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Research Article

Strong Convergence Theorems for Equilibrium Problems and Fixed Point Problems in Hilbert Spaces

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We introduce an Ishikawa iterative scheme by the viscosity approximate method for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in Hilbert space. Then, we prove some strong convergence theorems which extend and generalize S. Takahashi and W. Takahashi's results (2007).

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R, where R is the set of real numbers. The equilibrium problem for $F : C \times C \to R$ is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{EP}(F)$. Given a mapping $T : C \to H$, let $F(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $z \in \mathrm{EP}(F)$ if and only if $\langle Tz, y - z\rangle \geq 0$ for all $y \in C$. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1); for more details, see [1, 2].

Recall that a self-mapping S of a closed convex subset C of H is nonexpansive [3] if there holds that

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

We denote the set of fixed points of S by $F(S)$. There are some methods for approximation of fixed points of a nonexpansive mapping. In 2000, Moudafi [4] introduced the viscosity approximation method for nonexpansive mappings (see [5] for further developments in both Hilbert and Banach spaces). Some methods have been proposed to solve the equilibrium problem; see, for instance, [1, 2, 6, 7]. Recently, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $\mathrm{EP}(F)$ is nonempty and proved a strong convergence theorem. S. Takahashi and W. Takahashi [7] introduced a Mann iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem.

On the other hand, Ishikawa [8] introduced the following iterative process defined recursively by

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n) S y_n, \\ y_n &= \beta_n x_n + (1-\beta_n) S x_n, \quad \forall n \in N, \end{aligned} \tag{1.3}$$

where the initial guess x_0 is taking in C arbitrarily, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in the interval $[0,1]$.

In this paper, motivated by the ideas in [4–8], we introduce an Ishikawa iterative scheme by the viscosity approximation method for finding a common element of the set of solution (1.1) and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ by

$$\begin{aligned} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle &\geq 0, \quad \forall y \in C, \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n) S y_n, \\ y_n &= \beta_n x_n + (1-\beta_n) S u_n, \quad \forall n \in N, \end{aligned} \tag{1.4}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ and $\{r_n\} \subset (0,\infty)$.

We will prove in Section 3 that if the sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{r_n\}$ of parameters satisfy appropriate conditions, then the sequences $\{x_n\}$, $\{y_n\}$, and $\{u_n\}$ generated by (1.4) converge strongly to $z \in F(S) \cap \mathrm{EP}(F)$. The results in this paper extend and generalize S. Takahashi and W. Takahashi's results [7].

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. $x_n \to x$ implies that $\{x_n\}$ converges strongly to x and $x_n \rightharpoonup x$ means that $\{x_n\}$ converges weakly to x. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in R$; see [9].

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \leq \|x - y\|$ for all $y \in C$. Such a P_C is called the metric projection of H onto C. It is also known that $y = P_C(x)$ is equivalent to $\langle x - y, y - z\rangle \geq 0$ for all $z \in C$.

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for any $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t \downarrow 0} F\big(tz + (1-t)x, y\big) \leq F(x, y); \tag{2.2}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

We recall some lemmas needed later.

Lemma 2.1 (see [2]). *Let C be a nonempty closed convex subset of H and let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{2.3}$$

Lemma 2.2 (see [5]). *Let C be a nonempty closed convex subset of H, and let F be a bifunction from $C \times C$ to R satisfying (A1)–(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\} \tag{2.4}$$

for all $x \in H$. Then, the following statements hold:

(1) *T_r is single-valued;*

(2) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\|T_r(x) - T_r(y)\right\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle; \tag{2.5}$$

(3) $F(T_r) = \mathrm{EP}(F)$;

(4) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 2.3 (see [10]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - c_n)a_n + b_n, \quad \forall n \in N, \tag{2.6}$$

where $\{b_n\}$ *is a sequence of real numbers and* $\{c_n\}$ *is a sequence in* $(0,1)$ *such that*

(i) $\sum_{n=1}^{\infty} c_n = \infty$,

(ii) $\limsup_{n\to\infty}(b_n/c_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |b_n| < \infty$.

Then, $\lim_{n\to\infty} a_n = 0$.

3. Strong Convergence Theorem

In this section, we show a strong convergence theorem which solves the problem of finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of a nonexpansive mapping in a Hilbert space.

