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Integrable systems with BMS_3 Poisson structure and the dynamics of locally flat spacetimes

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Abstract: We construct a hierarchy of integrable systems whose Poisson structure corresponds to the BMS_3 algebra, and then discuss its description in terms of the Riemannian geometry of locally flat spacetimes in three dimensions.

The analysis is performed in terms of two-dimensional gauge fields for $isl(2,\mathbb{R})$, being isomorphic to the Poincaré algebra in 3D. Although the algebra is not semisimple, the formulation can still be carried out *à la* Drinfeld-Sokolov because it admits a nondegenerate invariant bilinear metric. The hierarchy turns out to be bi-Hamiltonian, labeled by a nonnegative integer k, and defined through a suitable generalization of the Gelfand-Dikii polynomials. The symmetries of the hierarchy are explicitly found. For $k \geq 1$, the corresponding conserved charges span an infinite-dimensional Abelian algebra without central extensions, so that they are in involution; while in the case of $k = 0$, they generate the BMS_3 algebra. In the special case of $k = 1$, by virtue of a suitable field redefinition and time scaling, the field equations are shown to be equivalent to the ones of a specific type of the Hirota-Satsuma coupled KdV systems. For $k \geq 1$, the hierarchy also includes the so-called perturbed KdV equations as a particular case. A wide class of analytic solutions is also explicitly constructed for a generic value of k.

Remarkably, the dynamics can be fully geometrized so as to describe the evolution of spacelike surfaces embedded in locally flat spacetimes. Indeed, General Relativity in 3D can

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be endowed with a suitable set of boundary conditions, so that the Einstein equations precisely reduce to the ones of the hierarchy aforementioned. The symmetries of the integrable systems then arise as diffeomorphisms that preserve the asymptotic form of the spacetime metric, and therefore, they become Noetherian. The infinite set of conserved charges is then recovered from the corresponding surface integrals in the canonical approach.

Keywords: Conformal and W Symmetry, Space-Time Symmetries, Integrable Hierarchies, Gauge-gravity correspondence

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Contents

1 Introduction

It has been recently shown that General Relativity on AdS_3 can be endowed with a suitable set of boundary conditions, so that the Einstein equations on the reduced phase space precisely reduce to the ones of the KdV hierarchy in two spacetime dimensions [1]. In turn, the dynamics of the KdV hierarchy can then be understood in terms of the geometry of spacelike surfaces that evolve within a three-dimensional spacetime of negative constant curvature. In the absence of cosmological constant, it is then natural to wonder whether General Relativity in three spacetime dimensions might also be linked with some sort of integrable systems. Thus, one of the main purposes of this work is constructing a new hierarchy of integrable systems, whose geometrical interpretation in terms of locally flat spacetimes in 3D can be precisely established.

Let us then begin considering a Galilean conformal algebra which can be understood as the nonrelativistic limit of the algebra of the conformal group (see, e.g. [2–4]). In two spacetime dimensions, the Galilean conformal algebra (GCA_2) is then obtained from a suitable Inönü-Wigner contraction of two copies of the Virasoro algebra, where the parameter of the contraction is the speed of light ($c \to \infty$). Remarkably, GCA_2 is isomorphic to the Bondi-Metzner-Sachs algebra in three spacetime dimensions (BMS_3), which spans the diffeomorphisms that preserve the asymptotic form of the metric for General Relativity [5–7], possibly endowed with parity-odd terms [8]. The Poisson bracket algebra is given by

$$\begin{aligned} i\left\{\mathcal{J}_m,\mathcal{J}_n\right\} &= (m-n)\,\mathcal{J}_{m+n} + 2\pi c_{\mathcal{J}}\, m^3\delta_{m+n,0}\,, \\ i\left\{\mathcal{J}_m,\mathcal{P}_n\right\} &= (m-n)\,\mathcal{P}_{m+n} + 2\pi c_{\mathcal{P}}\, m^3\delta_{m+n,0}\,, \\ i\left\{\mathcal{P}_m,\mathcal{P}_n\right\} &= 0\,, \end{aligned} \tag{1.1}$$

with m and n arbitrary integers. The central extensions $c_{\mathcal{P}}$ and $c_{\mathcal{J}}$ are related to the Newton constant and to the coupling of the parity-odd terms, respectively. The BMS_3 algebra (1.1) is then described by the semi-direct sum of a Virasoro algebra, spanned by $\mathcal{J}_m$, with the Abelian ideal generated by $\mathcal{P}_m$. Note that the Poincaré algebra in three dimensions is manifestly seen as the subalgebra of (1.1) spanned by the subset of generators with m, $n = -1,0,1$ (after suitable trivial shifts of $\mathcal{J}_0$ and $\mathcal{P}_0$). Induced and coadjoint representations of the BMS_3 algebra have been considered in [9, 10].

