defined and bounded. Furthermore, from Lemma 11, we can transform problem (1) into an operator equation $u = \mathcal{F}u$ and it is clear to see that u is a solution of problem (1) which is equivalent to a fixed point of $\mathcal{F}$.

Remark 13. Setting $\overline{u} = \overline{v} = \overline{w} = \overline{\varpi} = 0$ in (C_2'), we have

$$|f(t,u,v,w,\varpi)| \leq q(t)\left(a_1|u| + a_2|v| + a_3|w| + a_4|\varpi|\right) + |f(t,0,0,0,0)| \tag{38}$$

for $(t,u,v,w,\varpi) \in \mathbb{N}_{\alpha-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$, which implies that condition (C_2') is stronger than (C_2). So under assumptions (C_1) and (C_2'), the operator $\mathcal{F}: X \to X$ defined by (34) is also well defined.

Now, we are in the position to give the main results of this work.

Theorem 14. *Assume that $f: \mathbb{N}_{\alpha-1}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous, and suppose that conditions (C_1) and (C_2) hold. Then problem (1) has at least one solution $u \in X$.*

Proof. In what follows, we divide this proof into three steps.

Step 1. Choose

$$R \geq \frac{|u_\infty| + p_5^*}{\Gamma(\alpha) - p^*} \tag{39}$$

and let

$$U = \{u \in X : \|u\|_X \leq R\}. \tag{40}$$

Then, for any $u \in U$, by (35), (37), and the fact $\Gamma(\alpha) \in (0,1]$, we can verify that $\|\mathcal{F}u\|_X \leq R$, which implies $\mathcal{F}: U \to U$.

Step 2. Let V be s subset of U. We employ Lemma 9 to verify that $\mathcal{F}V$ is relatively compact.

In view of Lemma 10 and the boundness of V, there exists $M > 0$ such that

$$\sum_{t=0}^{\infty}|g_u(t)| \leq M \quad \text{for any } u \in V. \tag{41}$$

By (34) and (36), we have

$$\left|\frac{(\mathcal{F}u)(t_2)}{1+t_2^{\underline{\alpha-1}}} - \frac{(\mathcal{F}u)(t_1)}{1+t_1^{\underline{\alpha-1}}}\right| \leq \frac{1}{\Gamma(\alpha)}\left|u_\infty + \sum_{\tau=0}^{\infty}g_u(\tau)\right| \cdot \left|\frac{t_2^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}} - \frac{t_1^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}}\right| + \frac{1}{\Gamma(\alpha)}\left|\sum_{\tau=0}^{t_2-\alpha}\frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}}g_u(\tau) - \sum_{\tau=0}^{t_1-\alpha}\frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}}g_u(\tau)\right|, \quad t_1, t_2 \in \mathbb{N}_{\alpha-2},$$

$$\left|\left(\Delta^{\alpha-1}\mathcal{F}u\right)(s_2) - \left(\Delta^{\alpha-1}\mathcal{F}u\right)(s_1)\right| \leq \sum_{\varrho=s_1}^{s_2-1}|g_u(\varrho)|, \quad \text{for } s_1, s_2 \in \mathbb{N}_0 \text{ with } s_1 < s_2. \tag{42}$$

Observing (42), together with $\lim_{t\to+\infty}(t^{\underline{\alpha-1}}/(1+t^{\underline{\alpha-1}})) = 1$ and the conditions of Lemma 9, we only need to show that, for any $\epsilon > 0$, there exists sufficiently large positive integer T such that, for any $t_1, t_2 \in \mathbb{N}_{\alpha+T}$,

$$\left|\sum_{\tau=0}^{t_2-\alpha}\frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}}g_u(\tau) - \sum_{\tau=0}^{t_1-\alpha}\frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}}g_u(\tau)\right| < \epsilon, \tag{43}$$

and for any $s_1, s_2 \in \mathbb{N}_{T+1}$ with $s_2 > s_1$,

$$\sum_{\varrho=s_1}^{s_2-1}|g_u(\varrho)| \leq \epsilon. \tag{44}$$

Relation (41) yields that there exists a positive number $L \in \mathbb{N}_0$ such that

$$\sum_{t=L+1}^{\infty} |g_u(t)| \leq \frac{\epsilon}{3} \quad \text{uniformly with respect to } u \in V. \quad (45)$$