Theorem 3.1. *Let* C *be a nonempty closed convex subset of* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying (A1)–(A4) and let* S *be a nonexpansive mapping of* C *into* H *such that* $F(S) \cap \mathrm{EP}(F) \neq \emptyset$. *Let* f *be a contraction of* H *into itself and let* $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *be sequences generated by* $x_1 \in H$ *and* (1.4). *If* $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ *and* $\{r_n\} \subset (0,\infty)$ *satisfy the following conditions:*

$$\lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty,$$
$$0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \qquad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \tag{3.1}$$
$$\liminf_{n\to\infty} r_n > 0, \qquad \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty,$$

then, $\{x_n\}$, $\{y_n\}$, *and* $\{u_n\}$ *converge strongly to* $z \in F(S) \cap \mathrm{EP}(F)$, *where* $z = P_{F(S)\cap \mathrm{EP}(F)} f(z)$.

Proof. Let $Q = P_{F(S)\cap\mathrm{EP}(F)}$. Then Qf is a contraction of H into itself. In fact, there exists $a \in [0,1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in H$. So, we have that

$$\|Qf(x) - Qf(y)\| \leq \|f(x) - f(y)\| \leq a\|x - y\| \tag{3.2}$$

for all $x, y \in H$. Since H is complete, there exists a unique element $z \in H$ such that $z = Qf(z)$. Such a $z \in H$ is an element of C.

Let $v \in F(S) \cap \mathrm{EP}(F)$. Then from $u_n = T_{r_n} x_n$, we have

$$\|u_n - v\| = \|T_{r_n} x_n - T_{r_n} v\| \leq \|x_n - v\| \tag{3.3}$$

for all $n \in N$. Put $M = \max\{\|x_1 - v\|,\ (1/(1-a))\|f(v) - v\|\}$. It is obvious that $\|x_1 - v\| \leq M$.

Suppose $\|x_n - v\| \leq M$. Then, we have

$$\begin{aligned} \|x_{n+1} - v\| &\leq \alpha_n \|f(x_n) - v\| + (1-\alpha_n)\|Sy_n - v\| \\ &\leq \alpha_n \|f(x_n) - f(v)\| + \alpha_n \|f(v) - v\| + (1-\alpha_n)\|Sy_n - v\| \\ &\leq a\alpha_n \|x_n - v\| + \alpha_n \|f(v) - v\| + (1-\alpha_n)\|y_n - v\|. \end{aligned} \tag{3.4}$$

On the other hand

$$\begin{aligned} \|y_n - v\| &\leq \beta_n \|x_n - v\| + (1-\beta_n)\|Su_n - v\| \\ &\leq \beta_n \|x_n - v\| + (1-\beta_n)\|u_n - v\| \\ &\leq \beta_n \|x_n - v\| + (1-\beta_n)\|x_n - v\| \\ &= \|x_n - v\|. \end{aligned} \tag{3.5}$$

Putting (3.5) into (3.4), we have

$$\begin{aligned} \|x_{n+1} - v\| &\leq a\alpha_n \|x_n - v\| + \alpha_n \|f(v) - v\| + (1-\alpha_n)\|x_n - v\| \\ &= [1 - \alpha_n(1-a)]\|x_n - v\| + \alpha_n(1-a)\frac{\|f(v) - v\|}{1-a} \\ &\leq [1 - \alpha_n(1-a)]M + \alpha_n(1-a)M = M. \end{aligned} \tag{3.6}$$

So, we have that $\|x_{n+1} - v\| \leq M$ for any $n \in N$. And hence $\{x_n\}$ is bounded. We also obtain that $\{u_n\}, \{y_n\}, \{Su_n\}, \{Sy_n\}$, and $\{f(x_n)\}$ are bounded. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. In fact,

$$\begin{aligned} \|y_n - y_{n-1}\| &= \|\beta_n x_n + (1-\beta_n)Su_n - [\beta_{n-1}x_{n-1} + (1-\beta_{n-1})Su_{n-1}]\| \\ &= \|\beta_n(x_n - x_{n-1}) + (\beta_n - \beta_{n-1})x_{n-1} + (1-\beta_n)(Su_n - Su_{n-1}) + (\beta_{n-1} - \beta_n)Su_{n-1}\| \\ &\leq |\beta_n - \beta_{n-1}|\|x_{n-1}\| + \beta_n\|x_n - x_{n-1}\| + (1-\beta_n)\|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|\|Su_{n-1}\|, \end{aligned} \tag{3.7}$$