The BMS_3 algebra also naturally arises in diverse contexts of physical interest. For instance, it describes the worldsheet symmetries of the bosonic sector in the tensionless limit of closed string theory [11–18], and it is then expected to be relevant for the description of interacting higher spin fields [19–23] (for a review, see e.g. [24]). In two spacetime dimensions, the BMS_3 algebra also describes the symmetries of a "flat analog" of Liouville theory [25, 26], while on Minkowski spacetime in 3D, in the absence of central extensions, the algebra (1.1) manifests itself through nonlocal symmetries of a free massless Klein-Gordon field [27]. The algebra (1.1) also plays a key role in nonrelativistic and flat holography [3, 7, 28–40]. Furthermore, by virtue of a Sugawara-like construction, their generators have been recently seen to emerge as composite operators of the affine currents that describe the asymptotic symmetries of the "soft hairy" boundary conditions in [41–43]. In the sense of [44–46], this might shed light in the resolution of the information loss paradox [47]. Similar results also hold for the analysis of the near horizon symmetries of non-extremal black holes, so that (twisted) warped conformal algebras also lead to BMS_3 [48–50]. The minimal supersymmetric extension of the BMS_3 algebra has been shown to generate the asymptotic symmetries of $\mathcal{N} = 1$ supergravity in 3D [51–53], for a suitable set of boundary conditions [8, 54], and it is then isomorphic to the minimal

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supersymmetric extension of GCA_2 [55, 56] (see also [57]). Supersymmetric extensions of BMS_3 with $\mathcal{N} > 1$ have also been discussed along diverse lines in [14, 15, 56, 58–62]. Interestingly, nonlinear extensions of the BMS_3 algebra are known to exist when higher spin bosonic or fermionic generators are included, see [63–67] and [68, 69] respectively. Further generalizations of the BMS_3 algebra can also be found from suitable expansions of the Virasoro algebra [70].

One of the main purposes of our work is exploring whether the BMS_3 algebra could be further linked with some sort of integrable systems. There are some hints that can be borrowed from CFTs in two dimensions that suggest to look towards this direction. Indeed, it is known that CFTs in 2D admit an infinite set of conserved charges that commute between themselves, which can be constructed out from suitable nonlinear combinations of the generators of the Virasoro algebra and their derivatives (see e.g. [71]). Remarkably, these composite operators turn out to be precisely the conserved charges of the KdV equation, which also correspond to the Hamiltonians of the KdV hierarchy. Therefore, since the BMS_3 algebra can be seen as a limiting case of the conformal algebra in 2D, it is natural to wonder about the possibility of performing a similar construction in that limit. Specifically, one would like to know about the existence of an infinite number of commuting conserved charges that could be suitably recovered from the BMS_3 generators, as well as its possible relation with some integrable system, or even to an entire hierarchy of them. Here we show that this is certainly the case.

Furthermore, and noteworthy, the dynamics of this class of integrable systems can be equivalently understood in terms of Riemannian geometry. Indeed, following similar strategy as the one in [1], here we show that General Relativity without cosmological constant in 3D can be endowed with a suitable set of boundary conditions, so that in the reduced phase space, the Einstein equations precisely agree with the ones of the hierarchy aforementioned. In other words, the dynamics of our class of integrable system can be fully geometrized, since it can be seen to emerge from the evolution of spacelike surfaces embedded in locally flat spacetimes. As a remarkable consequence, in the geometric picture, the symmetries of the integrable systems correspond to diffeomorphisms that maintain the asymptotic form of the spacetime metric, so that they manifestly become Noetherian. Hence, the infinite set of conserved charges can be readily obtained from the surface integrals associated to the asymptotic symmetries in the canonical approach.

