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Research Article

On IVF Approximating Spaces

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We introduce the concept of IVF approximating spaces and obtain decision conditions that every IVF topological space is an IVF approximating space.

1. Introduction

Rough set theory was proposed by Pawlak [1] as a mathematical tool to handle imprecision and uncertainty in data analysis. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields [2–5].

The basic structure of rough set theory is an approximation space. Based on it, lower and upper approximations can be induced. Using these approximations, knowledge hidden in information systems may be revealed and expressed in the form of decision rules (see [2]).

As a generalization of Zadeh's fuzzy set, interval-valued fuzzy (IVF, for short) sets were introduced by Gorzałczany [6] and Türksen [7]. Mondal and Samanta [8] defined topology of IVF sets and studied their properties.

By replacing crisp relations with IVF relations, Sun et al. [9] introduced IVF rough sets based on an IVF approximation space, defined IVF information systems, and discussed their attribute reduction. Gong et al. [10] presented IVF rough sets based on approximation spaces and studied the knowledge discovery in IVF information systems.

Topological structure is an important base for knowledge extraction and processing. Therefore, an interesting and natural research topic in rough set theory is to study the relationship between rough sets and topologies.

The purpose of this paper is to investigate IVF approximating space, that is, a particular type of IVF topological spaces where the given IVF topology coincides with the IVF topology induced by some reflexive IVF relation.

2. Preliminaries

Throughout this paper, "interval-valued fuzzy" is denoted briefly by "IVF." U denotes a nonempty set called the universe. I denotes $[0,1]$, and $[I]$ denotes $\{[a,b] : a, b \in I \text{ and } a \le b\}$. $F^{(i)}(U)$ denotes the family of all IVF sets in U. $\overline{a}$ denotes $[a,a]$ for each $a \in [0,1]$.

For any $[a_j, b_j] \in [I]$ $(j = 1, 2)$, we define

$$\begin{gathered}
[a_1,b_1] = [a_2,b_2] \Longleftrightarrow a_1 = a_2, \quad b_1 = b_2, \\
[a_1,b_1] \le [a_2,b_2] \Longleftrightarrow a_1 \le a_2, \quad b_1 \le b_2, \\
[a_1,b_1] < [a_2,b_2] \Longleftrightarrow [a_1,b_1] \le [a_2,b_2], \\
[a_1,b_1] \ne [a_2,b_2], \\
\overline{1} - [a_1,b_1] \quad \text{or} \quad [a_1,b_1]^c = [1-b_1, 1-a_1].
\end{gathered} \tag{1}$$

Obviously, $([a,b]^c)^c = [a,b]$ for each $[a,b] \in [I]$.

Definition 1 (see [6, 7]). For each $\{[a_j, b_j] : j \in J\} \subseteq [I]$, one define

$$\bigvee_{j\in J} [a_j, b_j] = \left[\bigvee_{j\in J} a_j, \bigvee_{j\in J} b_j\right], \qquad \bigwedge_{j\in J} [a_j, b_j] = \left[\bigwedge_{j\in J} a_j, \bigwedge_{j\in J} b_j\right], \tag{2}$$

where $\bigvee_{j\in J} a_j = \sup\{a_j : j \in J\}$ and $\bigwedge_{j\in J} a_j = \inf\{a_j : j \in J\}$.

Definition 2 (see [6, 7]). An IVF set A in U is defined by a mapping $A : U \to [I]$.

Denote

$$A(x) = [A^-(x), A^+(x)] \quad (x \in U). \tag{3}$$

Then $A^-(x)$ (resp., $A^+(x)$) is called the lower (resp., upper) degree at which x belongs to A. A^- (resp., A^+) is called the lower (resp., upper) IVF set of A.

The set of all IVF sets in U is denoted by $F^{(i)}(U)$.

Let $a, b \in I$. $\widetilde{[a,b]}$ represents the IVF set which satisfies $\widetilde{[a,b]}(x) = [a,b]$ for each $x \in U$. We denoted $\widetilde{[a,a]}$ by $\tilde{a}$.

We recall some basic operations on $F^{(i)}(U)$ as follows [6, 7]: for any $A, B \in F^{(i)}(U)$ and $[a,b] \in [I]$,

(1) $A = B \Longleftrightarrow A(x) = B(x)$ for each $x \in U$,

(2) $A \subseteq B \Longleftrightarrow A(x) \le B(x)$ for each $x \in U$,

(3) $A = B^c \Longleftrightarrow A(x) = B(x)^c$ for each $x \in U$,

(4) $(A \cap B)(x) = A(x) \wedge B(x)$ for each $x \in U$,

(5) $(A \cup B)(x) = A(x) \vee B(x)$ for each $x \in U$.

Moreover,

$$\left(\bigcup_{j\in J} A\right)(x) = \bigvee_{j\in J} A(x), \qquad \left(\bigcap_{j\in J} A\right)(x) = \bigwedge_{j\in J} A(x), \tag{4}$$

where $\{A_j : j \in J\} \subseteq F^{(i)}(U)$.

(6) $([a,b]A)(x) = [a,b] \wedge [A^-(x), A^+(x)]$ for each $x \in U$.

Obviously,

$$A = B \Longleftrightarrow A^- = B^-, \qquad A^+ = B^+, \qquad \left(\widetilde{[a,b]}\right)^c = \widetilde{[a,b]^c} \quad ([a,b] \in [I]). \tag{5}$$

Definition 3 (see [8]). $A \in F^{(i)}(U)$ is called an IVF point in U, if there exist $[a,b] \in [I] - \{\overline{0}\}$ and $x \in U$ such that

$$A(y) = \begin{cases} [a,b], & y = x, \\ \overline{0}, & y \ne x. \end{cases} \tag{6}$$

We denote A by $x_{[a,b]}$.

If $[a,b] = \overline{1}$, then

$$x_{\overline{1}}(y) = \begin{cases} \overline{1}, & y = x, \\ \overline{0}, & y \ne x. \end{cases} \tag{7}$$

Remark 4. $A = \bigcup_{x\in U} (A(x) x_{\overline{1}}) (A \in F^{(i)}(U))$.