On the other hand, from the monotonicity of $t^{\underline{\alpha-1}}$ and $\lim_{t\to+\infty}((t-L-1)^{\underline{\alpha-1}}/(1+t^{\underline{\alpha-1}})) = 1$, there exist $T \in \mathbb{N}_{L+1}$ such that, for any $t_1, t_2 \in \mathbb{N}_{\alpha+T}$ and $\tau \in \mathbb{N}_0^L$,

$$\begin{aligned} &\left| \frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}} - \frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}} \right| \\ &\quad \leq \left[1 - \frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}}\right] + \left[1 - \frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}}\right] \\ &\quad \leq \left[1 - \frac{(t_2-L-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}}\right] + \left[1 - \frac{(t_1-L-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}}\right] \\ &\quad < \frac{\epsilon}{3M}. \end{aligned} \quad (46)$$

Now taking $t_1, t_2 \in \mathbb{N}_{\alpha+T}$, by virtue of (41), (45), and (46), we get

$$\begin{aligned} &\left| \sum_{\tau=0}^{t_2-\alpha} \frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}} g_u(\tau) - \sum_{\tau=0}^{t_1-\alpha} \frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}} g_u(\tau) \right| \\ &\quad \leq \sum_{\tau=0}^{L} \left| \frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}} - \frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}} \right| |g_u(\tau)| \\ &\qquad + \sum_{\tau=L+1}^{t_2-\alpha} \frac{(t_2-\tau-1)^{\underline{\alpha-1}}}{1+t_2^{\underline{\alpha-1}}} |g_u(\tau)| \\ &\qquad + \sum_{\tau=L+1}^{t_1-\alpha} \frac{(t_1-\tau-1)^{\underline{\alpha-1}}}{1+t_1^{\underline{\alpha-1}}} |g_u(\tau)| \\ &\quad < \frac{\epsilon}{3M} \sum_{\tau=0}^{\infty} |g_u(\tau)| + 2 \sum_{\tau=L+1}^{\infty} |g_u(\tau)| < \epsilon. \end{aligned} \quad (47)$$

Moreover, from (45), we have

$$\sum_{\varrho=s_1}^{s_2-1} |g_u(\varrho)| \leq \sum_{\varrho=L+1}^{\infty} |g_u(\varrho)| < \epsilon \quad (48)$$

which holds for any $s_1, s_2 \in \mathbb{N}_{T+1}$ with $s_2 > s_1$ and arbitrary $u \in V$. Moreover, it follows from (47) and (48) that (43) and (44) hold. Consequently, by Lemma 9, $\mathcal{F}V$ is relatively compact.

Step 3. $\mathcal{F} : U \to U$ is a continuous operator.

Let u_n, $u \in U$, $n \in \mathbb{N}_1$ such that $\|u_n - u\|_X \to 0$ as $n \to \infty$. Then by (C_2), for any $\epsilon > 0$ there exists a positive integer L such that

$$\begin{aligned} &\sum_{t=\alpha+L}^{\infty} \left\{ \left(1 + t^{\underline{\alpha-1}}\right) [p_1(t) + p_3(t) k^* + p_4(t) h^*] \right. \\ &\quad \left. + p_2(t) \right\} < \frac{\Gamma(\alpha)}{6R} \epsilon, \\ &\sum_{t=\alpha+L}^{\infty} p_2(t) < \frac{\Gamma(\alpha)}{6} \epsilon. \end{aligned} \quad (49)$$

On the other hand, from the continuity of f, we know that there exists $N \in \mathbb{N}_1$ such that, for any $n > N$ and $t \in \mathbb{N}_0^L$,

$$\left| g_{u_n}(t) - g_u(t) \right| \leq \frac{\Gamma(\alpha)}{3(L+1)} \epsilon. \quad (50)$$

