

A power law for the lowest eigenvalue in localized massive gravity

André Miemiec^{*1}

*Department of Physics
Queen Mary and Westfield College
Mile End Rd
London E1 4NS, UK*

Abstract

This short note contains a detailed analysis to find the right power law the lowest eigenvalue of a localized massive graviton bound state in a four dimensional AdS background has to satisfy. In contrast to a linear dependence of the cosmological constant we find a quadratic one.

^{*1}a.miemiec@qmw.ac.uk

1 Definition of the Problem

The consistent embedding of four dimensional physics, in particular four dimensional gravity, into a higher dimensional spacetime progresses significantly, when the authors of [1] observed that the high dimensional graviton can be localized within the extra dimensions. This opens a door for realizing the observable physics on a hypersurface in higher dimensional spacetime. After this discovery several attempts were made to generalize this construction to four dimensional spacetime with a non vanishing cosmological constant. The relevant solutions are constructed in [2, 3, 4, 5] and include, beside the original Minkowski space, de Sitter (dS₄) and anti de Sitter (AdS₄) spacetimes. In the papers [6] and [7] the authors focus independently on the localization of massive gravity fluctuations around the cosmological AdS brane solution, i.e. they investigate the perturbation h_{ij} around an AdS₄ background g_{ij} . The metric reads^{*2}:

$$ds^2 = e^{2A(x)} \left[(g_{ij} + h_{ij}) d\xi^i d\xi^j - dx^2 \right]$$

and the warp factor $A(x)$ in the conformal coordinate x is given by

$$A(x) = \ln \frac{L\sqrt{-\Lambda}}{\sin [\sqrt{-\Lambda} (|x| + x_0)]}. \quad (1.1)$$

The contribution of the fifth dimension to the lowest order fluctuations of the gravitational field [8, 9] is encoded in the next equation

$$\left[-\frac{d^2}{dx^2} + \frac{9}{4}A'(x)^2 + \frac{3}{2}A''(x) \right] \psi(x) = \underbrace{m^2}_E \psi(x) \quad (1.2)$$

and we are left with the problem to determine the spectrum of this differential operator and in particular the quantitative behaviour of the lowest eigenvalue E_0 . This is precisely the question we are concerned with in this note. See [10] for numerical investigations of the same question and the similar results of [11] in a slightly modified setting.

It is well known that such an operator can be factorized by aiding what is called the superpotential W . It is simply given through the warp factor $A(x)$ and reads $W = \frac{3}{2}A'(x)$. Introducing the operators A and A^\dagger below^{*3}

$$A = \frac{d}{dx} + W, \quad A^\dagger = -\frac{d}{dx} + W$$

the differential equation (1.2) can be written as $AA^\dagger \psi = m^2 \psi$. We want to remind that this type of problem associate the spectra of two Hamiltonians $H_1 = A^\dagger A$ and $H_2 = AA^\dagger$ to each other as shown in Fig. 1. Both Hamiltonians are of the form $Hu = u'' + V(x)u$ and the potentials generated by W read:

$$W = -\frac{3}{2} \alpha \operatorname{sgn} x \cot(\alpha|x| + \beta) \quad \begin{array}{l} \nearrow V_1 = 3\alpha \cot \beta \delta(x) + \alpha^2 \left(\frac{3}{4} \frac{1}{\sin^2(\alpha|x| + \beta)} - \frac{9}{4} \right) \\ \searrow V_2 = -3\alpha \cot \beta \delta(x) + \alpha^2 \left(\frac{15}{4} \frac{1}{\sin^2(\alpha|x| + \beta)} - \frac{9}{4} \right) \end{array}$$

^{*2} My conventions follow [7]. There the action is given this metric belongs to.

^{*3} Since the warp factor does not appear anymore it is unambiguous to use the symbol A for the operators.

Here $\alpha = \sqrt{-\Lambda}$ and $\beta = \sqrt{-\Lambda} x_0$ are convenient abbreviations used in the analysis. The spectrum we are primarily concerned with belongs to the Hamiltonian H_2 .