Definition 5 (see [8]). $\tau \subseteq F^{(i)}(U)$ is called an IVF topology on U, if

(i) $\tilde{0}, \tilde{1} \in \tau$,

(ii) $A, B \in \tau \Rightarrow A \cap B \in \tau$,

(iii) $\{A_j : j \in J\} \subseteq \tau \Rightarrow \bigcup_{j\in J} A_j \in \tau$.

The pair (U, τ) is called an IVF topological space. Every member of τ is called an IVF open set in U. Its complement is called an IVF closed set in U.

An IVF topology τ is called Alexandrov, if (ii) in Definition 5 is replaced by

(ii)′ $\{A_j : j \in J\} \subseteq \tau \Rightarrow \bigcap_{j\in J} A_j \in \tau$.

We denote $\tau^c = \{A : A^c \in \tau\}$.

The interior and closure of $A \in F^{(i)}(U)$ denoted, respectively, by $\text{int}(A)$ and $\text{cl}(A)$, are defined as follows:

$$\text{int}(A) \quad \text{or} \quad \text{int}_\tau(A) = \bigcup \{B \in \tau : B \subseteq A\}, \qquad \text{cl}(A) \quad \text{or} \quad \text{cl}_\tau(A) = \bigcap \{B \in \tau^c : B \supseteq A\}. \tag{8}$$

Proposition 6 (see [8]). *Let τ be an IVF topology on U. Then, for any $A, B \in F^{(i)}(U)$,*

(1) $\text{int}(\tilde{1}) = \tilde{1}$, $\text{cl}(\tilde{0}) = \tilde{0}$,

(2) $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$,

(3) $A \subseteq B \Rightarrow \text{int}(A) \subseteq \text{int}(B)$, $\text{cl}(A) \subseteq \text{cl}(B)$,

(4) $\text{int}(A^c) = (\text{cl}(A))^c$, $\text{cl}(A^c) = (\text{int}(A))^c$,

(5) $\text{int}(A \cap B) = \text{int}(A) \cap \text{int}(B)$, $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$,

(6) $\text{int}(\text{int}(A)) = \text{int}(A)$, $\text{cl}(\text{cl}(A)) = \text{cl}(A)$.

3. IVF Approximation Spaces and IVF Rough Sets

Recall that R is called an IVF relation on U if $R \in F^{(i)}(U \times U)$.

Definition 7 (see [9]). Let R be an IVF relation on U. Then, R is called

(1) reflexive, if $R(x,x) = \overline{1}$ for each $x \in U$,

(2) symmetric, if $R(x,y) = R(y,x)$ for any $x, y \in U$,

(3) transitive, if $R(x,z) \ge R(x,y) \wedge R(y,z)$ for any $x, y, z \in U$.

Let R be an IVF relation on U. R is called preorder if R is reflexive and transitive (see [11]).

Definition 8 (see [9]). Let R be an IVF relation on U. The pair (U, R) is called an IVF approximation space. For each $A \in F^{(i)}(U)$, the IVF lower and the IVF upper approximations of A with respect to (U, R), denoted by $\underline{R}(A)$ and $\overline{R}(A)$, are two IVF sets and are, respectively, defined as follows:

$$\underline{R}(A)(x) = \bigwedge_{y \in U} \left(A(y) \vee \left(\bar{1} - R(x, y) \right) \right) \quad (x \in U),$$
$$\overline{R}(A)(x) = \bigvee_{y \in U} \left(A(y) \wedge R(x, y) \right) \quad (x \in U). \tag{9}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the IVF rough set of A with respect to (U, R).

Remark 9. Let (U, R) be an IVF approximation space. Then,

(1) for each $x, y \in U$,

$$\overline{R}(x_{\bar{1}})(y) = R(y, x), \qquad \underline{R}\left((x_{\bar{1}})^c \right)(y) = \bar{1} - R(y, x); \tag{10}$$

(2) for each $[a, b] \in [I]$, $\underline{R}(\widetilde{[a, b]}) \supseteq \widetilde{[a, b]} \supseteq \overline{R}(\widetilde{[a, b]})$.

Proposition 10 (see [9]). *Let (U, R) be an IVF approximation space. Then, for each $A \in F^{(i)}(U)$,*

$$\left(\underline{R}(A) \right)^- = \underline{R^+}(A^-), \qquad \left(\underline{R}(A) \right)^+ = \underline{R^-}(A^+),$$
$$\left(\overline{R}(A) \right)^- = \overline{R^-}(A^-), \qquad \left(\overline{R}(A) \right)^+ = \overline{R^+}(A^+). \tag{11}$$

Proposition 11. *Let (U, R) be an IVF approximation space. Then, for any $A, B \in F^{(i)}(U)$, $\{A_j : j \in J\} \subseteq F^{(i)}(U)$, and $[a, b] \in [I]$,*

(1) $\underline{R}(\tilde{1}) = \tilde{1}$, $\overline{R}(\tilde{0}) = \tilde{0}$,

(2) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$,

(3) $\underline{R}(A^c) = (\overline{R}(A))^c$, $\overline{R}(A^c) = (\underline{R}(A))^c$,

(4) $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}(A_j)$, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{R}(A_j)$,

(5) $\underline{R}(\widetilde{[a, b]} \cup A) = \widetilde{[a, b]} \cup \underline{R}(A)$, $\overline{R}([a, b]A) = [a, b]\overline{R}(A)$.

Proof. (1) and (2) are obvious.