In the next section we explicitly construct dynamical (Hamiltonian) systems whose Poisson structure corresponds to the BMS_3 algebra. In order to analyze their symmetries and conserved charges, in section 3 we show how the Drinfeld-Sokolov formulation can be adapted to our case, through the use of suitable flat connections for $isl(2,\mathbb{R})$. Section 4 is devoted to a thorough construction and the analysis of an entire bi-Hamiltonian hierarchy of integrable systems with BMS_3 Poisson structure, labeled by a nonnegative integer k. We start with a very simple dynamical system ($k = 0$) from which the bi-Hamiltonian structure can be naturally unveiled. The case of $k = 1$ is described in section 4.2, where the Abelian infinite-dimensional symmetries and conserved charges are explicitly identified in terms of a suitable generalization of the Gelfand-Dikii polynomials (see also appendices A and B). The equivalence between our field equations and the ones of the Hirota-Satsuma

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coupled KdV system of type ix, is shown to hold by virtue of a suitable field redefinition and time scaling in section 4.2.4. The hierarchy of integrable systems with BMS_3 Poisson structure is then explicitly discussed in section 4.3, where it is shown that the so-called "perturbed KdV equations" are included as a particular case. Section 4.4 is devoted to the construction of a wide interesting class of analytic solutions for a generic value of the label of the hierarchy k. In section 5, we show how the dynamics of the hierarchy of integrable systems can be fully geometrized in terms of locally flat three-dimensional spacetimes. The deep link with General Relativity in 3D is explicitly addressed. We conclude with some remarks about possible extensions of our results in section 6. Appendices A, B, C and D are devoted to some technical remarks.

2 Dynamical systems with BMS_3 Poisson structure

In order to construct dynamical systems whose Poisson structure is described by the BMS_3 algebra, let us consider two independent dynamical fields, $\mathcal{J}=\mathcal{J}(t,\phi)$ and $\mathcal{P}=\mathcal{P}(t,\phi)$, being defined on a cylinder whose coordinates range as $0\leq\phi<2\pi$, and $-\infty<t<\infty$. The Poisson structure we look for can then be defined in terms of the following operator

$$\mathcal{D}^{(2)}\equiv\begin{pmatrix}\mathcal{D}^{(\mathcal{J})} & \mathcal{D}^{(\mathcal{P})}\\ \mathcal{D}^{(\mathcal{P})} & 0\end{pmatrix}\,, \tag{2.1}$$

where $\mathcal{D}^{(\mathcal{J})}$ and $\mathcal{D}^{(\mathcal{P})}$ stand for Schwarzian derivatives, given by

$$\begin{aligned}\mathcal{D}^{(\mathcal{J})}&=2\mathcal{J}\partial_\phi+\partial_\phi\mathcal{J}-c_{\mathcal{J}}\partial_\phi^3\,,\\ \mathcal{D}^{(\mathcal{P})}&=2\mathcal{P}\partial_\phi+\partial_\phi\mathcal{P}-c_{\mathcal{P}}\partial_\phi^3\,,\end{aligned} \tag{2.2}$$

with arbitrary constants $c_{\mathcal{J}}$ and $c_{\mathcal{P}}$.

The operator $\mathcal{D}^{(2)}$ in eq. (2.1) then allows to write the Poisson brackets of two arbitrary functionals of the dynamical fields, $F=F[\mathcal{J},\mathcal{P}]$ and $G=G[\mathcal{J},\mathcal{P}]$, according to

$$\{F,G\}\equiv\int d\phi\begin{pmatrix}\frac{\delta F}{\delta\mathcal{J}} & \frac{\delta F}{\delta\mathcal{P}}\end{pmatrix}\begin{pmatrix}\mathcal{D}^{(\mathcal{J})} & \mathcal{D}^{(\mathcal{P})}\\ \mathcal{D}^{(\mathcal{P})} & 0\end{pmatrix}\begin{pmatrix}\frac{\delta G}{\delta\mathcal{J}}\\ \frac{\delta G}{\delta\mathcal{P}}\end{pmatrix}. \tag{2.3}$$

Therefore, the brackets of the dynamical fields read

$$\begin{aligned}\{\mathcal{J}(\phi),\mathcal{J}(\bar\phi)\}&=\mathcal{D}^{(\mathcal{J})}\delta\left(\phi-\bar\phi\right)\,,\\ \{\mathcal{J}(\phi),\mathcal{P}(\bar\phi)\}&=\mathcal{D}^{(\mathcal{P})}\delta\left(\phi-\bar\phi\right)\,,\\ \{\mathcal{P}(\phi),\mathcal{P}(\bar\phi)\}&=0\,,\end{aligned} \tag{2.4}$$

so that expanding in Fourier modes as $X=\frac{1}{2\pi}\sum_n X_n e^{-in\phi}$, the algebra of the Poisson brackets in (2.4) precisely reduces to the BMS_3 algebra in eq. (1.1).