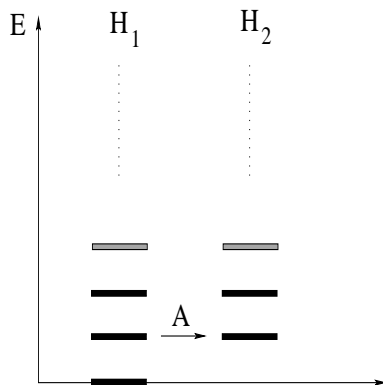


Fig.1. Spectra of H_1 and H_2

The additional structure present by this factorization is very useful for constructing the solutions. On the one side for special parameters, W depend on, it becomes possible to solve the Hamiltonian purely algebraically, i.e. the potential becomes self similar [12]. For the potentials at hand the mismatch is parametrized by β . Only if $\beta = \pi/2$ they display this property in full beauty.

On the other side a nice property of supersymmetric quantum mechanics is the fact that for parity conserving potentials the operator A intertwines the parity and maps even to odd eigenfunctions of the dual problem and vice versa. Since we are interested in the lowest eigenvalue of the operator H_2 , the eigenfunction of which is an even one, we can translate it into the problem to find the second eigenvalue of H_1 , the eigenfunction of which is an odd one. The advantage of this is that we can neglect the whole trouble arising from the δ -function. Since an odd eigenfunction is zero at the origin the product $\delta(x)\psi(x) = \delta(x)\psi(0)$ does not contribute and drops out. For this reason we can restrict our considerations to the potential below without the δ -term.

$$\tilde{V}_1 = \alpha^2 \left(\frac{3}{4} \frac{1}{\sin^2(\alpha|x| + \beta)} - \frac{9}{4} \right)$$

This potential stays finite only in the interval

$$-\pi + \beta \leq \alpha x \leq \pi - \beta$$

which forces us to require a vanishing wave function at the boundaries. A typical picture the potential look like is given in Fig. 3. Since we want to consider the odd solutions of the problem above, we also require the vanishing of the wave function at the origin. This leads by symmetry to the boundary conditions below:

$$\psi^{(1)}(0) = \psi^{(1)}\left(\frac{\pi}{\alpha} - \frac{\beta}{\alpha}\right) = 0. \quad (1.3)$$

The full solution of the problem will be obtained stepwise. The first one is the solution of the differential equation for $\beta = 0$. The second is to justify the arguments in a way, that the shifted

potential coincides on the positive x -axis with the given potentials of interest. Furthermore we implement the boundary conditions for the odd solutions of H_1 . In the last step we transform the odd solution of H_1 to an even solution of H_2 .

After all we discuss the validity of an expansion of the lowest eigenvalue E_0 of H_2 in terms of β .

2 The Power Law

2.1 The Differential Equation

After some mild manipulations including a rescaling of x corresponding to $\xi = \alpha x$ and renaming ξ to x afterwards we arrive at the problem to find the solutions of:

$$-\frac{d^2 u(x)}{dx^2} + \left(\frac{3}{4} \frac{1}{\sin^2 x} - \frac{9}{4} \right) \cdot u(x) = E u(x)$$

Here E stands for E/α^2 as an side effect of this rescaling.

Introducing the new variable $y = \cos^2 x$ and rescaling the resulting function like $u(y) = \frac{v(y)}{(1-y)^{1/4}}$ we obtain

$$4y(y-1)v'' + 2(y-1)v' - (E+2)v(y) = 0.$$

This is the hypergeometric differential equation. Since the argument y varies along $0 \leq y \leq 1$ the solution can be written as:

$$\begin{aligned} v(y) &= c_1 F\left(-\frac{1}{4} + \frac{\sqrt{4E+9}}{4}, -\frac{1}{4} - \frac{\sqrt{4E+9}}{4}, \left[\frac{1}{2}\right], y\right) \\ &+ c_2 \sqrt{y} F\left(\frac{1}{4} + \frac{\sqrt{4E+9}}{4}, \frac{1}{4} - \frac{\sqrt{4E+9}}{4}, \left[\frac{3}{2}\right], y\right) \end{aligned}$$