(3) For each $x \in U$, by Proposition 10,

$$\begin{aligned}
&\underline{R}(A^c)(x) \\
&= \left[\bigwedge_{y \in U} \left((1 - A^+(y)) \vee (1 - R^+(x, y)) \right), \bigwedge_{y \in U} \left((1 - A^-(y)) \vee (1 - R^-(x, y)) \right) \right] \\
&= \left[\bigwedge_{y \in U} \left(1 - A^+(y) \wedge R^+(x, y) \right), \bigwedge_{y \in U} \left(1 - A^-(y) \wedge R^-(x, y) \right) \right] \\
&= \left[1 - \bigvee_{y \in U} \left(A^+(y) \wedge R^+(x, y) \right), 1 - \bigvee_{y \in U} \left(A^-(y) \wedge R^-(x, y) \right) \right] \\
&= \left[1 - \left(\overline{R}(A) \right)^+(x), 1 - \left(\overline{R}(A) \right)^-(x) \right] \\
&= \bar{1} - \left[\left(\overline{R}(A) \right)^-(x), \left(\overline{R}(A) \right)^+(x) \right] \\
&= \bar{1} - \overline{R}(A)(x) = \left(\overline{R}(A) \right)^c(x).
\end{aligned} \tag{12}$$

Then, $\underline{R}(A^c) = (\overline{R}(A))^c$.

Pick $A = B^c$. Since $\underline{R}(B^c) = (\overline{R}(B))^c$,

$$\overline{R}(A^c) = \overline{R}(B) = \left(\left(\overline{R}(B) \right)^c \right)^c = \left(\underline{R}(B^c) \right)^c = \left(\underline{R}(A) \right)^c. \tag{13}$$

(4) For each $x \in U$, by

$$\begin{aligned}
&\underline{R}\left(\bigcap_{j \in J} A_j \right)(x) \\
&= \bigwedge_{y \in U} \left(\left(\bigcap_{j \in J} A_j \right)(y) \vee \left(\bar{1} - R(x, y) \right) \right) \\
&= \bigwedge_{y \in U} \left(\left(\bigwedge_{j \in J} A_j(y) \right) \vee \left(\bar{1} - R(x, y) \right) \right) \\
&= \bigwedge_{y \in U} \left(\bigwedge_{j \in J} \left(A_j(y) \vee \left(\bar{1} - R(x, y) \right) \right) \right) \\
&= \bigwedge_{j \in J} \left(\bigwedge_{y \in U} \left(A_j(y) \vee \left(\bar{1} - R(x, y) \right) \right) \right) \\
&= \bigwedge_{j \in J} \underline{R}(A)(x) = \left(\bigcap_{j \in J} \underline{R}(A_j) \right)(x),
\end{aligned} \tag{14}$$

we have $\underline{R}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}(A_j)$.

By (3) and $\underline{R}(\bigcap_{j \in J} (A_j)^c) = \bigcap_{j \in J} \underline{R}((A_j)^c)$, we have

$$\begin{aligned}
&\left(\overline{R}\left(\bigcup_{j \in J} A_j \right) \right)^c \\
&= \underline{R}\left(\left(\bigcup_{j \in J} A_j \right)^c \right) = \underline{R}\left(\bigcap_{j \in J} A_j^c \right) \\
&= \bigcap_{j \in J} \underline{R}\left(A_j^c \right) = \bigcap_{j \in J} \left(\overline{R}(A_j) \right)^c = \left(\bigcup_{j \in J} \overline{R}(A_j) \right)^c.
\end{aligned} \tag{15}$$

Then, $\overline{R}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{R}(A_j)$.

(5) For each $x \in U$, by Proposition 10,

$$\begin{aligned}
&\overline{R}([a,b]A)(x) \\
&= \bigvee_{y\in U}(([a,b]A)(y)\wedge R(x,y)) \\
&= \bigvee_{y\in U}(([a,b]\wedge A(y))\wedge R(x,y)) \\
&= \bigvee_{y\in U}([a,b]\wedge (A(y)\wedge R(x,y))) \\
&= [a,b]\wedge\left(\bigvee_{y\in U}(A(y)\wedge R(x,y))\right) \\
&= [a,b]\wedge \overline{R}(A)(x) = \left([a,b]\overline{R}(A)\right)(x).
\end{aligned} \tag{16}$$

Then, $\overline{R}([a,b]A) = [a,b]\overline{R}(A)$.

Similarly, we can prove that $\underline{R}(\widetilde{[a,b]}\cup A) = \widetilde{[a,b]}\cup\underline{R}(A)$. □

Theorem 12. *Let R be an IVF relation on U, and let τ be an IVF topology on U. If one of the following conditions is satisfied, then R is preorder.*

(1) *$\underline{R}$ is the interior operator of τ.*

(2) *$\overline{R}$ is the closure operator of τ.*

Proof. By Propositions 6(4) and 11(3), (1) and (2) are equivalent. We only need to prove that (2) implies the reflexivity and transitivity of R.

By Remark 9(1), $\overline{R}(x_{\overline{1}})(y) = R(y,x)$ for any $x, y \in U$. Note that $\overline{R}$ is the closure operator of τ. Then, for each $x \in U$,

$$R(x,x) = \overline{R}(x_{\overline{1}})(x) = \mathrm{cl}_\tau(x_{\overline{1}})(x) \ge x_{\overline{1}}(x) = \overline{1}. \tag{17}$$

Thus, R is reflexive.

For any $x, y, z \in U$, denote $\mathrm{cl}_\tau(z_{\overline{1}})(y) = [a,b]$, and by Remark 4, Remark 9(1) and, Proposition 11(5),

$$\begin{aligned}
R(x,y)\wedge R(y,z) &= \overline{R}(y_{\overline{1}})(x)\wedge\overline{R}(z_{\overline{1}})(y) \\
&= \overline{R}(y_{\overline{1}})(x)\wedge \mathrm{cl}_\tau(z_{\overline{1}})(y) \\
&= \overline{R}(y_{\overline{1}})(x)\wedge[a,b] \\
&= [a,b]\overline{R}(y_{\overline{1}})(x) = \overline{R}([a,b]y_{\overline{1}})(x) \\
&= \mathrm{cl}_\tau([a,b]y_{\overline{1}})(x) \\
&= \mathrm{cl}_\tau(\mathrm{cl}_\tau(z_{\overline{1}})(y)y_{\overline{1}})(x) \\
&\le cl_\tau\left(\bigcup_{t\in U}(\mathrm{cl}_\tau(z_{\overline{1}})(t)t_{\overline{1}})\right)(x) \\
&= \mathrm{cl}_\tau(\mathrm{cl}_\tau(z_{\overline{1}}))(x) = \mathrm{cl}_\tau(z_{\overline{1}})(x) \\
&= R(x,z).
\end{aligned} \tag{18}$$