The field equations for the class of dynamical systems we were searching for can then be readily defined as follows

$$\begin{pmatrix}\dot{\mathcal{J}}\\ \dot{\mathcal{P}}\end{pmatrix}=\mathcal{D}^{(2)}\begin{pmatrix}\mu_{\mathcal{J}}\\ \mu_{\mathcal{P}}\end{pmatrix}\,, \tag{2.5}$$

where dot denotes the derivative in time, while $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ stand for arbitrary functions of the dynamical fields and their derivatives along ϕ. When these functions are defined in terms of the variation of a functional $H = H[\mathcal{J}, \mathcal{P}]$, so that

$$\mu_{\mathcal{J}} = \frac{\delta H}{\delta \mathcal{J}}\,, \qquad \mu_{\mathcal{P}} = \frac{\delta H}{\delta \mathcal{P}}\,, \tag{2.6}$$

the dynamical system is Hamiltonian; and hence, by virtue of (2.3), the field equations can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}} \\ \mu_{\mathcal{P}} \end{pmatrix} = \begin{pmatrix} \{\mathcal{J}, H\} \\ \{\mathcal{P}, H\} \end{pmatrix} . \tag{2.7}$$

Note that unwrapping the angular coordinate to range as $-\infty < \phi < \infty$, allows to extend this class of dynamical systems to $\mathbb{R}^2$, provided that the fall-off of the dynamical fields $\mathcal{J}$, $\mathcal{P}$ is sufficiently fast in order to get rid of boundary terms. Hereafter, for the sake of simplicity, we will assume that the dynamical systems are defined on a cylinder, with a single exception for an interesting particular solution that is described in section 4.4.2.

3 Zero-curvature formulation

In order to study the properties of the dynamical systems with BMS$_3$ Poisson structure that evolve according to eq. (2.5), including their symmetries and the corresponding conserved charges, it turns out to be useful to reformulate them in terms of a flat connection for a suitable Lie algebra (see e.g. [72, 73]). In the standard approach of Drinfeld and Sokolov [74], the Lie algebra is assumed to be semisimple. Here we slightly extend this approach in a sense explained right below.

For our purposes, the relevant Lie algebra to be considered corresponds to $isl(2, \mathbb{R})$, which is isomorphic to the Poincaré algebra in 3D. Their commutation relations can then be written as

$$[J_a, J_b] = \epsilon_{abc} J^c\,, \qquad [J_a, P_b] = \epsilon_{abc} P^c\,, \qquad [P_a, P_b] = 0\,, \tag{3.1}$$

where J_a stand for the generators of $sl(2, \mathbb{R}) \simeq so(2, 1)$. It is worth emphasizing that the algebra (3.1) is not semisimple; but nonetheless, it admits an invariant bilinear metric whose nonvanishing components read

$$\langle J_a, J_b \rangle = c_{\mathcal{J}} \eta_{ab}\,, \qquad \langle J_a, P_b \rangle = c_{\mathcal{P}} \eta_{ab}\,, \tag{3.2}$$

where $c_{\mathcal{J}}$ and $c_{\mathcal{P}}$ are arbitrary constants. Note that the invariant bilinear metric is nondegenerate provided that $c_{\mathcal{P}} \neq 0$, which will be assumed afterwards.[1]

Hence, by virtue of (3.2), the analysis of the class of dynamical systems defined in section 2 can still be performed *à la* Drinfeld-Sokolov, provided that the field equations are able to be reproduced in terms of a flat connection for $isl(2, \mathbb{R})$.

[1]We choose the orientation according to $\epsilon_{012} = 1$, while the Minkowski metric η_{ab} is assumed to be non-diagonal, whose non-vanishing components are given by $\eta_{01} = \eta_{10} = \eta_{22} = 1$.

