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Research Article

Analysis of the Error in a Numerical Method Used to Solve Nonlinear Mixed Fredholm-Volterra-Hammerstein Integral Equations

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This work presents an analysis of the error that is committed upon having obtained the approximate solution of the nonlinear Fredholm-Volterra-Hammerstein integral equation by means of a method for its numerical resolution. The main tools used in the study of the error are the properties of Schauder bases in a Banach space.

1. Introduction

In this paper we consider the following nonlinear mixed Fredholm-Volterra-Hammerstein integral equation:

$$x(t) = y_0(t) + \int_{\alpha}^{\alpha+\beta} k_1(t,s)g_1(s,x(s))ds + \int_{\alpha}^{t} k_2(t,s)g_2(s,x(s))ds, \quad t \in [\alpha, \alpha+\beta], \tag{1.1}$$

where $y_0 : [\alpha, \alpha+\beta] \to \mathbb{R}$, $g_1, g_2 : [\alpha, \alpha+\beta] \times \mathbb{R} \to \mathbb{R}$ and the kernels $k_1, k_2 : [\alpha, \alpha+\beta]^2 \to \mathbb{R}$ are assumed to be known continuous functions, and $x : [\alpha, \alpha+\beta] \to \mathbb{R}$ is the unknown function to be determined.

Equation (1.1) arises in a variety of applications in many fields, including continuum mechanics, potential theory, electricity and magnetism, three-dimensional contact problems, and fluid mechanics, and so forth (see, e.g., [1–4]). Several numerical methods for approximating the solution of integral, and integrodifferential equations are known (see, e.g., [5–8]). For Fredholm-Volterra-Hammerstein integral equations, the classical method of successive approximations was introduced in [9]. An optimal control problem method was presented in [10], and a collocation-type method was developed in [11–13]. Computational methods based on Bernstein operational matrices and the Chebyshev approximation method were presented in [14, 15], respectively.

The use of fixed point techniques and Schauder bases, in the field of numerical resolution of differential, integral and integro-differential equations, allows for the development of new methods providing significant improvements upon other known methods (see [16–23]).

In this work we make an analysis of the error committed upon having obtained the approximate solution of the nonlinear Fredholm-Volterra-Hammerstein integral equation, using the theorem of Banach fixed point and Schauder bases (see [21], for a detailed description of the numerical method used in a more general equation).

In order to recall the aforementioned numerical method, let $C([\alpha,\alpha+\beta])$ and $C([\alpha,\alpha+\beta]^2)$ be the Banach spaces of all continuous and real-valued functions on $[\alpha,\alpha+\beta]$ and $[\alpha,\alpha+\beta]^2$ endowed with their usual supnorms. Throughout this paper we will make the following assumptions on k_i and g_i for $i \in \{1,2\}$.

(i) Since $k_i \in C([\alpha,\alpha+\beta]^2)$, there exists $M_{k_i} \geq 0$ such that $|k_i(t,s)| \leq M_{k_i}$ for all $(t,s) \in [\alpha,\alpha+\beta]^2$.

(ii) $g_i : [\alpha,\alpha+\beta] \times \mathbb{R} \to \mathbb{R}$ are functions such that there exists $L_{g_i} > 0$ such that $|g_i(s,y) - g_i(s,z)| \leq L_{g_i}|y-z|$ for $s \in [\alpha,\alpha+\beta]$ and for all $y,z \in \mathbb{R}$.

(iii) $\beta \sum_{i=1}^{2} M_{k_i} L_{g_i} < 1$.

We organize this paper as follows. In Section 2, we reformulate (1.1) in terms of a convenient integral operator T and we describe the numerical method used. The study of the error is described in Section 3. Finally, in Section 4 we show some illustrative examples.

2. Analytical Preliminaries

In this section we recall, in a summarized form, the concepts and results relative to the numerical method used for the study of the error that we carried out.

Let us start by observing that (1.1) is equivalent to the problem of finding fixed points of the operator $T : C([\alpha,\alpha+\beta]) \to C([\alpha,\alpha+\beta])$ defined by

$$\begin{aligned}(Tx)(t) &:= y_0(t) + \int_{\alpha}^{\alpha+\beta} k_1(t,s) g_1(s,x(s))\,ds \\ &\quad + \int_{\alpha}^{t} k_2(t,s) g_2(s,x(s))\,ds, \quad t \in [\alpha,\alpha+\beta],\ x \in C([\alpha,\alpha+\beta]).\end{aligned} \tag{2.1}$$

A direct calculation over T leads to

$$\|Ty_1 - Ty_2\| \leq M \|y_1 - y_2\| \tag{2.2}$$

for all $y_1, y_2 \in C([\alpha,\alpha+\beta])$, where we denote $M := \beta \sum_{i=1}^{2} M_{k_i} L_{g_i}$. As the operator T defined in (2.1) satisfies (2.2), under condition (iii) and from the Banach fixed-point theorem, it follows that there exists a unique fixed point $x \in C([\alpha, \alpha+\beta])$ for T that is the unique solution of (1.1). In addition, for each $\tilde{x} \in C([\alpha, \alpha+\beta])$, we have

$$\|T^m \tilde{x} - x\| \leq \frac{M^m}{1-M} \|T\tilde{x} - \tilde{x}\| \tag{2.3}$$

and in particular $x = \lim_m T^m \tilde{x}$.