Then, R is transitive. □

Theorem 13. *Let (U,R) be an IVF approximation space. Then,*

$$\begin{aligned}
(1)\ R \text{ is reflexive} &\Longleftrightarrow (ILR)\quad \forall A\in F^{(i)}(U), \\
&\quad \underline{R}(A)\subseteq A \\
&\Longleftrightarrow (IUR)\quad \forall A\in F^{(i)}(U), \\
&\quad A\subseteq\overline{R}(A),
\end{aligned} \tag{19}$$

$$\begin{aligned}
(2)\ R \text{ is symmetric} &\Longleftrightarrow (ILS)\quad \forall(x,y)\in U\times U, \\
&\quad \underline{R}\left((x_{\overline{1}})^c\right)(y) = \underline{R}\left((y_{\overline{1}})^c\right)(x) \\
&\Longleftrightarrow (IUS)\quad \forall(x,y)\in U\times U, \\
&\quad \overline{R}(x_{\overline{1}})(y) = \overline{R}(y_{\overline{1}})(x),
\end{aligned} \tag{20}$$

$$\begin{aligned}
(3)\ R \text{ is transitive} &\Longleftrightarrow (ILT)\quad \forall A\in F^{(i)}(U), \\
&\quad \underline{R}(A)\subseteq\underline{R}(\underline{R}(A)) \\
&\Longleftrightarrow (IUT)\quad \forall A\in F^{(i)}(U), \\
&\quad \overline{R}\left(\overline{R}(A)\right)\subseteq\overline{R}(A).
\end{aligned} \tag{21}$$

Proof. (1) By Proposition 11(3), (ILR) and (IUR) are equivalent. We only need to prove that the reflexivity of R is equivalent to (IUR).

Assume that R is reflexive. For any $A \in F^{(i)}(U)$ and $x \in U$, by the reflexivity of R, $R(x,x) = \overline{1}$. Then,

$$\begin{aligned}
\left(\overline{R}(A)\right)(x) &= \bigvee_{y\in U}(A(y)\wedge R(x,y)) \\
&\ge A(x)\wedge R(x,x) = A(x).
\end{aligned} \tag{22}$$

Thus, $A \subseteq \overline{R}(A)$.

Conversely, assume that (IUR) holds. For each $x \in U$, pick $A = x_{\overline{1}}$. By (IUR), we have $x_{\overline{1}} \subseteq \overline{R}(x_{\overline{1}})$. By Remark 9(1),

$$\overline{1} = x_{\overline{1}}(x) \le \overline{R}(x_{\overline{1}})(x) = R(x,x) \le \overline{1}. \tag{23}$$

Then, $R(x,x) = \overline{1}$. Thus, R is reflexive.

(2) By Proposition 11(3), (ILS) and (IUS) are equivalent. We only need to prove that the symmetry of R is equivalent to (IUS).

For any $x, y \in U$, by Remark 9(1), $\overline{R}(y_{\overline{1}})(x) = R(x,y)$ and $\overline{R}(x_{\overline{1}})(y) = R(y,x)$. So, the symmetry of R is equivalent to (IUS).

(3) By Proposition 11(3), (ILT) and (IUT) are equivalent. We only need to prove that the transitivity of R is equivalent to (IUT).

Assume that R is transitive. Then, $R(x,z) \geq \bigvee_{y\in U}(R(x,y)\wedge R(y,z))$ for any $x, y, z \in U$. Denote $a_{xz} = \bigvee_{y\in U}(R(x,y)\wedge R(y,z))$. Then, for any $A \in F^{(i)}(U)$ and $x \in U$,

$$\begin{aligned}\overline{R}(\overline{R}(A))(x) &= \bigvee_{y\in U}(\overline{R}(A)(y)\wedge R(x,y)) \\ &= \bigvee_{y\in U}\left(\left(\bigvee_{z\in U}(A(z)\wedge R(y,z))\right)\wedge R(x,y)\right) \\ &= \bigvee_{y\in U}\left(\bigvee_{z\in U}((A(z)\wedge R(y,z))\wedge R(x,y))\right) \\ &= \bigvee_{y\in U}\left(\bigvee_{z\in U}(A(z)\wedge(R(y,z)\wedge R(x,y)))\right) \\ &\leq \bigvee_{y\in U}\left(\bigvee_{z\in U}(A(z)\wedge a_{xz})\right) = \bigvee_{z\in U}(A(z)\wedge a_{xz}) \\ &\leq \bigvee_{z\in U}(A(z)\wedge R(x,z)) = \overline{R}(A)(x).\end{aligned} \tag{24}$$

So, $\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

Conversely, assume that (IUT) holds. For any $x, y, z \in U$, by (IUT),

$$\overline{R}(\overline{R}(z_{\tilde{1}})) \subseteq \overline{R}(z_{\tilde{1}}). \tag{25}$$

By Remark 9(1),

$$\begin{aligned}R(x,y)\wedge R(y,z) &\leq \bigvee_{t\in U}(R(x,t)\wedge R(t,z)) \\ &= \bigvee_{t\in U}(R(x,t)\wedge \overline{R}(z_{\tilde{1}})(t)) \\ &= \overline{R}(\overline{R}(z_{\tilde{1}}))(x) \leq \overline{R}(z_{\tilde{1}})(x) \\ &= R(x,z).\end{aligned} \tag{26}$$

Hence, R is transitive. □

Corollary 14. *Let (U, R) be an IVF approximation space. If R is preorder, then*

$$\begin{aligned}&\underline{R}(\underline{R}(A)) = \underline{R}(A), \\ &\overline{R}(\overline{R}(A)) = \overline{R}(A), \\ &(A \in F^{(i)}(U)).\end{aligned} \tag{27}$$

Proof. This holds by Theorem 13. □

4. Relationships between IVF Relations and IVF Topologies

Let R be an IVF relation on U. We denote

$$\begin{aligned}\tau_R &= \{A \in F^{(i)}(U) : \underline{R}(A) = A\}, \\ \theta_R &= \{\underline{R}(A) : A \in F^{(i)}(U)\}.\end{aligned} \tag{28}$$

4.1. IVF Topologies Induced by IVF Relations

Theorem 15. *Let R be an IVF relation on U. If R is reflexive, then τ_R is an IVF topology on U.*

Proof. (i) By Proposition 11(1), $\underline{R}(\tilde{1}) = \tilde{1}$. Then, $\tilde{1} \in \tau_R$.