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We then propose that the spacelike component of the $isl(2,\mathbb{R})$-valued gauge field $a = a_\mu dx^\mu$ is given by

$$a_\phi = J_1 + \frac{1}{c_\mathcal{P}}\left[\left(\mathcal{J} - \frac{c_\mathcal{J}}{c_\mathcal{P}}\mathcal{P}\right) P_0 + \mathcal{P} J_0\right] , \tag{3.3}$$

with $\mathcal{J} = \mathcal{J}(t,\phi)$ and $\mathcal{P} = \mathcal{P}(t,\phi)$, while the timelike component reads

$$a_t = \Lambda\left(\mu_\mathcal{J}, \mu_\mathcal{P}\right) , \tag{3.4}$$

where

$$\begin{aligned}\Lambda\left(\mu_\mathcal{J},\mu_\mathcal{P}\right) =& \mu_\mathcal{P} P_1 + \mu_\mathcal{J} J_1 - \mu_\mathcal{P}{}' P_2 - \mu_\mathcal{J}{}' J_2 \\ &+ \left(\frac{1}{c_\mathcal{P}}\mathcal{P}\mu_\mathcal{J} - \mu_\mathcal{J}{}''\right) J_0 + \left[\frac{1}{c_\mathcal{P}}\left(\mathcal{J} - \frac{c_\mathcal{J}}{c_\mathcal{P}}\mathcal{P}\right)\mu_\mathcal{J} + \frac{1}{c_\mathcal{P}}\mathcal{P}\mu_\mathcal{P} - \mu_\mathcal{P}{}''\right] P_0 .\end{aligned} \tag{3.5}$$

Here $\mu_\mathcal{J}$ and $\mu_\mathcal{P}$ can be assumed to be given by some arbitrary functions of $\mathcal{J}$, $\mathcal{P}$, and their derivatives along ϕ (being denoted by a prime here and afterwards). Therefore, requiring the field strength for the gauge field defined in (3.3) and (3.4) to vanish, i.e.,

$$f = da + a^2 = 0 , \tag{3.6}$$

implies that the field equations for the dynamical system with BMS_3 Poisson structure in (2.5) hold. It is worth mentioning that (a_t, a_ϕ) can then be interpreted as the components of an $isl(2,\mathbb{R})$-valued Lax pair.

3.1 Symmetries and conserved charges

One of the advantages of formulating the field equations in terms of a flat connection, is that the symmetries of the dynamical system in (2.5) turn out to correspond to gauge transformations, $\delta_\lambda a = d\lambda + [a,\lambda]$, that preserve the form of the gauge field defined through (3.3) and (3.4).

Hence, requiring the form of the spacelike component of the connection a_ϕ in (3.3) to be preserved under gauge transformations, restricts the Lie-algebra-valued parameter to be of the form

$$\lambda = \Lambda\left(\varepsilon_\mathcal{J}, \varepsilon_\mathcal{P}\right) , \tag{3.7}$$

where Λ is precisely given by eq. (3.5), but now depends on two arbitrary functions $\varepsilon_\mathcal{J} = \varepsilon_\mathcal{J}(t,\phi)$ and $\varepsilon_\mathcal{P} = \varepsilon_\mathcal{P}(t,\phi)$, while the dynamical fields have to transform according to

$$\begin{pmatrix}\delta\mathcal{J}\\ \delta\mathcal{P}\end{pmatrix} = \mathcal{D}^{(2)}\begin{pmatrix}\varepsilon_\mathcal{J}\\ \varepsilon_\mathcal{P}\end{pmatrix} . \tag{3.8}$$

Analogously, preserving the timelike component of the gauge field a_t in (3.4), implies that the transformation law of the functions $\mu_\mathcal{J}$ and $\mu_\mathcal{P}$ is given by

$$\delta\mu_\mathcal{J} = \dot{\varepsilon}_\mathcal{J} + \varepsilon_\mathcal{J}\mu_\mathcal{J}{}' - \mu_\mathcal{J}\varepsilon_\mathcal{J}{}' , \tag{3.9}$$

$$\delta\mu_\mathcal{P} = \dot{\varepsilon}_\mathcal{P} + \varepsilon_\mathcal{J}\mu_\mathcal{P}{}' + \varepsilon_\mathcal{P}\mu_\mathcal{J}{}' - \mu_\mathcal{J}\varepsilon_\mathcal{P}{}' - \mu_\mathcal{P}\varepsilon_\mathcal{J}{}' . \tag{3.10}$$

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However, $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ generically depend on the dynamical fields and their spatial derivatives, which means that eqs. (3.9), (3.10) actually become a consistency condition to be fulfilled by the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries of the dynamical system.