and hence

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \left\|\alpha_n f(x_n)+(1-\alpha_n)Sy_n-\alpha_{n-1}f(x_{n-1})-(1-\alpha_{n-1})Sy_{n-1}\right\| \\
&= \left\|\alpha_n f(x_n)-\alpha_n f(x_{n-1})+\alpha_n f(x_{n-1})-\alpha_{n-1}f(x_{n-1})\right. \\
&\qquad \left.+(1-\alpha_n)Sy_n-(1-\alpha_n)Sy_{n-1}+(1-\alpha_n)Sy_{n-1}-(1-\alpha_{n-1})Sy_{n-1}\right\| \\
&\le \alpha_n a\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\|+(1-\alpha_n)\left\|y_n-y_{n-1}\right\| \\
&\quad +|\alpha_n-\alpha_{n-1}|\left\|Sy_{n-1}\right\| \\
&\le \alpha_n a\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left\|f(x_{n-1})\right\|+(1-\alpha_n) \\
&\quad \times \left[\left|\beta_n-\beta_{n-1}\right|\|x_{n-1}\|+\beta_n\|x_n-x_{n-1}\|+(1-\beta_n)\|u_n-u_{n-1}\|+\left|\beta_n-\beta_{n-1}\right|\|Su_{n-1}\|\right] \\
&\quad +|\alpha_n-\alpha_{n-1}|\left\|Sy_{n-1}\right\| \\
&= \left[\beta_n-\alpha_n(\beta_n-a)\right]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|\left[\left\|f(x_{n-1})\right\|+\left\|Sy_{n-1}\right\|\right] \\
&\quad +(1-\alpha_n)\left|\beta_n-\beta_{n-1}\right|\left[\|x_{n-1}\|+\|Su_{n-1}\|\right]+(1-\alpha_n)(1-\beta_n)\|u_n-u_{n-1}\| \\
&\le \left[\beta_n-\alpha_n(\beta_n-a)\right]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|K_1+(1-\alpha_n)\left|\beta_n-\beta_{n-1}\right|K_2 \\
&\quad +(1-\alpha_n)(1-\beta_n)\|u_n-u_{n-1}\|,
\end{aligned}
\tag{3.8}
$$

where $K_1=\sup\{\|f(x_n)\|+\|Sy_n\|:n\in N\}$ and $K_2=\sup\{\|x_n\|+\|Su_n\|:n\in N\}$.

On the other hand, from $u_n=T_{r_n}x_n$ and $u_{n+1}=T_{r_{n+1}}x_{n+1}$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C, \tag{3.9}$$

$$F(u_{n+1},y)+\frac{1}{r_{n+1}}\langle y-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0,\quad \forall y\in C. \tag{3.10}$$

Putting $y=u_{n+1}$ in (3.9) and $y=u_n$ in (3.10), we have

$$
\begin{aligned}
F(u_n,u_{n+1})+\frac{1}{r_n}\langle u_{n+1}-u_n,u_n-x_n\rangle\ge 0, \\
F(u_{n+1},u_n)+\frac{1}{r_{n+1}}\langle u_n-u_{n+1},u_{n+1}-x_{n+1}\rangle\ge 0.
\end{aligned}
\tag{3.11}
$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1}-u_n,\frac{u_n-x_n}{r_n}-\frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle\ge 0, \tag{3.12}$$

and hence

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \geq 0. \tag{3.13}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then, we have

$$\begin{aligned} \|u_{n+1} - u_n\|^2 &\leq \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\ &\leq \|u_{n+1} - u_n\| \left\{ \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \right\}, \end{aligned} \tag{3.14}$$

and hence

$$\begin{aligned} \|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|u_{n+1} - x_{n+1}\| \\ &\leq \|x_{n+1} - x_n\| + \frac{1}{b} |r_{n+1} - r_n| L, \end{aligned} \tag{3.15}$$

where $L = \sup\{\|u_n - x_n\| : n \in N\}$. So from (3.8), we have

$$\begin{aligned} \|x_{n+1} - x_n\| \leq{}& [\beta_n - \alpha_n(\beta_n - a)] \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K_1 \\ &+ (1 - \alpha_n) |\beta_n - \beta_{n-1}| K_2 + (1 - \alpha_n)(1 - \beta_n) \left[\|x_n - x_{n-1}\| + \frac{1}{b} |r_n - r_{n-1}| L \right] \\ ={}& (1 - \alpha_n(1 - a)) \|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}| K_1 \\ &+ (1 - \alpha_n) |\beta_n - \beta_{n-1}| K_2 + (1 - \alpha_n)(1 - \beta_n) \frac{1}{b} |r_n - r_{n-1}| L. \end{aligned} \tag{3.16}$$