By Theorem 13(1), $\underline{R}(\tilde{0}) \subseteq \tilde{0}$. Then, $\underline{R}(\tilde{0}) = \tilde{0}$. So, $\tilde{0} \in \tau_R$.

(ii) Let $A, B \in \tau_R$. By Proposition 11(4),

$$\underline{R}(A\cap B) = \underline{R}(A)\cap\underline{R}(B). \tag{29}$$

Then, $\underline{R}(A\cap B) = A\cap B$. Thus, $A\cap B \in \tau_R$.

(iii) Let $\{A_j : j \in J\} \subseteq \tau_R$. Then, $\underline{R}(A_j) = A_j$ for each $j \in J$. By Proposition 11(2),

$$\underline{R}\left(\bigcup_{j\in J}A_j\right) \supseteq \bigcup_{j\in J}\underline{R}(A_j) = \bigcup_{j\in J}A_j. \tag{30}$$

By Theorem 13(1), $\underline{R}(\bigcup_{j\in J}A_j) \subseteq \bigcup_{j\in J}A_j$.

Then, $\underline{R}(\bigcup_{j\in J}A_j) = \bigcup_{j\in J}A_j$, and so $\bigcup_{j\in J}A_j \in \tau_R$.

Thus, τ_R is an IVF topology on U. □

Definition 16. Let R be an IVF relation on U. If R is reflexive, then τ_R is called the IVF topology induced by R on U.

Theorem 17. *Let R be a reflexive IVF relation on U, and let τ_R be the IVF topology induced by R on U. Then, the following properties hold:*

(1) $\tau_R \subseteq \theta_R$,

(2) *for each* $A \in F^{(i)}(U)$,

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \text{cl}_{\tau_R}(A), \tag{31}$$

(3) *for each* $[a,b] \in [I]$, $\widetilde{[a,b]} \in \tau_R \cap \tau_R^c$.

Proof. (1) This is obvious.

(2) For each $A \in F^{(i)}(U)$, by Proposition 11(2), we have

$$\begin{aligned}\text{int}_{\tau_R}(A) &= \bigcup\{B : B \in \tau_R, B \subseteq A\} \\ &= \bigcup\{\underline{R}(B) : B \in \tau_R, B \subseteq A\} \subseteq \underline{R}(A).\end{aligned} \tag{32}$$

By Propositions 6(4) and 11(3),

$$\text{cl}_{\tau_R}(A) = (\text{int}_{\tau_R}(A^c))^c \supseteq (\underline{R}(A^c))^c = \overline{R}(A). \tag{33}$$

By Theorem 13(1),

$$\text{int}_{\tau_R}(A) \subseteq \underline{R}(A) \subseteq A \subseteq \overline{R}(A) \subseteq \text{cl}_{\tau_R}(A). \tag{34}$$

(3) For each $[a,b] \in [I]$, by Remark 9(2) and Theorem 13(1), $\underline{R}(\widetilde{[a,b]}) = \widetilde{[a,b]}$. Then, $\widetilde{[a,b]} \in \tau_R$. By Proposition 6(4),

$$\begin{aligned} \mathrm{cl}_{\tau_R}\left(\widetilde{[a,b]}\right) &= \left(\mathrm{int}_{\tau_R}\left(\left(\widetilde{[a,b]}\right)^c\right)\right)^c \\ &= \left(\mathrm{int}_{\tau_R}\left(\widetilde{[a,b]^c}\right)\right)^c \\ &= \left(\widetilde{[a,b]^c}\right)^c = \widetilde{[a,b]}. \end{aligned} \tag{35}$$

So, $\widetilde{[a,b]} \in \tau_R^c$. □

Theorem 18. *Let R be a reflexive IVF relation on U, and let τ_R be the IVF topology induced by R on U. If R is transitive, then*

(1) $\tau_R = \theta_R$,

(2) $\underline{R}$ *is the interior operator of* τ_R,

(3) $\overline{R}$ *is the closure operator of* τ_R.

Proof. (1) Obviously,

$$\tau_R \subseteq \left\{\underline{R}(A) : A \in F^{(i)}(U)\right\}. \tag{36}$$

By Corollary 14, $\tau_R \supseteq \{\underline{R}(A) : A \in F^{(i)}(U)\}$. Then, $\tau_R = \{\underline{R}(A) : A \in F^{(i)}(U)\} = \theta_R$.

(2) It suffices to show that for each $A \in F^{(i)}(U)$,

$$\underline{R}(A) = \mathrm{int}_{\tau_R}(A), \tag{37}$$

where $\mathrm{int}_{\tau_R}(A) = \bigcup\{B \in \tau_R : B \subseteq A\}$.

Since $\underline{R}(A) \in \theta_R$, by (1), $\underline{R}(A) \in \tau_R$.

By Theorem 13(1), $\underline{R}(A) \subseteq A$. Then, $\underline{R}(A) \subseteq \mathrm{int}_{\tau_R}(A)$.

By (1), $\mathrm{int}_{\tau_R}(A) \subseteq \underline{R}(A)$. Then, $\underline{R}(A) = \mathrm{int}_{\tau_R}(A)$.