In the case of Hamiltonian systems, $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ are determined by the corresponding functional variations of the Hamiltonian as in (2.6), and consequently, the consistency condition for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$, that arises from (3.9), (3.10), can be compactly written as

$$\begin{pmatrix} \dot{\varepsilon}_{\mathcal{J}}\,(t,\phi) \\ \dot{\varepsilon}_{\mathcal{P}}\,(t,\phi) \end{pmatrix} = -\begin{pmatrix} \frac{\delta}{\delta\mathcal{J}(t,\phi)} \\ \frac{\delta}{\delta\mathcal{P}(t,\phi)} \end{pmatrix} \int d\varphi \left(\mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}} \\ \mu_{\mathcal{P}} \end{pmatrix} \right)^{T} \begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} . \tag{3.11}$$

In sum, the functions that parametrize the symmetries of the Hamiltonian system with BMS_3 Poisson structure must fulfill the consistency condition in (3.11), which for an arbitrary choice of Hamiltonian, implies that $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ generically acquire an explicit dependence on the dynamical fields and their spatial derivatives.

The variation of the canonical generators associated to the symmetries spanned by $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ can then be readily found by virtue of eqs. (3.2), (3.3) and (3.7), which reduces to the following simple expression

$$\delta Q\left[\varepsilon_{\mathcal{J}},\varepsilon_{\mathcal{P}}\right] = -\int d\phi \left\langle \lambda\delta a_{\phi}\right\rangle = -\int d\phi \left(\varepsilon_{\mathcal{J}}\delta\mathcal{J} + \varepsilon_{\mathcal{P}}\delta\mathcal{P}\right) . \tag{3.12}$$

As a cross-check, it is simple to verify that the variation of the canonical generators is conserved ($\delta\dot{Q} = 0$) provided that the consistency condition for the symmetry parameters in (3.11) is satisfied.

It is also worth emphasizing that the integrability conditions of (3.12) require that the allowed parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ must correspond to the variation of a functional, since $\varepsilon_{\mathcal{J}} = -\frac{\delta Q}{\delta\mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = -\frac{\delta Q}{\delta\mathcal{P}}$.

Nevertheless, it must be highlighted that finding the explicit form of the conserved charges Q is not so simple, because it amounts to know the general solution of the consistency condition for the parameters in (3.11). Indeed, although the consistency condition is linear in the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$, the generic solution manifestly depends on the dynamical fields and their spatial derivatives that fulfill a nonlinear field equation. Thus, for a generic choice of the Hamiltonian, solving eq. (3.11) is actually an extremely difficult task.

However, for a generic Hamiltonian that is independent of the coordinates, the conserved charges associated to translations along space and time can be directly constructed. In fact, for a flat connection, diffeomorphisms spanned by $\xi = \xi^{\mu}\partial_{\mu}$ are equivalent to gauge transformations generated by $\lambda = -\xi^{\mu}a_{\mu}$, since $\mathcal{L}_{\xi}a = d\lambda + [a,\lambda]$. Therefore, for $\xi = \partial_{\phi}$, the linear momentum on the cylinder readily integrates as

$$Q[\partial_{\phi}] = Q[-1,0] = \int d\phi\mathcal{J} . \tag{3.13}$$

Analogously, the variation of the energy is obtained for $\xi = \partial_t$, so that

$$\delta Q[\partial_t] = \delta Q[-\mu_{\mathcal{J}},-\mu_{\mathcal{P}}] = \int d\phi \left(\mu_{\mathcal{J}}\delta\mathcal{J} + \mu_{\mathcal{P}}\delta\mathcal{P}\right) , \tag{3.14}$$

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which by virtue of (2.6), integrates as expected, i.e.,

$$Q[\partial_t] = H\,. \tag{3.15}$$

Note that, generically, there might be additional nontrivial solutions of eq. (3.11) that would lead to further conserved charges.