But it is not possible, in an explicit way, to calculate the sequence of iterations $\{T^m\}_{m\geq 1}$, to obtain the unique sequence x of (1.1), for which reason a numerical method is needed in order to approximate the fixed point of T.

Now we recall the concrete Schauder bases in the spaces $C([\alpha, \alpha+\beta])$ and $C([\alpha, \alpha+\beta]^2)$. Let $\{t_n\}_{n\geq 1}$ be a dense sequence of distinct points in $[\alpha, \alpha+\beta]$ such that $t_1 = \alpha$ and $t_2 = \alpha+\beta$. We set $b_1(t) := 1$ for $t \in [\alpha, \alpha+\beta]$, and for $n \geq 1$, and we let b_n be a piecewise linear continuous function on $[\alpha, \alpha+\beta]$ with nodes at $\{t_j : 1 \leq j \leq n\}$, uniquely determined by the relations $b_n(t_n) = 1$ and $b_n(t_k) = 0$ for $k < n$. We denote by $\{P_n\}_{n\geq 1}$ the sequence of associated projections and $\{b_n^*\}_{n\geq 1}$ the coordinate functionals. It is easy to check that $\{b_n\}_{n\geq 1}$ is a Schauder basis in $C([\alpha, \alpha+\beta])$ (see [24]).

From the Schauder basis $\{b_n\}_{n\geq 1}$ in $C([\alpha, \alpha+\beta])$, we can build another Schauder basis $\{B_n\}_{n\geq 1}$ of $C([\alpha, \alpha+\beta]^2)$ (see [25, 26]). It is sufficient to consider $B_n(t,s) := b_i(t)b_j(s)$ for all $t, s \in [\alpha, \alpha+\beta]$, with $\tau(n) = (i,j)$, where for a real number p, $[p]$ will denote its integer part and $\tau = (\tau_1, \tau_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is the bijective mapping defined by

$$\tau(n) := \begin{cases} (\sqrt{n}, \sqrt{n}), & \text{if } [\sqrt{n}] = \sqrt{n}, \\ \left(n - [\sqrt{n}]^2, [\sqrt{n}] + 1\right), & \text{if } 0 < n - [\sqrt{n}]^2 \leq [\sqrt{n}], \\ \left([\sqrt{n}] + 1, n - [\sqrt{n}]^2 - [\sqrt{n}]\right), & \text{if } [\sqrt{n}] < n - [\sqrt{n}]^2. \end{cases} \tag{2.4}$$

We denote by $\{Q_n\}_{n\geq 1}$ the sequence of associated projections and by $\{B_n^*\}_{n\geq 1}$ the coordinate functionals. The Schauder basis $\{B_n\}_{n\geq 1}$ of $C([\alpha, \alpha+\beta]^2)$ has similar properties to the ones for the one-dimensional case. See Table 1 and note under some weak conditions (see the last row, which is derived easily from the third row of Table 1, resp., and the Mean-Value theorems for one and two variables) we can estimate the rate of the convergence of the sequence of projections in the one and two-dimensional cases, where we consider the dense subset $\{t_i\}_{i\geq 1}$ of distinct points in $[\alpha, \alpha+\beta]$, T_n as the set $\{t_1, \ldots, t_n\}$ ordered in an increasing way for $n \geq 2$, and ΔT_n denotes the maximum distance between two consecutive points of T_n.

Let us consider the continuous integral operator $T : C([\alpha, \alpha+\beta]) \to C([\alpha, \alpha+\beta])$ defined in (2.1). Let $\tilde{x} \in C([\alpha, \alpha+\beta])$, and the functions $\phi_1, \phi_2 \in C([\alpha, \alpha+\beta]^2)$, defined for $\phi_1(t,s) = k_1(t,s)g_1(s, \tilde{x}(s))$, $\phi_2(t,s) = k_2(t,s)g_2(s, \tilde{x}(s))$. Let $\{\lambda_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ be the sequences of scalars satisfying $\phi_1 = \sum_{n\geq 1} \lambda_n B_n$, $\phi_2 = \sum_{n\geq 1} \mu_n B_n$. Then for all $t \in [\alpha, \alpha+\beta]$, we have that

$$(T\tilde{x})(t) = y_0(t) + \sum_{n\geq 1} \lambda_n \int_{\alpha}^{\alpha+\beta} B_n(t,s)\,ds + \sum_{n\geq 1} \mu_n \int_{\alpha}^{t} B_n(t,s)\,ds. \tag{2.5}$$

The equality (2.5) enables us to determine, in an elemental way, the image of any continuous function under the operator T. However, it does not seem to be a usable expression due to the two infinite sums appearing in it. For this reason, the aforementioned sums are truncated.