Therefore, for $t \in \mathbb{N}_{\alpha-2}$ and $n > N$, by (49)-(50) and Remark 12, we can obtain that

$$\begin{aligned} &\frac{\left|(\mathcal{F}u_n)(t) - (\mathcal{F}u)(t)\right|}{1 + t^{\underline{\alpha-1}}} \leq \sum_{s=0}^{\infty} \frac{G(t,s)}{1 + t^{\underline{\alpha-1}}} \left| g_{u_n}(s) - g_u(s) \right| \\ &\quad < \frac{1}{\Gamma(\alpha)} \left\{ \sum_{s=0}^{L} \left| g_{u_n}(s) - g_u(s) \right| + \sum_{s=L+1}^{\infty} \left| g_{u_n}(s) \right. \right. \\ &\quad \left. \left. - g_u(s) \right| \right\} \leq \frac{1}{\Gamma(\alpha)} \left\{ \sum_{s=0}^{L} \left| g_{u_n}(s) - g_u(s) \right| \right. \\ &\quad + 2R \sum_{s=\alpha+L}^{\infty} \left\{ \left(1 + s^{\underline{\alpha-1}}\right) \right. \\ &\quad \left. \cdot [p_1(s) + p_3(s) k^* + p_4(s) h^*] + p_2(s) \right\} \\ &\quad \left. + 2 \sum_{s=\alpha+L}^{\infty} p_5(s) \right\} < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned} \quad (51)$$

Meanwhile, for $t \in \mathbb{N}_0$ and $n > N$, applying (49)-(50) again, we can easily verify that

$$\begin{aligned} &\left| \left(\Delta^{\alpha-1} \mathcal{F} u_n\right)(t) - \left(\Delta^{\alpha-1} \mathcal{F} u\right)(t) \right| \\ &\quad \leq \sum_{s=t}^{\infty} \left| g_{u_n}(t) - g_u(t) \right| \leq \sum_{s=0}^{\infty} \left| g_{u_n}(t) - g_u(t) \right| \\ &\quad < \Gamma(\alpha) \epsilon < \epsilon. \end{aligned} \quad (52)$$

Then, by virtue of (51) and (52), we conclude that $\|\mathcal{F}u_n - \mathcal{F}u\|_X \leq \epsilon$ as $n > N$, which asserts the continuity of $\mathcal{F}$.

Therefore, by Schauder's fixed point theorem, we obtain that problem (1) has at least one solution in U and the proof is finished. □

Theorem 15. *Suppose that conditions (C_1) and (C_2') hold. Then problem (1) has a unique solution $u \in X$.*

Proof. For any $u, v \in X$, in view of (C_2') and Remark 12, we have

$$\begin{aligned}\frac{|(\mathcal{F}u)(t)-(\mathcal{F}v)(t)|}{1+t^{\underline{\alpha-1}}} &\le \sum_{s=0}^{\infty}\frac{G(t,s)}{1+t^{\underline{\alpha-1}}}\left|g_u(s)-g_v(s)\right| \\ &\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{\infty}q(s+\alpha-1) \\ &\cdot\Big(a_1\left|u(s+\alpha-1)-v(s+\alpha-1)\right| \\ &+a_2\left|\Delta^{\alpha-1}u(s)-\Delta^{\alpha-1}v(s)\right| \\ &+a_3\left|(Tu)(s)-(Tv)(s)\right|+a_4\left|(Su)(s)-(Sv)(s)\right|\Big) \\ &\le \frac{1}{\Gamma(\alpha)}q^*\|u-v\|_X, \quad t\in\mathbb{N}_{\alpha-2}.\end{aligned} \tag{53}$$

On the other hand, by (36) and using (C_2') again, we have

$$\begin{aligned}\left|\left(\Delta^{\alpha-1}\mathcal{F}u\right)(t)-\left(\Delta^{\alpha-1}\mathcal{F}v\right)(t)\right| &\le \sum_{s=t}^{\infty}\left|g_u(s)-g_v(s)\right| \\ &\le q^*\|u-v\|_X, \quad t\in\mathbb{N}_0.\end{aligned} \tag{54}$$

So, from (53), (54) and the facts that $q^* < \Gamma(\alpha)$ and $\Gamma(\alpha) \in (0,1]$ when $\alpha \in (1,2]$, we know that $\mathcal{F}$ is a contraction mapping. By means of Banach contraction mapping principle, we get that $\mathcal{F}$ has a unique fixed point in X; that is, problem (1) has a unique solution. This completes the proof. □

4. Examples

In this section, we will illustrate the possible applications of the above established analytical results with the following two concrete examples.