(3) This holds by (2), Proposition 6(4), and Proposition 11(3). □

Example 19. Let $U = \{x, y, z\}$, and let R be a reflexive IVF relation on U. R is defined as follows:

$$\begin{gathered} R(x,y) = R(x,z) = R(z,x) = \overline{0}, \\ R(y,x) = [0.2, 0.7], \\ R(y,z) = \overline{1}, \qquad R(z,y) = [0.3, 0.8]. \end{gathered} \tag{38}$$

Pick

$$A = \frac{\overline{0}}{x} + \frac{[0.4, 0.5]}{y} + \frac{\overline{1}}{z}, \qquad B = \frac{\overline{1}}{x} + \frac{[0.5, 0.6]}{y} + \frac{\overline{0}}{z}. \tag{39}$$

(1) We have

$$R(z,y) \wedge R(y,x) = [0.2, 0.7] \nleq \overline{0} = R(z,x). \tag{40}$$

Then, R is not transitive.

(2) Since

$$\begin{aligned} \underline{R}(A) &= \frac{\overline{0}}{x} + \frac{[0.3, 0.5]}{y} + \frac{[0.4, 0.7]}{z}, \\ \underline{R}(\underline{R}(A)) &= \frac{\overline{0}}{x} + \frac{[0.3, 0.5]}{y} + \frac{[0.3, 0.7]}{z}, \end{aligned} \tag{41}$$

we have $\underline{R}(\underline{R}(A)) \neq \underline{R}(A)$. Then, $\underline{R}(A) \notin \tau_R$. Thus,

$$\tau_R \neq \left\{\underline{R}(A) : A \in F^{(i)}(U)\right\}, \qquad \mathrm{int}_{\tau_R}(A) \neq \underline{R}(A). \tag{42}$$

Obviously, $B^c = A$. By Proposition 11(3),

$$\left(\overline{R}(B)\right)^c = \underline{R}(B^c) = \underline{R}(A) \notin \tau_R. \tag{43}$$

Then, $\overline{R}(B) \notin \tau_R^c$. Thus, $\mathrm{cl}_{\tau_R}(B) \neq \overline{R}(B)$.

4.2. IVF Relations Induced by IVF Topologies

Definition 20. Let τ be an IVF topology on U. Define an IVF relation R_τ on U by

$$R_\tau(x,y) = \mathrm{cl}_\tau(y_{\overline{1}})(x) \tag{44}$$

for each $(x,y) \in U \times U$. Then, R_τ is called the IVF relation induced by τ on U.

An IVF topology τ on U is said to satisfy the following:

(C_1) *axiom:* $\mathrm{cl}_\tau([a,b]A) = [a,b]\mathrm{cl}_\tau(A)$ *for any* $[a,b] \in [I]$ *and* $A \in F^{(i)}(U)$,

(C_2) *axiom:* $\mathrm{cl}_\tau(\bigcup_{j\in J} A_j) = \bigcup_{j\in J}\mathrm{cl}_\tau(A_j)$ *for any* $\{A_j : j \in J\} \subseteq F^{(i)}(U)$.

Theorem 21. *Let τ be an IVF topology on U, and let R_τ be the IVF relation induced by τ on U. Then, the following properties hold.*

(1) R_τ *is reflexive.*

(2) *If τ satisfies (C_2) axiom and $\{\widetilde{[a,b]} : [a,b] \in [I]\} \subseteq \tau$, then*

$$\underline{R_\tau}(A) \subseteq \mathrm{int}_\tau(A) \subseteq A \subseteq \mathrm{cl}_\tau(A) \subseteq \overline{R_\tau}(A) \quad \left(A \in F^{(i)}(U)\right). \tag{45}$$

Proof. (1) For each $x \in U$,

$$R_\tau(x,x) = \mathrm{cl}_\tau(x_{\overline{1}})(x) \geq (x_{\overline{1}})(x) = \overline{1}. \tag{46}$$

Then, R_τ is reflexive.

(2) Since $\{\widetilde{[a,b]} : [a,b] \in [I]\} \subseteq \tau$, we have $\{\widetilde{[a,b]} : [a,b] \in [I]\} \subseteq \tau^c$. For each $A \in F^{(i)}(U)$, by Remark 4, (C_2) axiom, and Proposition 11,

$$\begin{aligned} \mathrm{cl}_\tau(A) &= \mathrm{cl}_\tau\left(\bigcup_{y\in U}(A(y)\, y_{\overline{1}})\right) \\ &= \bigcup_{y\in U}\mathrm{cl}_\tau(A(y)\, y_{\overline{1}}) \\ &= \bigcup_{y\in U}\mathrm{cl}_\tau\left(\widetilde{A(y)} \cap y_{\overline{1}}\right) \\ &\subseteq \bigcup_{y\in U}\left(\mathrm{cl}_\tau\left(\widetilde{A(y)}\right) \cap \mathrm{cl}_\tau(y_{\overline{1}})\right) \\ &= \bigcup_{y\in U}\left(\widetilde{A(y)} \cap \mathrm{cl}_\tau(y_{\overline{1}})\right). \end{aligned} \tag{47}$$

Then, for each $x \in U$,

$$\begin{aligned} \mathrm{cl}_\tau(A)(x) &\le \bigvee_{y\in U}\left(\widetilde{A(y)}(x) \wedge \mathrm{cl}_\tau(y_{\bar{1}})(x)\right) \\ &= \bigvee_{y\in U}(A(y) \wedge R_\tau(x,y)) = \overline{R_\tau}(A)(x). \end{aligned} \tag{48}$$

Hence, $\mathrm{cl}_\tau(A) \subseteq \overline{R_\tau}(A)$.