Using Lemma 2.1 in [10], we obtain

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{3.17}$$

From (3.15) and $|r_{n+1} - r_n| \to 0$, we have

$$\lim_{n \to \infty} \|u_{n+1} - u_n\| = 0. \tag{3.18}$$

It follows from (3.7) that

$$\lim_{n \to \infty} \|y_{n+1} - y_n\| = 0. \tag{3.19}$$

Since $x_n = \alpha_{n-1} f(x_{n-1}) + (1-\alpha_{n-1}) S y_{n-1}$, we have

$$\begin{aligned} \|x_n - Sy_n\| &\le \|x_n - Sy_{n-1}\| + \|Sy_{n-1} - Sy_n\| \\ &\le \alpha_{n-1}\|f(x_{n-1}) - Sy_{n-1}\| + \|y_{n-1} - y_n\|. \end{aligned} \tag{3.20}$$

From $\alpha_n \to 0$, we have $\|x_n - Sy_n\| \to 0$. For $v \in F(S) \cap \mathrm{EP}(F)$, we have

$$\begin{aligned} \|u_n - v\|^2 &= \|T_{r_n} x_n - T_{r_n} v\|^2 \\ &\le \langle T_{r_n} x_n - T_{r_n} v, x_n - v\rangle \\ &= \langle u_n - v, x_n - v\rangle \\ &= \frac{1}{2}\left(\|u_n - v\|^2 + \|x_n - v\|^2 - \|x_n - u_n\|^2\right), \end{aligned} \tag{3.21}$$

and hence

$$\|u_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - u_n\|^2. \tag{3.22}$$

Therefore, from the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \|y_n - v\|^2 &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\|Su_n - v\|^2 \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\|u_n - v\|^2 \\ &\le \beta_n\|x_n - v\|^2 + (1-\beta_n)\left[\|x_n - v\|^2 - \|x_n - u_n\|^2\right] \\ &= \|x_n - v\|^2 - (1-\beta_n)\|x_n - u_n\|^2, \end{aligned} \tag{3.23}$$

and hence

$$\begin{aligned} \|x_{n+1} - v\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n) Sy_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|Sy_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\|y_n - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + (1-\alpha_n)\left[\|x_n - v\|^2 - (1-\beta_n)\|x_n - u_n\|^2\right] \\ &\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2. \end{aligned} \tag{3.24}$$

So, we have

$$\begin{aligned} (1-\alpha_n)(1-\beta_n)\|x_n - u_n\|^2 &\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\ &\le \alpha_n\|f(x_n) - v\|^2 + \|x_n - x_{n+1}\|(\|x_n - v\| + \|x_{n+1} - v\|). \end{aligned} \tag{3.25}$$

Without loss of generality, let us assume that there exists two real numbers β^* and $\overline{\beta}$ such that $1 > \overline{\beta} \geq \beta_n \geq \beta^* > 0$ for all $n \in N$. Hence,

$$\begin{aligned}(1-\alpha_n)\left(1-\overline{\beta}\right)\|x_n-u_n\|^2 &\leq (1-\alpha_n)(1-\beta_n)\|x_n-u_n\|^2 \\ &\leq \alpha_n\left\|f(x_n)-v\right\|^2+\|x_n-x_{n+1}\|(\|x_n-v\|+\|x_{n+1}-v\|).\end{aligned} \tag{3.26}$$

It follows that $\|x_n-u_n\| \to 0$. We also have

$$\begin{aligned}\|Su_n-x_n\| &\leq \left\|Sy_n-x_n\right\|+\left\|Su_n-Sy_n\right\| \\ &\leq \left\|Sy_n-x_n\right\|+\left\|u_n-y_n\right\| \\ &\leq \left\|Sy_n-x_n\right\|+\|u_n-x_n\|+\left\|x_n-y_n\right\| \\ &= \left\|Sy_n-x_n\right\|+\|u_n-x_n\|+(1-\beta_n)\|x_n-Su_n\|.\end{aligned} \tag{3.27}$$