3. Study of the Error

In this section we realize a new study of the error, obtaining one bound of it. Supposing conditions of regularity in the functions data, we improve and complete the study realized in [21].

Let $\tilde{x} \in C([\alpha, \alpha+\beta])$ and consider

$$x_0(t) := \tilde{x}(t) \in C([\alpha, \alpha+\beta]), \tag{3.1}$$

and for $m \in \mathbb{N}$, define inductively for $r \in \{1, \ldots, m\}$ the following functions:

$$\sigma_{r-1}(t,s) := k_1(t,s) g_1(s, x_{r-1}(s)), \tag{3.2}$$

$$\psi_{r-1}(t,s) := k_2(t,s) g_2(s, x_{r-1}(s)), \tag{3.3}$$

$$x_r(t) := y_0(t) + \int_\alpha^{\alpha+\beta} Q_{n_r^2}(\sigma_{r-1}(t,s))\,ds + \int_\alpha^t Q_{n_r^2}(\psi_{r-1}(t,s))\,ds, \tag{3.4}$$

where $t, s \in [\alpha, \alpha+\beta]$ and $n_r \in \mathbb{N}$.

Proposition 3.1. *The sequence $\{x_r\}_{r\geq 1}$ is uniformly bounded.*

Proof. Let $R = \max\{|g_1(s,0)| : s \in [\alpha, \alpha+\beta]\}$, $S = \max\{|g_2(s,0)| : s \in [\alpha, \alpha+\beta]\}$, and we have for all $r \geq 1$ and $(t,s) \in [\alpha, \alpha+\beta]^2$

$$\begin{aligned}
|\sigma_{r-1}(t,s)| &= |k_1(t,s)| \left| g_1(s, x_{r-1}(s)) \right| \\
&\leq M_{k_1}\left(\left| g_1(s, x_{r-1}(s)) - g_1(s,0) \right| + \left| g_1(s,0) \right| \right) \\
&\leq M_{k_1}\left(L_{g_1} |x_{r-1}(s)| + R \right), \\
\left|\psi_{r-1}(t,s)\right| &= |k_2(t,s)| \left| g_2(s, x_{r-1}(s)) \right| \\
&\leq M_{k_2}\left(\left| g_2(s, x_{r-1}(s)) - g_2(s,0) \right| + \left| g_2(s,0) \right| \right) \\
&\leq M_{k_2}\left(L_{g_2} |x_{r-1}(s)| + S \right).
\end{aligned} \tag{3.5}$$

For the monotonicity of the Schauder basis, we have

$$\begin{aligned}|x_r(t)| &\leq |y_0(t)| + \int_\alpha^{\alpha+\beta} \left|Q_{n_r^2}(\sigma_{r-1}(t,s))\right| ds + \int_\alpha^t \left|Q_{n_r^2}(\psi_{r-1}(t,s))\right| ds \\ &\leq |y_0(t)| + \int_\alpha^{\alpha+\beta} \|\sigma_{r-1}\| ds + \int_\alpha^t \|\psi_{r-1}\| ds \\ &\leq |y_0(t)| + \beta(M_{k_1}R + M_{k_2}S) + M_{k_1}L_{g_1}\int_\alpha^{\alpha+\beta} \|x_{r-1}\| ds + M_{k_2}L_{g_2}\int_\alpha^t \|x_{r-1}\| ds.\end{aligned} \tag{3.6}$$

Therefore,

$$\|x_r\| \leq \|y_0\| + \beta(M_{k_1}R + M_{k_2}S) + M\|x_{r-1}\|. \tag{3.7}$$

Applying recursively this process we get

$$\begin{aligned}\|x_r\| &\leq (\|y_0\| + \beta(M_{k_1}R + M_{k_2}S))\left(1 + M + \cdots + M^{r-1}\right) + M^r\|x_0\| \\ &\leq (\|y_0\| + \beta(M_{k_1}R + M_{k_2}S))\frac{1-M^r}{1-M} + M^r\|x_0\|\end{aligned} \tag{3.8}$$

for all $r \geq 1$. Then $\{x_r\}_{r\geq 1}$ is uniformly bounded. □

Remark 3.2. For $i \in \{1,2\}$, the sequence $\{g_i(\cdot, x_r(\cdot)\}_{r\geq 1}$ is uniformly bounded, as it follows Proposition 3.1 and the fact that g_i for $i \in \{1,2\}$ is Lipschitz in its second variable.

Proposition 3.3. *Let $y_0 \in C^1([\alpha, \alpha+\beta])$, and for $i \in \{1,2\}$, $k_i \in C^1([\alpha, \alpha+\beta]^2)$, $g_i \in C^1([\alpha, \alpha+\beta] \times \mathbb{R})$ such that $\partial g_i/\partial s$ and $\partial g_i/\partial x$ satisfy a global Lipschitz condition in the last variable. Let $x_0(t) := \tilde{x}(t) \in C^1([\alpha, \alpha+\beta])$, and define inductively as in* (3.2), (3.3), *and* (3.4) *the functions σ_{r-1}, ψ_{r-1} and x_r, respectively. Then*

$$\left\{\frac{\partial \sigma_{r-1}}{\partial t}\right\}_{r\geq 1}, \quad \left\{\frac{\partial \sigma_{r-1}}{\partial s}\right\}_{r\geq 1}, \quad \left\{\frac{\partial \psi_{r-1}}{\partial t}\right\}_{r\geq 1}, \quad \left\{\frac{\partial \psi_{r-1}}{\partial s}\right\}_{r\geq 1} \tag{3.9}$$

are uniformly bounded.