Transforming back to the original problem we now obtain:

$$\begin{aligned} u(x) &= c_1 \frac{1}{\sqrt{\sin x}} F\left(-\frac{1}{4} + \frac{\sqrt{4E+9}}{4}, -\frac{1}{4} - \frac{\sqrt{4E+9}}{4}, \left[\frac{1}{2}\right], \cos^2 x\right) \\ &+ c_2 \frac{\cos x}{\sqrt{\sin x}} F\left(\frac{1}{4} + \frac{\sqrt{4E+9}}{4}, \frac{1}{4} - \frac{\sqrt{4E+9}}{4}, \left[\frac{3}{2}\right], \cos^2 x\right) \end{aligned}$$

2.2 The odd solutions of H_1

After shifting the argument like $x \rightarrow x + \beta$ we have to implement the boundary conditions of eq. (1.3). The first of the two in eq. (1.3) leads to the following choice for the constants c_1 and c_2 :

$$\begin{aligned} c_1 &= \frac{\cos \beta}{\sqrt{\sin \beta}} F\left(\frac{1}{4} - \frac{\sqrt{9+4E}}{4}, \frac{1}{4} + \frac{\sqrt{9+4E}}{4}, \left[\frac{3}{2}\right], \cos^2 \beta\right) \\ c_2 &= -\frac{1}{\sqrt{\sin \beta}} F\left(-\frac{1}{4} + \frac{\sqrt{9+4E}}{4}, -\frac{1}{4} - \frac{\sqrt{9+4E}}{4}, \left[\frac{1}{2}\right], \cos^2 \beta\right) \end{aligned}$$

The second boundary condition finally produces the transcendental equation, which determines the eigenvalues. To do that we have to keep in mind that the hypergeometric function evaluated at the point 1 can be expressed through the Γ -function as shown below:

$$F([a, b], [c], 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad \text{if } c-a-b > 0$$

Then one obtains:

$$0 = \frac{\cos \beta}{\Gamma(\frac{3}{4} - \frac{\sqrt{4E+9}}{4})\Gamma(\frac{3}{4} + \frac{\sqrt{4E+9}}{4})} F([\frac{1}{4} - \frac{\sqrt{9+4E}}{4}, \frac{1}{4} + \frac{\sqrt{9+4E}}{4}], [\frac{3}{2}], \cos^2 \beta) + \frac{1/2}{\Gamma(\frac{5}{4} - \frac{\sqrt{4E+9}}{4})\Gamma(\frac{5}{4} + \frac{\sqrt{4E+9}}{4})} F([\frac{1}{4} + \frac{\sqrt{9+4E}}{4}, -\frac{1}{4} - \frac{\sqrt{9+4E}}{4}], [\frac{1}{2}], \cos^2 \beta) \quad (2.1)$$

This equation can be analyzed without too much effort. The zeros of the graph on the right hand side in eq. (2.1) are the odd eigenvalues of H_1 or the even eigenvalues of H_2 , respectively. The graph for $\beta = 1/2$ is shown in Fig. 2.

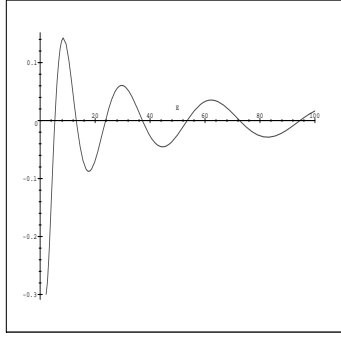


Fig.2. Odd eigenvalues of H_1 for $\beta = \frac{1}{2}$

In Tab. 1 we have collected some of the odd eigenvalues of H_1 for different values of β . If $\beta = \pi/2$ we obtain half of the integer spectrum of an algebraically solvable self similar potential. For very small β we see the ‘almost zero mode’.

n	$E_n^{\beta=0.1}$	$E_n^{\beta=0.5}$	$E_n^{\beta=\pi/2}$
1	0.01471523	0.32986726	4
3	4.07116545	5.35248438	18
5	10.19138101	13.16289911	40
7	18.39233651	23.78670947	70
9	28.68676887	37.23243928	108

Tab. 1: Some eigenvalues

It is quite interesting to emphasize that for very small β the eigenvalues converge to the spectrum of a self similar potential, too. The reason for this is the β -dependence of the shape of the potentials. In Fig. 3 and Fig. 4 we have shown the basic structure of the potential V_1 for $\beta = 0.5$ and $\beta = 0.1$. As β approaches zero the potential splits into two independent self similar potentials with integer spectrum $E_n = n(n+3)$.