It follows that

$$\beta^*\|Su_n-x_n\| \leq \beta_n\|Su_n-x_n\| \leq \left\|Sy_n-x_n\right\|+\|u_n-x_n\|. \tag{3.28}$$

Hence, $\|Su_n-x_n\| \to 0$. Since

$$\|Su_n-u_n\| \leq \|Su_n-x_n\|+\|x_n-u_n\|, \tag{3.29}$$

we also have $\lim_{n\to\infty}\|Su_n-u_n\|=0$. Next, we show that

$$\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle \leq 0, \tag{3.30}$$

where $z=P_{F(S)\cap \mathrm{EP}(F)}f(z)$. To show this inequality, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty}\langle f(z)-z,x_{n_i}-z\rangle=\limsup_{n\to\infty}\langle f(z)-z,x_n-z\rangle. \tag{3.31}$$

Since $\{u_{ni}\}$ is bounded, there exists a subsequence $\{u_{ni_j}\}$ of $\{u_{ni}\}$ which converges weakly to w. Without loss of generality, we can assume that $\{u_{ni}\} \rightharpoonup w$. From $\|Su_n-u_n\| \to 0$, we obtain $Su_{n_i} \rightharpoonup w$. Let us show $w \in \mathrm{EP}(F)$. By $u_n=T_{r_n}x_n$, we have

$$F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0, \quad \forall y \in C. \tag{3.32}$$

From (A2), we also have

$$\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq F(y,u_n), \tag{3.33}$$

and hence,

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq F(y, u_{n_i}). \tag{3.34}$$

Since $(u_{n_i} - x_{n_i})/r_{n_i} \rightarrow 0$ and $u_{n_i} \rightharpoonup w$, from (A4), we have

$$f(y, w) \leq 0, \quad \forall y \in C. \tag{3.35}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) \leq 0$. So we have

$$0 = F(y_t, y_t) \leq tF(y_t, y) + (1-t)F(y_t, w) \leq tF(y_t, y). \tag{3.36}$$

Dividing by t, we get

$$F(y_t, y) \geq 0. \tag{3.37}$$

Letting $t \rightarrow 0$ and from (A3), we get

$$F(w, y) \geq 0 \tag{3.38}$$

for all $y \in C$ and hence $w \in \mathrm{EP}(F)$. We shall show that $w \in F(S)$. Assume $w \notin F(S)$. Since $u_{ni} \rightharpoonup w$ and $w \neq Sw$, from the Opial theorem [11] we have

$$\begin{aligned} \liminf_{i \to \infty} \|u_{n_i} - w\| &< \liminf_{i \to \infty} \|u_{n_i} - Sw\| \\ &\leq \liminf_{i \to \infty} \{ \|u_{n_i} - Su_{n_i}\| + \|Su_{n_i} - Sw\| \} \\ &\leq \liminf_{i \to \infty} \|u_{n_i} - w\|. \end{aligned} \tag{3.39}$$

This is a contradiction. So, we get $w \in F(S)$. Therefore, $w \in F(S) \cap \mathrm{EP}(F)$. Since $z = P_{F(S) \cap \mathrm{EP}(F)} f(z)$, we have

$$\begin{aligned} \limsup_{n \to \infty} \langle f(z) - z, x_n - z \rangle &= \lim_{i \to \infty} \langle f(z) - z, x_{n_i} - z \rangle \\ &= \lim_{i \to \infty} \langle f(z) - z, u_{n_i} - z \rangle \\ &= \langle f(z) - z, w - z \rangle \leq 0. \end{aligned} \tag{3.40}$$

From $x_{n+1} - z = \alpha_n (f(x_n) - z) + (1 - \alpha_n)(Sy_n - z)$, we have

$$(1-\alpha_n)^2 \left\|Sy_n - z\right\|^2 \geq \|x_{n+1} - z\|^2 - 2\alpha_n \langle f(x_n) - z, x_{n+1} - z\rangle, \tag{3.41}$$

$$\begin{aligned}\left\|y_n - z\right\|^2 &= \left\|\beta_n x_n + (1-\beta_n)Su_n - z\right\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|Su_n - z\|^2 \\ &\leq \beta_n \|x_n - z\|^2 + (1-\beta_n)\|u_n - z\|^2 \\ &\leq \|x_n - z\|^2.\end{aligned} \tag{3.42}$$