Proof. From (3.2) and (3.3), we have, respectively, that for all $r \geq 1$, $(\partial\sigma_{r-1}/\partial t)(t,s) = (\partial k_1/\partial t)(t,s)g_1(s, x_{r-1}(s))$, $(\partial\psi_{r-1}/\partial t)(t,s) = (\partial k_2/\partial t)(t,s)g_2(s, x_{r-1}(s))$, and therefore by the conditions over k_1, k_2, and Remark 3.2, $\{\partial\sigma_{r-1}/\partial t\}_{r\geq 1}$, $\{\partial\psi_{r-1}/\partial t\}_{r\geq 1}$ are uniformly bounded.

Observe that

$$\begin{aligned}|x_r'(t)| \leq |y_0'(t)| &+ \int_\alpha^{\alpha+\beta} \left|\frac{\partial}{\partial t} Q_{n_r^2}(\sigma_{r-1}(t,s))\right| ds \\ &+ \left|Q_{n_r^2}(\psi_{r-1}(t,t))\right| + \int_\alpha^t \left|\frac{\partial}{\partial t} Q_{n_r^2}(\psi_{r-1}(t,s))\right| ds.\end{aligned} \tag{3.10}$$

In view of the monotonicity of the Schauder basis, we have

$$\|x_r'\| \le \|y_0'\| + \|\varphi_{r-1}\| + \beta\left(\left\|\frac{\partial \sigma_{r-1}}{\partial t}\right\| + \left\|\frac{\partial \varphi_{r-1}}{\partial t}\right\|\right), \tag{3.11}$$

and hence the sequence $\{x_r'\}_{r\ge1}$ is uniformly bounded.

On the other hand from (3.2) and (3.3), respectively, we have

$$\begin{aligned}\frac{\partial \sigma_{r-1}}{\partial s}(t,s) &= \frac{\partial k_1}{\partial s}(t,s) g_1(s, x_{r-1}(s)) \\ &\quad + k_1(t,s)\left(\frac{\partial g_1}{\partial s}(s, x_{r-1}(s)) + \frac{\partial g_1}{\partial x}(s, x_{r-1}(s)) x_{r-1}'(s)\right), \\ \frac{\partial \varphi_{r-1}}{\partial s}(t,s) &= \frac{\partial k_2}{\partial s}(t,s) g_2(s, x_{r-1}(s)) \\ &\quad + k_2(t,s)\left(\frac{\partial g_2}{\partial s}(s, x_{r-1}(s)) + \frac{\partial g_2}{\partial x}(s, x_{r-1}(s)) x_{r-1}'(s)\right).\end{aligned} \tag{3.12}$$

For $i \in \{1,2\}$, let $U = \max\{|(\partial g_i/\partial s)(s,0)| : s \in [\alpha, \alpha+\beta]\}$, and we have for all $r \ge 1$ and $s \in [\alpha, \alpha+\beta]$

$$\left|\frac{\partial g_i}{\partial s}(s, x_{r-1}(s))\right| \le \left|\frac{\partial g_i}{\partial s}(s, x_{r-1}(s)) - \frac{\partial g_i}{\partial s}(s,0)\right| + \left|\frac{\partial g_i}{\partial s}(s,0)\right| \le l_{g_i}|x_{r-1}(s)| + U \tag{3.13}$$

with l_{g_i} as the Lipschitz constant of $\partial g_i/\partial s$ in the last variable.

By repeating the previous argument, we have

$$\left|\frac{\partial g_i}{\partial x}(s, x_{r-1}(s))\right| \le q_{g_i}|x_{r-1}(s)| + V, \tag{3.14}$$

where $V = \max\{|(\partial g_i/\partial x)(s,0)| : s \in [\alpha, \alpha+\beta]\}$, and q_{g_i} is the Lipschitz constant of $\partial g_i/\partial x$ in the last variable.

Therefore by the conditions over k_1, k_2, Proposition 3.1, Remark 3.2, and (3.11),

$$\left\{\frac{\partial \sigma_{r-1}}{\partial s}\right\}_{r\ge1}, \qquad \left\{\frac{\partial \varphi_{r-1}}{\partial s}\right\}_{r\ge1} \tag{3.15}$$

are uniformly bounded. □

Proposition 3.4. *With the previous notation and the same hypothesis as in Proposition 3.3, there is* $\rho_1, \rho_2 > 0$ *such that for all* $r \ge 1$ *and* $n_r \ge 2$, *we have*

$$\begin{aligned}\left\|\sigma_{r-1} - Q_{n_r^2}(\sigma_{r-1})\right\| &\le \rho_1 \Delta T_{n_r}, \\ \left\|\varphi_{r-1} - Q_{n_r^2}(\varphi_{r-1})\right\| &\le \rho_2 \Delta T_{n_r}.\end{aligned} \tag{3.16}$$

Table 1: Properties of the univariate and bivariate Schauder bases.