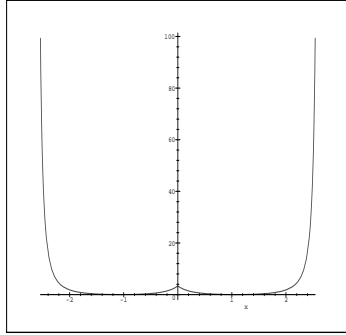


Fig.3. V_1 for $\beta = 0.5$

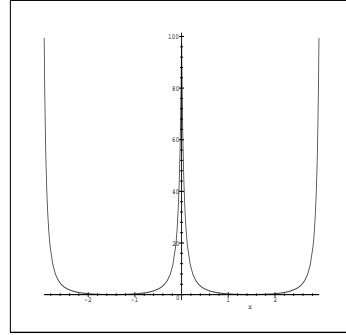


Fig.4. V_1 for $\beta = 0.1$

2.3 Obtaining solutions of H_2

The back transformation follows the pattern sketched in Fig. 1. The eigenvalues and eigenfunctions of the Hamiltonian H_2 can be obtained from those of H_1 by:

$$\psi_n^{(2)} = \frac{1}{\sqrt{E_{n+1}^{(1)}}} A\psi_{n+1}^{(1)} \quad E_n^{(2)} = E_{n+1}^{(1)}$$

In the next two figures we show the lowest normalized eigenfunctions. In Fig. 5 we pictured only the first odd eigenfunction of H_1 , since for all higher ones there is no characteristic change in shape. In Fig. 6 we show the two lowest even eigenfunctions of the Hamiltonian H_2 .

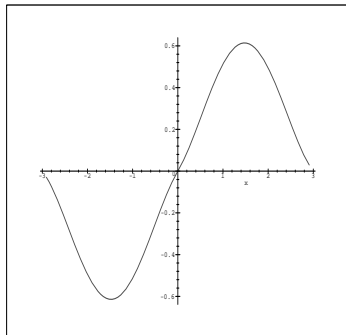


Fig.5. Odd eigenfunctions of H_1

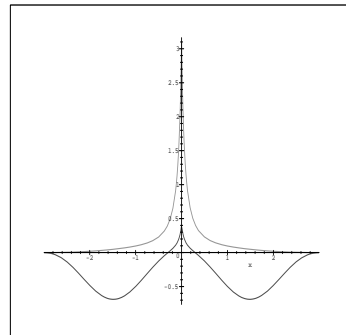


Fig.6. Even eigenfunctions of H_2

2.4 The lowest Eigenvalue

To estimate the lowest eigenvalue E_0 in eq. (2.1) more precisely, we try to find an expansion of E_0 in terms of β . The expansion is around the small quantities E and $1 - \cos^2 \beta$. We start by expanding the square roots in eq. (2.1) and obtain:

$$0 = 2 \cos \beta F([-1/2 - E/6, 1 + E/6], [3/2], \cos^2 \beta) \Gamma(2 + E/6) \Gamma(1/2 - E/6) \\ + F([1/2 + E/6, -1 - E/6], [1/2], \cos^2 \beta) \Gamma(-E/6) \Gamma(3/2 + E/6)$$

Now we carefully rewrite each single term as an expansion in the small quantities. The first term we consider is the hypergeometric function, which appears in the second term of eq. (2.1). We are looking for a power expansion in the deviation E of the parameters. This leads to:

$$F([1/2 + E/6, -1 - E/6], [1/2], x) = (1 - x) + \left[-\frac{x}{3} + \frac{(1-x)}{6} \ln |x-1| \right] \cdot E + \mathcal{O}(E^2)$$

Inserting this into the equation above we obtain:

$$0 = 2 \cos \beta F([-1/2 - E/6, 1 + E/6], [3/2], \cos^2 \beta) \Gamma(2 + E/6) \Gamma(1/2 - E/6) \\ + \left[\sin^2 \beta + E \left(-\frac{\cos^2 \beta}{3} + \frac{\sin^2 \beta}{6} \ln \sin^2 \beta \right) \right] \Gamma(-E/6) \Gamma(3/2 + E/6)$$