It follows that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq (1-\alpha_n)^2 \left\|Sy_n - z\right\|^2 + 2\alpha_n \langle f(x_n) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2 \left\|y_n - z\right\|^2 + 2\alpha_n \langle f(x_n) - f(z), x_{n+1} - z\rangle \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2 \|x_n - z\|^2 + 2\alpha_n a \|x_n - z\| \|x_{n+1} - z\| \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle \\ &\leq (1-\alpha_n)^2 \|x_n - z\|^2 + \alpha_n a \left\{\|x_n - z\|^2 + \|x_{n+1} - z\|^2\right\} \\ &\quad + 2\alpha_n \langle f(z) - z, x_{n+1} - z\rangle.\end{aligned} \tag{3.43}$$

Hence

$$\|x_{n+1} - z\|^2 \leq \frac{(1-\alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} \|x_n - z\|^2 + \frac{2\alpha_n}{1 - \alpha_n a} \langle f(z) - z, x_{n+1} - z\rangle. \tag{3.44}$$

From $\alpha_n \to 0$, we know that there exists a positive integer n_0, such that $1 > 1 - \alpha_n a > 1/2$ for all $n \geq n_0$. Then

$$\begin{aligned}\frac{(1-\alpha_n)^2 + \alpha_n a}{1 - \alpha_n a} &= \frac{1 - 2\alpha_n + \alpha_n a}{1 - \alpha_n a} + \frac{\alpha_n^2}{1 - \alpha_n a} \\ &= 1 - \frac{2(1-a)\alpha_n}{1 - \alpha_n a} + \frac{\alpha_n^2}{1 - \alpha_n a} \\ &\leq 1 - 2(1-a)\alpha_n + 2\alpha_n^2, \quad \forall n \geq n_0.\end{aligned} \tag{3.45}$$

Putting above inequality into (3.44), we get

$$\|x_{n+1} - z\|^2 \leq (1 - 2(1 - a)\alpha_n)\|x_n - z\|^2 + 2\overline{M}\alpha_n^2 + \frac{2\alpha_n}{1 - \alpha_n a}\sigma_n, \quad \forall n \geq n_0, \tag{3.46}$$

where $\overline{M} = \sup\{\|x_n - z\|^2 : n \in N\}$, and $\sigma_n = \langle f(z) - z, x_{n+1} - z\rangle$.

It follows from Lemma 2.3 that

$$x_n \longrightarrow z \in F(S) \cap \mathrm{EP}(F). \tag{3.47}$$

It follows from $\|x_n - u_n\| \rightarrow 0$ and (3.42) that $u_n \rightarrow z$ and $y_n \rightarrow z$. □

By Theorem 3.1, we can obtain the following new result.

Corollary 3.2. *Let C be a nonempty closed convex subset of H. Let S be a nonexpansive mapping of C into H such that $F(S) \neq \emptyset$. Let f be a contraction of H into itself and let $\{x_n\}$ and $\{y_n\}$ be sequences generated by $x_1 \in H$ and*

$$\begin{aligned} x_{n+1} &= \alpha_n f(x_n) + (1 - \alpha_n)Sy_n, \\ y_n &= \beta_n x_n + (1 - \beta_n)SP_C x_n, \quad \forall n \in N. \end{aligned} \tag{3.48}$$

If $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ satisfy the following conditions:

$$\begin{gathered} \lim_{n\to\infty} \alpha_n = 0, \qquad \sum_{n=1}^{\infty} \alpha_n = \infty, \qquad \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty, \\ 0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1, \qquad \sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty, \end{gathered} \tag{3.49}$$

then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S)$, where $z = P_{F(S)}f(z)$.

Proof. Put $F(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in N$ in Theorem 3.1. Then, we get $u_n = P_C x_n$. So from Theorem 3.1, the sequences $\{x_n\}$ and $\{y_n\}$ converge strongly to $z \in F(S)$, where $z = P_{F(S)}f(z)$. □

Remark 3.3. Theorem 3.1 and Corollary 3.2, respectively, extend and generalize Theorem 3.2 and Corollary 3.3 in [7] from the Mann iterative form to the Ishikawa iterative form.

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