$b_1(t) = 1$ $n \geq 2 \Rightarrow b_n(t_k) = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k < n \end{cases}$	$B_1(t,s) = 1$ $n \geq 2 \Rightarrow B_n(t_i, t_j) = \begin{cases} 1, & \text{if } \tau(n) = (i,j) \\ 0, & \text{if } \tau^{-1}(i,j) < n \end{cases}$
$y \in C([\alpha, \alpha+\beta])$ $\Downarrow$ $b_1^*(y) = y(t_1)$ $n \geq 2 \Rightarrow b_n^*(y) = y(t_n) - \sum_{k=1}^{n-1} b_k^*(y) b_k(t_n)$	$z \in C([\alpha, \alpha+\beta]^2)$ $\Downarrow$ $B_1^*(z) = z(t_1, t_1)$ $\left.\begin{array}{c} n \geq 2 \\ \tau(n) = (i,j) \end{array}\right\} \Rightarrow B_n^*(z) = z(t_i, t_j) - \sum_{k=1}^{n-1} B_k^*(z) B_k(t_i, t_j)$
$y \in C([\alpha, \alpha+\beta])$ $\Downarrow$ $k \leq n \Rightarrow P_n(y)(t_k) = y(t_k)$	$z \in C([\alpha, \alpha+\beta]^2)$ $\Downarrow$ $\tau^{-1}(i,j) \leq n \Rightarrow Q_n(z)(t_i, t_j) = z(t_i, t_j)$
$\{b_n\}_{n\geq 1}$ is monotone, that is, $\sup_{n\geq 1}\Vert P_n\Vert = 1$	$\{B_n\}_{n\geq 1}$ is monotone, that is, $\sup_{n\geq 1}\Vert Q_n\Vert = 1$
$y \in C^1([\alpha, \alpha+\beta]),\ n \geq 2$ $\Downarrow$ $\Vert y - P_n(y)\Vert \leq 2\Vert y'\Vert \Delta T_n$	$z \in C^1([\alpha, \alpha+\beta]^2),\ n \geq 2$ $\Downarrow$ $\Vert z - Q_{n^2}(z)\Vert \leq 4 \max\left\{ \left\Vert \frac{\partial z}{\partial t}\right\Vert, \left\Vert \frac{\partial z}{\partial s}\right\Vert \right\} \Delta T_n$

Proof. In the last property in Table 1, take $\rho_1 = 4\max\{\Vert\partial\sigma_{r-1}/\partial t\Vert, \Vert\partial\sigma_{r-1}/\partial s\Vert\}_{r\geq 1}$ and $\rho_2 = 4\max\{\Vert\partial\varphi_{r-1}/\partial t\Vert, \Vert\partial\varphi_{r-1}/\partial s\Vert\}_{r\geq 1}$. □

In the result below we show that the sequence defined in (3.4) approximates the exact solution of (1.1) as well as giving an upper bound of the error committed.

Theorem 3.5. *With the previous notation and the same hypothesis as in Proposition 3.3, let $m \in \mathbb{N}$, $n_r \in \mathbb{N}$, $n_r \geq 2$, and $\{\varepsilon_1, \ldots, \varepsilon_m\}$ be a set of positive numbers such that for all $r \in \{1, \ldots, m\}$ we have*

$$\Delta T_{n_r} \leq \frac{\varepsilon_r}{\beta(\rho_1 + \rho_2)}. \tag{3.17}$$

Then,

$$\Vert Tx_{r-1} - x_r\Vert \leq \varepsilon_r. \tag{3.18}$$

Moreover, if x is the exact solution of the integral equation (1.1)*, then the error $\Vert x - x_m\Vert$ is given by*

$$\Vert x - x_m\Vert \leq \frac{M^m}{1-M}\Vert T\tilde{x} - \tilde{x}\Vert + \sum_{r=1}^{m} M^{m-r}\varepsilon_r. \tag{3.19}$$

Proof. First we deal with proving (3.18). For all $r \in \{1, \dots, m\}$ and $t \in [\alpha, \alpha+\beta]$, Proposition 3.4 gives

$$\begin{aligned} |Tx_{r-1}(t) - x_r(t)| &\le \int_\alpha^{\alpha+\beta} \left|\sigma_{r-1}(t,s) - Q_{n_r^2}(\sigma_{r-1}(t,s))\right| ds \\ &\quad + \int_\alpha^t \left|\varphi_{r-1}(t,s) - Q_{n_r^2}(\varphi_{r-1}(t,s))\right| ds \\ &\le \rho_1 \Delta T_{n_r}\beta + \rho_2 \Delta T_{n_r}\beta = \Delta T_{n_r}\beta(\rho_1+\rho_2) \le \varepsilon_r. \end{aligned} \tag{3.20}$$