Example 1. Consider the discrete fractional difference boundary value problem:

$$\begin{aligned}&\Delta^{3/2}u(t)+\frac{3^{-(t+1)}}{\left[1+(t+1/2)^{\underline{1/2}}\right]^2}\sin\left[u(t+1/2)\right] \\ &+\frac{4^{-(t+1/2)}}{\left[1+(t+1/2)^{\underline{1/2}}\right]}\times\left\{1+u\left(t+\frac{1}{2}\right)\right. \\ &+\left[1+\left(t+\frac{1}{2}\right)^{\underline{1/2}}\right]\Delta^{1/2}u(t) \\ &+\sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}u\left(s+\frac{1}{2}\right) \\ &\left.+\sum_{s=0}^{\infty}\frac{\cos\left(t^2s\right)}{(s+2)^{\underline{2}}\left[1+(s+1/2)^{\underline{1/2}}\right]}u\left(s+\frac{1}{2}\right)\right\}^{1/4}=0, \\ &\qquad t\in\mathbb{N}_0, \\ &u\left(-\frac{1}{2}\right)=0, \\ &\Delta^{1/2}u(\infty)=u_\infty.\end{aligned} \tag{55}$$

Conclusion. Problem (55) has at least one solution $u : \mathbb{N}_{-1/2} \to \mathbb{R}$.

Proof. Corresponding to problem (1), we have $\alpha = 3/2$,

$$\begin{aligned}k(t,s)&=\frac{1}{(t+s+2)^{\underline{2}}}, \\ h(t,s)&=\frac{\cos\left(t^2s\right)}{(s+2)^{\underline{2}}\left[1+(s+1/2)^{\underline{1/2}}\right]}, \\ f(t,u,v,w,\varpi)&=\frac{3^{-t-1/2}}{\left(1+t^{\underline{1/2}}\right)^2}\sin u \\ &+\frac{4^{-t}}{\left(1+t^{\underline{1/2}}\right)}\left[1+u+\left(1+t^{\underline{1/2}}\right)v+w+\varpi\right]^{1/4},\end{aligned} \tag{56}$$

$(t,u,v,w,\varpi)\in\mathbb{N}_{1/2}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}$.

From the expression of f, it is easy to see that f is continuous. Furthermore, we can verify that

$$\begin{aligned}k^*&=\sup_{t\in\mathbb{N}_0}\sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}=\sup_{t\in\mathbb{N}_0}\frac{1}{2(t+1)}=\frac{1}{2}<\infty, \\ h^*&=\sup_{t\in\mathbb{N}_0}\frac{1}{1+(t+1/2)^{\underline{1/2}}} \\ &\cdot\sum_{s=0}^{\infty}\frac{\left|\cos\left(t^2s\right)\right|\left[1+(s+1/2)^{\underline{1/2}}\right]}{\left[1+(s+1/2)^{\underline{1/2}}\right](s+2)^{\underline{2}}} \\ &\le\sup_{t\in\mathbb{N}_0}\frac{1}{1+(t+1/2)^{\underline{1/2}}}\sum_{s=0}^{\infty}\frac{1}{(s+2)^{\underline{2}}}\le\frac{1}{1+\Gamma(3/2)} \\ &<\frac{3}{4}<\infty.\end{aligned} \tag{57}$$

So condition (C_1) is satisfied.

On the other hand, by using a simple inequality

$$(1+z)^\gamma\le 1+\gamma z, \quad \text{for } z\in[0,+\infty), \ \gamma\in(0,1), \tag{58}$$

we have

$$\begin{aligned} \left|f(t,u,v,w,\varpi)\right| &\le \frac{3^{-(t+1/2)}}{\left(1+t^{1/2}\right)^2}|\sin u| + \frac{4^{-t}}{\left(1+t^{1/2}\right)}\Big[1 \\ &\quad + |u| + \left(1+t^{1/2}\right)|v| + |w| + |\varpi|\Big]^{1/4} \\ &\le \frac{4^{-(t+1)}}{\left(1+t^{1/2}\right)}\left[|u| + \left(1+t^{1/2}\right)|v| + |w| + |\varpi|\right] \\ &\quad + \frac{3^{-(t+1/2)}}{\left(1+t^{1/2}\right)^2} + \frac{4^{-t}}{\left(1+t^{1/2}\right)}, \end{aligned} \tag{59}$$

and therefore

$$\begin{aligned} \left|f(t,u,v,w,\varpi)\right| &\le p_1(t)|u| + p_2(t)|v| + p_3(t)|w| \\ &\quad + p_4(t)|\varpi| + p_5(t), \end{aligned} \tag{60}$$

where

$$\begin{aligned} p_1(t) &= p_3(t) = p_4(t) = \frac{4^{-(t+1)}}{\left(1+t^{1/2}\right)}, \\ p_2(t) &= 4^{-(t+1)}, \\ p_5(t) &= \frac{3^{-(t+1/2)}}{\left(1+t^{1/2}\right)^2} + \frac{4^{-t}}{\left(1+t^{1/2}\right)}. \end{aligned} \tag{61}$$