To deal with the Γ -functions, we use the two expansions:

$$\Gamma(E/6) \Gamma(1/2 - E/6) = \frac{6}{E} \sqrt{\pi} + 2 \ln 2 \sqrt{\pi} + \mathcal{O}(E) \\ \Gamma(-E/6) \Gamma(1/2 + E/6) = -\frac{6}{E} \sqrt{\pi} + 2 \ln 2 \sqrt{\pi} + \mathcal{O}(E)$$

Since we are interested in the limit $E \rightarrow 0$, the dominant contribution comes from the first terms. Using this to further simplify the transcendental equation we obtain:

$$0 = 2 \cos \beta F([-1/2 - E/6, 1 + E/6], [3/2], \cos^2 \beta) (1 + E/6) \\ - \left[\sin^2 \beta + E \left(-\frac{\cos^2 \beta}{3} + \frac{\sin^2 \beta}{6} \ln \sin^2 \beta \right) \right] (3 + E)/E$$

The last term to consider is the remaining hypergeometric function. In principle we should treat this case completely analogous to the case before. But there is a problem in doing that. It is not obvious what the closed expression for the series in x to the order $\mathcal{O}(E)$ would be. All that can be done is to write down this expansion. By nice circumstances it is possible to find the value of this series at $x = 1$, i.e. the sum of all its coefficients:

$$F([-1/2 - E/6, 1 + E/6], [3/2], x) = \underbrace{F([-1/2, 1], [3/2], x)}_{\text{elementary fct.}} \\ - \underbrace{\left(\frac{1}{6} x + \frac{1}{60} x^2 + \frac{1}{180} x^3 + \frac{61}{22680} x^4 \dots \right)}_{\frac{1}{12} + \frac{1}{6} \ln 2 \text{ if } x = 1} \cdot E + \mathcal{O}(E^2)$$

But now our limit helps again. Since the term, not known precisely, contributes to the third power in E , we can neglect its effect in the expansion to lowest order in E . In the equation obtained after this manipulations E and β decouple completely and we are able to solve for E_0 . The resulting expression can be expanded to extract the right power of β to lowest order. This leads to:

$$E_0(\beta) = \frac{3}{2}\beta^2 + \mathcal{O}(\beta^4) \quad (2.2)$$

Some short remarks. The question arises: is it sensible to take the limit $\beta \rightarrow 0$ while keeping α fixed at the same time, i.e. is $E_0 = 0$ a solution in this limit? From the formula above it seems to be true. But in fact this limit is not a continuous one. The resolution of this puzzle is contained in what we had discussed before. If one looks at the potentials in Fig. 3 and Fig. 4, than all mystery disappears. In the limit $\beta \rightarrow 0$ the barrier of the potential at $x = 0$ grows to infinity. Thus a wave function localized at $x = 0$ has to vanish discontinuously. Is β small but finite the corresponding wave function, the ‘almost zero mode’, exists.

Last but not least we want to discuss the precision of our approximation, which can be determined by comparing the exact with the estimated lowest eigenvalues $E_0(\beta)$. This is done in Fig. 7. The upper curve corresponds to the asymptotic formula. The deviation in the interesting limit (E and β small) is around 1 percent, which is quite good.

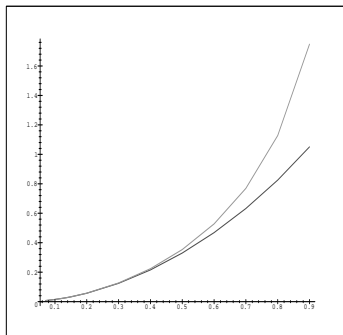


Fig.7. The precision of the asympt. formula

We conclude with the substitution of the technical α and β parameters defined in section 1 by the physical parameters and undo the scaling of E as introduced at the beginning of subsection 2.1. Then we obtain the power law for the lowest eigenvalue in the form:

$$m_0^2 = \frac{3}{2} |\Lambda|^2 x_0^2 + \mathcal{O}(|\Lambda|^3)$$

From the pure scaling argument of subsection 2.1 which applies to the spectrum of the selfsimilar cousin of the class of Hamiltonians considered here one would expect a linear dependence of the cosmological constant. But now we find that the true dependence is a quadratic one.

Acknowledgements:

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