By Propositions 6(4) and 11(3),

$$\mathrm{int}_\tau(A) = (\mathrm{cl}_\tau(A^c))^c \supseteq \left(\overline{R_\tau}(A^c)\right)^c = \underline{R_\tau}(A), \tag{49}$$

so

$$\underline{R_\tau}(A) \subseteq \mathrm{int}_\tau(A) \subseteq A \subseteq \mathrm{cl}_\tau(A) \subseteq \overline{R_\tau}(A). \tag{50}$$

□

Proposition 22. *Let τ be an IVF topology on U. If τ satisfies (C_1) and (C_2) axioms, then*

(1) *$\overline{R_\tau}$ is the closure operator of τ,*

(2) *$\underline{R_\tau}$ is the interior operator of τ,*

(3) *for each $[a,b] \in [I]$, $\widetilde{[a,b]} \in \tau$,*

(4) *τ is Alexandrov.*

Proof. (1) For each $A \in F^{(i)}(U)$, by Remark 4, (C_1) axiom, and (C_2) axiom,

$$\begin{aligned} \mathrm{cl}_\tau(A) &= \mathrm{cl}_\tau\left(\bigcup_{y\in U}(A(y)\,y_{\bar{1}})\right) \\ &= \bigcup_{y\in U}\mathrm{cl}_\tau(A(y)\,y_{\bar{1}}) = \bigcup_{y\in U}(A(y)\,\mathrm{cl}_\tau(y_{\bar{1}})). \end{aligned} \tag{51}$$

Then, for each $x \in U$,

$$\begin{aligned} \mathrm{cl}_\tau(A)(x) &= \bigvee_{y\in U}(A(y)(x) \wedge \mathrm{cl}_\tau(y_{\bar{1}})(x)) \\ &= \bigvee_{y\in U}(A(y) \wedge R_\tau(x,y)) = \overline{R_\tau}(A)(x). \end{aligned} \tag{52}$$

Hence, $\overline{R_\tau}(A) = \mathrm{cl}_\tau(A)$. Thus, $\overline{R_\tau}$ is the closure operator of τ.

(2) This holds by (1), Proposition 6(4) and, Proposition 11(3).

(3) For each $[a,b] \in [I]$, by (2), Remark 9(2), and Proposition 6(2),

$$\widetilde{[a,b]} \supseteq \mathrm{int}_\tau\left(\widetilde{[a,b]}\right) = \underline{R}\left(\widetilde{[a,b]}\right) \supseteq \widetilde{[a,b]}. \tag{53}$$

Then $\mathrm{int}_\tau(\widetilde{[a,b]}) = \widetilde{[a,b]}$, and so $\widetilde{[a,b]} \in \tau$.

(4) Let $\{A_j : j \in J\} \subseteq \tau$. By (2), for each $j \in J$,

$$A_j = \mathrm{int}_\tau(A_j) = \underline{R}(A_j). \tag{54}$$

By (2) and Proposition 11(4),

$$\bigcap_{j\in J}A_j = \bigcap_{j\in J}\underline{R}(A_j) = \underline{R}\left(\bigcap_{j\in J}A_j\right) = \mathrm{int}_\tau\left(\bigcap_{j\in J}A_j\right). \tag{55}$$

So $\bigcap_{j\in J}A_j \in \tau$.

Hence, τ is Alexandrov. □

Proposition 23. *Let R be a preorder IVF relation on U. Then, τ_R satisfies (C_1) and (C_2) axioms.*

Proof. For any $[a,b] \in [I]$ and $A \in F^{(i)}(U)$, by Theorem 18(3) and Proposition 11(5),

$$\mathrm{cl}_{\tau_R}([a,b]\,A) = \overline{R}([a,b]\,A) = [a,b]\,\overline{R}(A) = [a,b]\,\mathrm{cl}_{\tau_R}(A). \tag{56}$$

Thus, τ_R satisfies (C_1) axiom.

For any $\{A_j : j \in J\} \subseteq F^{(i)}(U)$, by Proposition 11(4) and Theorem 18,

$$\begin{aligned} \mathrm{cl}_{\tau_R}\left(\bigcup_{j\in J}A_j\right) &= \mathrm{cl}_{\theta_R}\left(\bigcup_{j\in J}A_j\right) \\ &= \overline{R}\left(\bigcup_{j\in J}A_j\right) = \bigcup_{j\in J}\overline{R}(A_j) \\ &= \bigcup_{j\in J}\mathrm{cl}_{\theta_R}(A_j) = \bigcup_{j\in J}\mathrm{cl}_{\tau_R}(A_j). \end{aligned} \tag{57}$$

Thus, τ_R satisfies (C_2) axiom. □

5. IVF Approximating Spaces

As can be seen from Section 4, a reflexive IVF relation yields an IVF topology. In this section, we consider the reverse problem; that is, under which conditions can an IVF topology be associated with an IVF relation which produces the given IVF topology?

Definition 24. Let (U,τ) be an IVF topological space. If there exists a reflexive IVF relation on U such that $\tau_R = \tau$, then (U,τ) is called an IVF approximating space.

Theorem 25. *Let τ be an IVF topology on U. Let R_τ be the IVF relation induced by (U,τ), and let τ_{R_τ} be the IVF topology induced by R_τ on U. If τ satisfies (C_1) and (C_2) axioms, then $\tau_{R_\tau} = \tau$.*

Proof. By Theorem 21(1), R_τ is reflexive. For any $x, y, z \in U$, put $\mathrm{cl}(z_{\bar{1}})(y) = [a,b]$. By Remark 4 and Proposition 11(2),

$$\begin{aligned} [a,b]\,\mathrm{cl}_\tau(y_{\bar{1}}) &= \mathrm{cl}_\tau([a,b]\,y_{\bar{1}}) \\ &= \mathrm{cl}_\tau(cl_\tau(z_{\bar{1}})(y)\,y_{\bar{1}}) \\ &\subseteq \mathrm{cl}_\tau\left(\bigcup_{t\in U}(cl_\tau(z_{\bar{1}})(t)\,t_{\bar{1}})\right) \\ &= \mathrm{cl}_\tau(cl_\tau(z_{\bar{1}})) = \mathrm{cl}_\tau(z_{\bar{1}}). \end{aligned} \tag{58}$$

Then,

$$\begin{aligned} R_\tau(x,y)\wedge R_\tau(y,z) &= \mathrm{cl}_\tau(y_{\bar 1})(x)\wedge \mathrm{cl}_\tau(z_{\bar 1})(y) \\ &= \mathrm{cl}_\tau(y_{\bar 1})(x)\wedge [a,b] \\ &= [a,b]\wedge \mathrm{cl}_\tau(y_{\bar 1})(x) \\ &= ([a,b]\,\mathrm{cl}_\tau(y_{\bar 1}))(x) \\ &\le \mathrm{cl}_\tau(z_{\bar 1})(x) = R_\tau(x,z). \end{aligned} \tag{59}$$

So, R is transitive.