To conclude the proof, we derive (3.19). From (2.3), we have

$$\|x - T^m \tilde{x}\| \le \frac{M^m}{1-M}\|T\tilde{x} - \tilde{x}\|, \tag{3.21}$$

and in addition, on the other hand, applying recursively (2.2) and (3.18), we obtain

$$\begin{aligned} \|T^m\tilde{x} - x_m\| &\le \sum_{r=1}^{m} \left\|T^{m-r+1}x_{r-1} - T^{m-r}x_r\right\| \\ &= \sum_{r=1}^{m} \left\|T^{m-r}Tx_{r-1} - T^{m-r}x_r\right\| \\ &\le \sum_{r=1}^{m} M^{m-r}\|Tx_{r-1} - x_r\| \le \sum_{r=1}^{m} M^{m-r}\varepsilon_r. \end{aligned} \tag{3.22}$$

Then we use the triangular inequality

$$\|x - x_m\| \le \|x - T^m\tilde{x}\| + \|T^m\tilde{x} - x_m\|, \tag{3.23}$$

and the proof is complete in view of (3.21) and (3.22). □

Remark 3.6. Under the hypotheses of Theorem 3.5, let us observe that by the inequality (3.19) we have

$$\|x - x_m\| \le \frac{M^m}{1-M}\|T\tilde{x} - \tilde{x}\| + \frac{1-M^m}{1-M}\max_{r\ge1}\{\varepsilon_r\}. \tag{3.24}$$

The first sumand on the right hand side approximates zero when m increases; with respect to the second sumand, since the points of the partition can be chosen in such a way that ΔT_{n_r} becomes so close to zero as we desire, the ε_r's can become so small as we desire, arriving in this way at an explicit control of the error committed.

Therefore, given $\varepsilon > 0$, there exists $m \ge 1$ such that $\|x - x_m\| < \varepsilon$ when choosing ε_r sufficiently small.

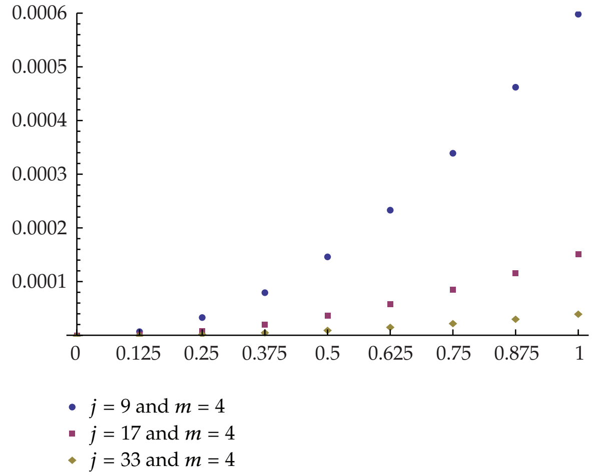

Figure 1: The plot of absolute errors for Example 4.1.

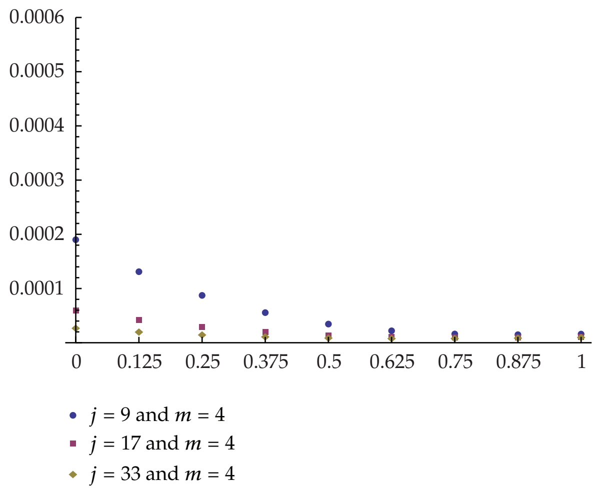

Figure 2: The plot of absolute errors for Example 4.2.

4. Numerical Examples

In this last section we illustrate the results previously developed, stressing the significance of inequality (3.19) in Theorem 3.5, as mentioned in Remark 3.6.

First of all, we show how the numerical method works, because we use it later in the estimation of the error. For solving the numerical example, Mathematica 7 is used, and to construct the Schauder basis in $C([0,1]^2)$, we considered the particular choice $t_1 = 0$, $t_2 = 1$ and for $n \in \mathbb{N} \cup \{0\}$, $t_{i+1} = (2k+1)/2^{n+1}$ if $i = 2^n + k + 1$ where $0 \leq k < 2^n$ are integers. To define the sequence $\{x_r\}_{r \geq 1}$, we take $x_0(t) = y_0(t)$ and $n_r = j$ (for all $r \geq 1$). In Tables 2 and 3, we exhibit, for $j = 9$, 17, and 33, the absolute errors committed in eight representative points of $[0,1]$ when we approximate the exact solution x by the iteration x_4. Its numerical results are also given in Figures 1 and 2, respectively.