As a closing remark of this section, it must be emphasized that in order to construct an integrable system with the BMS$_3$ Poisson structure, one should at least specify the precise form of the Hamiltonian, so that the general solution of the consistency condition for the parameters in (3.11) could be obtained. Explicit examples of integrable systems of this sort that actually belong to an infinite hierarchy of them are discussed in the next section.

4 Hierarchy of integrable systems with BMS$_3$ Poisson structure

In this section we introduce a bi-Hamiltonian hierarchy of integrable systems with BMS$_3$ Poisson structure in a constructive way. We start from an extremely simple case, which nonetheless, possesses the key ingredients in order to propose a precise nontrivial integrable system of this type, that can be extended to an entire hierarchy labeled by a nonnegative integer k. The contact with some known results in the literature for certain particular cases is also addressed. Furthermore, a wide class of analytic solutions are explicitly constructed for an arbitrary representative of the hierarchy, including a couple of simple and interesting particular examples.

4.1 Warming up with a simple dynamical system ($k = 0$)

Let us begin with one of the simplest possible examples of a dynamical system with BMS$_3$ Poisson structure. The field equations can be obtained from (2.7) with $\mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(0)}$ and $\mu_{\mathcal{P}} = \mu_{\mathcal{P}}^{(0)}$ constants, given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(0)} \\ \mu_{\mathcal{P}}^{(0)} \end{pmatrix} = \begin{pmatrix} 1 \\ a \end{pmatrix}\,, \tag{4.1}$$

so that, according to (2.6), the Hamiltonian is given by $H = H^{(0)}$, with

$$H^{(0)} = \int d\phi \left(\mathcal{J} + a\mathcal{P}\right)\,. \tag{4.2}$$

The field equations then explicitly read

$$\begin{aligned} \dot{\mathcal{J}} &= \mathcal{J}' + a\mathcal{P}'\,, \\ \dot{\mathcal{P}} &= \mathcal{P}'\,, \end{aligned} \tag{4.3}$$

which are trivially integrable. Indeed, the general solution of (4.3) on the cylinder is described by left movers and it can be expressed in terms of periodic functions $\mathcal{M} = \mathcal{M}(t+\phi)$ and $\mathcal{N} = \mathcal{N}(t+\phi)$, so that it reads

$$\begin{aligned} \mathcal{P} &= \mathcal{M}\,, \\ \mathcal{J} &= \mathcal{N} + at\mathcal{M}'\,. \end{aligned} \tag{4.4}$$

Besides, since the field equations are very simple in this case, the consistency condition for the parameters of their symmetries in (3.11) becomes independent of the dynamical fields and their spatial derivatives, which explicitly reduces to

$$\begin{aligned} \dot{\varepsilon}_{\mathcal{J}} &= \varepsilon_{\mathcal{J}}{}' \,, \\ \dot{\varepsilon}_{\mathcal{P}} &= \varepsilon_{\mathcal{P}}{}' + a\varepsilon_{\mathcal{J}}{}' \,. \end{aligned} \tag{4.5}$$

Note that (4.5) coincides with the field equations in (4.3) for $\varepsilon_{\mathcal{J}} = \mathcal{P}$ and $\varepsilon_{\mathcal{P}} = \mathcal{J}$, and hence, if one assumes that the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ depend only on the coordinates t, ϕ, and not on the dynamical fields, the general solution of the consistency conditions for the parameters is also given by chiral (left mover) functions as in (4.4). Therefore, the variation of the canonical generators in (3.12) readily integrates as

$$Q\left[\varepsilon_{\mathcal{J}}, \varepsilon_{\mathcal{P}}\right] = -\int d\phi \left(\varepsilon_{\mathcal{J}}\mathcal{J} + \varepsilon_{\mathcal{P}}\mathcal{P}\right) \,. \tag{4.6}$$

The algebra of the conserved charges (4.6) can then be directly obtained from their corresponding Poisson brackets. As a shortcut, by virtue of

$$\left\{Q\left[\varepsilon_1\right], Q\left[\varepsilon_2\right]\right\} = \delta_{\varepsilon_2} Q\left[\varepsilon_1\right] \,, \tag{4.7}$$

the algebra can also be read from the transformation law of the fields in (3.8), and it is then found to be given precisely by the BMS_3 algebra in (2.4), which once expanded in modes reads as in eq. (1.1).