By directly calculation, we have

$$\begin{aligned} p^* &< \frac{13}{24} < \Gamma\left(\frac{3}{2}\right), \\ p_5^* &< \frac{7}{6}. \end{aligned} \tag{62}$$

Thus, condition (C_2) holds. So, by Theorem 14, our conclusion follows. □

Example 2. Consider the following problem:

$$\begin{aligned} &\Delta^{4/3}u(t) + \frac{2^{-(t+1)}}{8\left[1+(t+1/3)^{\underline{1/3}}\right]}\left\{\cos\left[u(t+1/3)\right] + \sin\left[\Delta^{1/3}u(t)\right]\right\} + \frac{3^{-(t+1)}}{e^2\left[1+(t+1/3)^{\underline{1/3}}\right]}\left[\sum_{s=0}^{t}\frac{1}{(t+s+2)^{\underline{2}}}u\left(s+\frac{1}{3}\right)\right] \\ &\quad + \frac{e^{-(t+1)}}{e^3\left[2+\cos(t+1/3)+(t+1/3)^{\underline{1/3}}\right]}\left\{\sum_{s=0}^{\infty}\frac{\sin\left(t+e^s\right)}{(s+2)^{\underline{2}}\left[1+(s+1/3)^{\underline{1/3}}\right]}u\left(s+\frac{1}{3}\right)\right\} = 0, \quad t \in \mathbb{N}_0, \\ &u\left(-\frac{2}{3}\right) = 0, \\ &\Delta^{1/3}u(\infty) = u_\infty. \end{aligned} \tag{63}$$

Conclusion. Problem (63) has a unique solution $u : \mathbb{N}_{-2/3} \to \mathbb{R}$.

Proof. It is easy to see that problem (63) is the form of problem (1), where $\alpha = 4/3$,

$$\begin{aligned} k(t,s) &= \frac{1}{(t+s+2)^{\underline{2}}}, \\ h(t,s) &= \frac{\sin\left(t+e^s\right)}{(s+2)^{\underline{2}}\left[1+(s+1/3)^{\underline{1/3}}\right]}, \\ f(t,u,v,w,\varpi) &= \frac{2^{-(t+2/3)}}{8\left[1+t^{\underline{1/3}}\right]}(\cos u + \sin v) \\ &\quad + \frac{3^{-(t+2/3)}}{e^2\left[1+t^{\underline{1/3}}\right]}w \\ &\quad + \frac{e^{-(t+2/3)}}{e^3\left[2+\cos t + t^{\underline{1/3}}\right]}\varpi, \end{aligned} \tag{64}$$

for $(t,u,v,w,\varpi) \in \mathbb{N}_{1/3} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Choosing $a_1 = a_2 = 1/8$, $a_3 = 1/e^2$, $a_4 = 1/e^3$, and $q(t) = 2^{-(t+2/3)}/(1+t^{\underline{1/3}})$, $t \in \mathbb{N}_{1/3}$, then we can verify that $f^* < 1/4$, $k^* = 1/2$, $h^* < 0.5283$, $q^* < 0.3454 < 0.8930 \approx \Gamma(4/3)$, and

$$\begin{aligned} &\left|f(t,u,v,w,\varpi) - f(t,\overline{u},\overline{v},\overline{w},\overline{\varpi})\right| \le q(t) \\ &\quad \cdot \left(a_1|u-\overline{u}| + a_2|v-\overline{v}| + a_3|w-\overline{w}| + a_4|\varpi-\overline{\varpi}|\right) \end{aligned} \tag{65}$$

holds for any $t \in \mathbb{N}_{1/3}$, $u, v, w, \varpi, \overline{u}, \overline{v}, \overline{w}, \overline{\varpi} \in \mathbb{R}$.

Clearly, all conditions of Theorem 15 are fulfilled. Therefore, we can conclude that problem (63) has a unique solution. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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