So, R_τ is preorder. For each $A\in F^{(i)}(U)$, by Theorem 18,

$$\mathrm{cl}_{\tau_{R_\tau}}(A) = \mathrm{cl}_{\theta_{R_\tau}}(A) = \overline{R_\tau}(A). \tag{60}$$

Since τ satisfies (C_1) and (C_2) axioms, by Proposition 22(1), $\overline{R_\tau}(A) = \mathrm{cl}_\tau(A)$. So, $\mathrm{cl}_{\tau_{R_\tau}}(A) = \mathrm{cl}_\tau(A)$.

Thus, $\tau_{R_\tau} = \tau$. □

Theorem 26. *Let τ be an IVF topology on U. Then, the following are equivalent.*

(1) *τ satisfies (C_1) and (C_2) axioms.*

(2) *For any $[a,b]\in[I]$, $A\in F^{(i)}(U)$ and $\{A_j : j\in J\}\subseteq F^{(i)}(U)$,*

$$\begin{aligned} \mathrm{int}_\tau\left(\widetilde{[a,b]}\cup A\right) &= \widetilde{[a,b]}\cup \mathrm{int}_\tau(A), \\ \mathrm{int}_\tau\left(\bigcap_{j\in J} A_j\right) &= \bigcap_{j\in J}\mathrm{int}_\tau(A_j). \end{aligned} \tag{61}$$

(3) *There exists a preorder IVF relation ρ on U such that $\overline{\rho}$ is the closure operator of τ.*

(4) *There exists a preorder IVF relation ρ on U such that $\underline{\rho}$ is the interior operator of τ.*

(5) *$\overline{R_\tau}$ is the closure operator of τ.*

(6) *$\underline{R_\tau}$ is the interior operator of τ.*

Proof. (1) $\Longleftrightarrow$ (2) is obvious.

(1) $\Longrightarrow$ (3). Suppose that τ satisfies (C_1) and (C_2) axioms. Pick $\rho = R_\tau$. By Proposition 22(1), $\overline{\rho}$ is the closure operator of τ. By Theorem 12(2), ρ is preorder.

(3) $\Longrightarrow$ (4). Let $\overline{\rho}$ be the closure operator of τ for some preorder IVF relation ρ on U. For each $A\in F^{(i)}(U)$, by Propositions 6(4) and 11(3),

$$\underline{\rho}(A) = (\overline{\rho}(A^c))^c = (\mathrm{cl}_\tau(A^c))^c = \mathrm{int}_\tau(A). \tag{62}$$

Thus, $\underline{\rho}$ is the interior operator of τ.

(4) $\Longrightarrow$ (6). Let $\underline{\rho}$ be the interior operator of τ for some preorder IVF relation ρ on U. For each $(x,y)\in U\times U$, by Remark 9(1),

$$\begin{aligned} \rho(x,y) &= \bar 1 - \underline{\rho}\left((y_{\bar 1})^c\right)(x) = \bar 1 - \mathrm{int}_\tau\left((y_{\bar 1})^c\right)(x) \\ &= \mathrm{cl}_\tau(y_{\bar 1})(x) = R_\tau(x,y). \end{aligned} \tag{63}$$

Then, $\rho = R_\tau$. Note that $\underline{\rho}$ is the interior operator of τ. Then, $\underline{R_\tau}$ is the interior operator of τ.

(6) $\Longleftrightarrow$ (5). This holds by Propositions 6(4) and 11(3).

(5) $\Longrightarrow$ (1). For any $[a,b]\in[I]$ and $A\in F^{(i)}(U)$, by Proposition 11(5),

$$\mathrm{cl}_\tau([a,b]\,A) = \overline{R_\tau}([a,b]\,A) = [a,b]\,\overline{R_\tau}(A) = [a,b]\,\mathrm{cl}_\tau(A). \tag{64}$$

Thus, τ satisfies (C_1) axiom.

For any $\{A_j : j\in J\}\subseteq F^{(i)}(U)$, by Proposition 11(4),

$$\mathrm{cl}_\tau\left(\bigcup_{j\in J}A_j\right) = \overline{R_\tau}\left(\bigcup_{j\in J}A_j\right) = \bigcup_{j\in J}\overline{R_\tau}(A_j) = \bigcup_{j\in J}\mathrm{cl}_\tau(A_j). \tag{65}$$

Thus, τ satisfies (C_2) axiom. □

Theorem 27. *Let (U,τ) be an IVF topological space. If one of the following conditions is satisfied, then (U,τ) is an IVF approximating space.*

(1) *τ satisfies (C_1) and (C_2) axioms.*

(2) *For any $[a,b]\in[I]$ and $A\in F^{(i)}(U)$,*

$$\mathrm{int}\left(\widetilde{[a,b]}\cup A\right) = \widetilde{[a,b]}\cup \mathrm{int}(A). \tag{66}$$

(3) *There exists a preorder IVF relation R on U such that $\overline{R}$ is the closure operator of τ.*

(4) *There exists a preorder IVF relation R on U such that $\underline{R}$ is the interior operator of τ.*

(5) *$\overline{R_\tau}$ is the closure operator of τ.*

(6) *$\underline{R_\tau}$ is the interior operator of τ.*

Proof. These hold by Theorems 25 and 26. □

Example 28. $\{\widetilde{[a,b]} : [a,b]\in[I]\}$ is an IVF approximating space.

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