Table 2: Absolute errors for Example 4.1.

t	$j = 9$ $\lvert x_4(t) - x(t)\rvert$	$j = 17$ $\lvert x_4(t) - x(t)\rvert$	$j = 33$ $\lvert x_4(t) - x(t)\rvert$
0.0	0.0	0.0	0.0
0.125	7.64×10^{-6}	2.13×10^{-6}	7.49×10^{-7}
0.250	3.40×10^{-5}	8.93×10^{-6}	2.65×10^{-6}
0.375	8.03×10^{-5}	2.07×10^{-5}	5.79×10^{-6}
0.5	1.47×10^{-4}	3.75×10^{-5}	1.02×10^{-5}
0.625	2.34×10^{-4}	5.95×10^{-5}	1.59×10^{-5}
0.750	3.40×10^{-4}	8.63×10^{-5}	2.29×10^{-5}
0.875	4.63×10^{-4}	1.17×10^{-4}	3.10×10^{-5}
1	5.99×10^{-4}	1.52×10^{-4}	4.04×10^{-5}

Table 3: Absolute errors for Example 4.2.

t	$j = 9$ $\lvert x_4(t) - x(t)\rvert$	$j = 17$ $\lvert x_4(t) - x(t)\rvert$	$j = 33$ $\lvert x_4(t) - x(t)\rvert$
0.	1.91×10^{-4}	6.05×10^{-5}	2.78×10^{-5}
0.125	1.32×10^{-4}	4.31×10^{-5}	2.08×10^{-5}
0.250	8.83×10^{-5}	3.00×10^{-5}	1.55×10^{-5}
0.375	5.65×10^{-5}	2.08×10^{-5}	1.19×10^{-5}
0.5	3.54×10^{-5}	1.48×10^{-5}	9.77×10^{-6}
0.625	2.30×10^{-5}	1.16×10^{-5}	8.82×10^{-6}
0.750	1.71×10^{-5}	1.05×10^{-5}	8.86×10^{-6}
0.875	1.58×10^{-5}	1.08×10^{-5}	9.64×10^{-6}
1	1.69×10^{-5}	1.20×10^{-5}	1.08×10^{-5}

Example 4.1. We solve (1.1) with $k_1(t,s) = ts/5$, $g_1(s,x(s)) = \cos(x(s))$, $k_2(t,s) = s/3$, $g_2(s,x(s)) = \sin(x(s))$, and $y_0(t) = 1 + t - (t/5)(\cos(2) - \cos(1) + \sin(2)) + (1/3)(t\cos(1 + t) - \sin(1 + t) + \sin(1))$ with the exact solution $x(t) = 1 + t$.

Example 4.2. We solve (1.1) with $k_1(t,s) = (1/4)(1-t)^3$, $g_1(s,x(s)) = \arctan(x(s))$, $k_2(t,s) = 1/8$, $g_2(s,x(s)) = x(s)$, and $y_0(t) = t - (t^2/16) - ((\pi - \ln(4))/16)(t-1)^3$ with the exact solution $x(t) = t$.

Now we realize that the choice of a particular j, determining the dyadic partition of the interval $[0,1]$ from the first $2^j + 1$ nodes, and in such a way that the error is less than a fixed positive ε, that is, $\|x - x_m\| < \varepsilon$, can be easily determined practically: it suffices to compute, once again by means of Mathematica 7, the error. To this end, since it is measured in terms of the supnorm, we consider the nodes 0, 0.125, 0.25, 0.375, 0.5, 0.625, 0.75, 0.875, 1 and maximum of the absolute values of the differences between the values of the exact solution and the approximation obtained for the third iteration ($m = 3$). The numerical tests are given in Table 4 and correspond to the nonlinear mixed Fredhol-Volterra-Hammerstein equations considered in Examples 4.1 and 4.2, respectively.

Table 4: Number of nodes (j) from error (ε) and for $m = 3$.

ε	Example 4.1	Example 4.2
10^{-2}	$j = 5$	$j = 5$
10^{-3}	$j = 9$	$j = 9$
10^{-4}	$j = 33$	$j = 33$

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References

[1] M. A. Abdou, "On a symptotic methods for Fredholm-Volterra integral equation of the second kind in contact problems," *Journal of Computational and Applied Mathematics*, vol. 154, no. 2, pp. 431–446, 2003.

[2] S. Jiang and V. Rokhlin, "Second kind integral equations for the classical potential theory on open surfaces. II," *Journal of Computational Physics*, vol. 195, no. 1, pp. 1–16, 2004.

[3] K. V. Kotetishvili, G. V. Kekelia, G. S. Kevanishvili, I. G. Kevanishvili, and B. G. Midodashvili, "Development and solution of the integral equation for axial current of a center-driven dipole," *Journal of Applied Electromagnetism*, vol. 12, no. 3, pp. 1–14, 2010.

[4] X. Zhang, J. Luo, and Z. Zhao, "The iterative solution for electromagnetic field coupling to buried wires," *Mathematical Problems in Engineering*, vol. 2011, Article ID 165032, 8 pages, 2011.

[5] P. Baratella, "A Nyström interpolant for some weakly singular nonlinear Volterra integral equations," *Journal of Computational and Applied Mathematics*, vol. 237, no. 1, pp. 542–555, 2013.

[6] M. B. Dhakne and G. B. Lamb, "On an abstract nonlinear second order integrodifferential equation," *Journal of Function Spaces and Applications*, vol. 5, no. 2, pp. 167–174, 2007.

[7] M. Hadizadeh and M. Mohamadsohi, "Numerical solvability of a class of Volterra-Hammerstein integral equations with noncompact kernels," *Journal of Applied Mathematics*, no. 2, pp. 171–181, 2005.

[8] J. Zhu, Y. Yu, and V. Postolică, "Initial value problems for first order impulsive integro-differential equations of Volterra type in Banach spaces," *Journal of Function Spaces and Applications*, vol. 5, no. 1, pp. 9–26, 2007.

[9] F. G. Tricomi, *Integral Equations*, Dover, New York, NY, USA, 1985.

[10] M. A. El-Ameen and M. El-Kady, "A new direct method for solving nonlinear Volterra-Fredholm-Hammerstein integral equations via optimal control problem," *Journal of Applied Mathematics*, vol. 2012, Article ID 714973, 10 pages, 2012.

[11] H. R. Marzban, H. R. Tabrizidooz, and M. Razzaghi, "A composite collocation method for the nonlinear mixed Volterra-Fredholm-Hammerstein integral equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1186–1194, 2011.

[12] Y. Ordokhani and M. Razzaghi, "Solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via a collocation method and rationalized Haar functions," *Applied Mathematics Letters*, vol. 21, no. 1, pp. 4–9, 2008.

[13] K. Parand and J. A. Rad, "Numerical solution of nonlinear Volterra-Fredholm-Hammerstein integral equations via collocation method based on radial basis functions," *Applied Mathematics and Computation*, vol. 218, no. 9, pp. 5292–5309, 2012.

[14] K. Maleknejad, E. Hashemizadeh, and B. Basirat, "Computational method based on Bernstein operational matrices for nonlinear Volterra-Fredholm-Hammerstein integral equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 1, pp. 52–61, 2012.

[15] E. Babolian, F. Fattahzadeh, and E. G. Raboky, "A Chebyshev approximation for solving nonlinear integral equations of Hammerstein type," *Applied Mathematics and Computation*, vol. 189, no. 1, pp. 641–646, 2007.

[16] M. I. Berenguer, A. I. Garralda-Guillem, and M. R. Galán, "Biorthogonal systems approximating the solution of the nonlinear Volterra integro-differential equation," *Fixed Point Theory and Applications*, vol. 2010, Article ID 470149, 9 pages, 2010.

[17] M. I. Berenguer, D. Gámez, A. I. Garralda-Guillem, and M. C. S. Pérez, "Nonlinear Volterra integral equation of the second kind and biorthogonal systems," *Abstract and Applied Analysis*, vol. 2010, Article ID 135216, 11 pages, 2010.

[18] M. I. Berenguer, D. Gámez, A. I. Garralda-Guillem, M. R. Galán, and M. C. S. Pérez, "Biorthogonal systems for solving Volterra integral equation systems of the second kind," *Journal of Computational and Applied Mathematics*, vol. 235, no. 7, pp. 1875–1883, 2011.

[19] M. I. Berenguer, D. Gámez, and A. J. López Linares, "Fixed-point iterative algorithm for the linear Fredholm-Volterra integro-differential equation," *Journal of Applied Mathematics*, vol. 2010, Article ID 370894, 12 pages, 2012.

[20] M. I. Berenguer, D. Gámez, and A. J. López Linares, "Fixed point techniques and Schauder bases to approximate the solution of the first order nonlinear mixed Fredholm-Volterra integro-differential equation," *Journal of Computational and Applied Mathematics*, 2012. In press.

[21] F. Caliò, A.I. Garralda-Guillem, E. Marchetti, and M. Ruiz Galán, "About some numerical approaches for mixed integral equations," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 464–474, 2012.

[22] D. Gámez, A. I. G. Guillem, and M. R. Galán, "High-order nonlinear initial-value problems countably determined," *Journal of Computational and Applied Mathematics*, vol. 228, no. 1, pp. 77–82, 2009.

[23] D. Gámez, A. I. G. Guillem, and M. R. Galán, "Nonlinear initial-value problems and Schauder bases," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 63, no. 1, pp. 97–105, 2005.

[24] G. J. O. Jameson, *Topology and Normed Spaces*, Chapman & Hall, London, UK, 1974.

[25] B. R. Gelbaum and J. G. de Lamadrid, "Bases of tensor products of Banach spaces," *Pacific Journal of Mathematics*, vol. 11, pp. 1281–1286, 1961.

[26] Z. Semadeni, "Product Schauder bases and approximation with nodes in spaces of continuous functions," *Bulletin de l'Académie Polonaise des Sciences*, vol. 11, pp. 387–391, 1963.