It is worth highlighting that this simple dynamical system actually turns out to be bi-Hamiltonian. This is so because the field equations can be expressed in terms of two different Poisson structures, so that (4.3) can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(0)} \\ \mu_{\mathcal{P}}^{(0)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} \,, \tag{4.8}$$

where $\mathcal{D}^{(2)}$ is the BMS_3 one in (2.1), while $\mathcal{D}^{(1)}$ stands for the "canonical" Poisson structure, defined through the following differential operator

$$\mathcal{D}^{(1)} \equiv \begin{pmatrix} 0 & \partial_\phi \\ \partial_\phi & 0 \end{pmatrix} \,. \tag{4.9}$$

In (4.8) the functions $\mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}}^{(1)}$ are then given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} = \begin{pmatrix} \mathcal{P} \\ \mathcal{J} + a\mathcal{P} \end{pmatrix} \,, \tag{4.10}$$

and thus, according to (2.6), the canonical Poisson structure (4.9) is associated to the following Hamiltonian

$$H^{(1)} = \int d\phi \left(\mathcal{J}\mathcal{P} + \frac{a}{2}\mathcal{P}^2\right) \,. \tag{4.11}$$

It is worth highlighting that the conserved charge $H^{(1)}$ can also be obtained from (3.12) provided that the parameters of the symmetries are given by $\varepsilon_{\mathcal{J}} = \mu_{\mathcal{J}}^{(1)} = \mathcal{P}$ and $\varepsilon_{\mathcal{P}} = \mu_{\mathcal{P}}^{(1)} = \mathcal{J} + a\mathcal{P}$. Indeed, if the parameters are allowed to depend only on the fields and their derivatives, but not explicitly on the coordinates t, ϕ, one is able to construct an infinite set of independent commuting conserved charges of this sort. This is shown in section 4.3.

In sum, the analysis of this extremely simple dynamical system with BMS_3 Poisson structure $\mathcal{D}^{(2)}$, being trivially integrable, allows to unveil a naturally related Poisson structure given by $\mathcal{D}^{(1)}$. The presence of both Poisson structures turns out to be the key in order to proceed with the construction of nontrivial integrable systems as well as an entire hierarchy associated to them. This can be seen as follows. One begins verifying that both Poisson structures are "compatible" in the sense that any linear combination of $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$ also defines a Poisson structure. It is then enough proving that the operator $\mathcal{D}^{(3)} = \mathcal{D}^{(2)} + \mathcal{D}^{(1)}$ also defines a Poisson structure (see e.g. [75]). This is so because the Poisson bracket constructed out from $\mathcal{D}^{(3)}$ is clearly antisymmetric, and furthermore, since $\mathcal{D}^{(3)}$ also fulfills the BMS_3 algebra, but just being shifted by the zero modes of $\mathcal{P}_0 \to \mathcal{P}_0 + \pi$, the Jacobi identity also holds. Besides, by virtue of the fact that our Poisson structure $\mathcal{D}^{(1)}$ is nondegenerate, the hypotheses of a strong theorem for bi-Hamiltonian systems in [76], and further elaborated in [75], are satisfied, which guarantees the existence of the type of hierarchy of integrable systems that we are searching for.

Furthermore, and remarkably, the simple dynamical system described in this section can be seen to be equivalent to the Einstein equations for the reduced phase space that is obtained from a suitable set of boundary conditions for General Relativity in three spacetime dimensions, including its extension with purely geometrical parity-odd terms in the action. This is discussed in section 5. It is also worth pointing out that our field equations (4.4), in the case of $c_{\mathcal{J}} = a = 0$, can be interpreted as the ones of a compressible Euler fluid [77].

In the next subsection we carry out the explicit construction and the analysis of a simple, but nontrivial, integrable system with BMS_3 Poisson structure.

4.2 Integrable bi-Hamiltonian system ($k = 1$)

The first nontrivial integrable system of our hierarchy is obtained from (2.7) with $\mu_{\mathcal{J}} = \mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}} = \mu_{\mathcal{P}}^{(1)}$, where $\mu_{\mathcal{J}}^{(1)}$ and $\mu_{\mathcal{P}}^{(1)}$ are given by eq. (4.10), so that the Hamiltonian corresponds to $H^{(1)}$ in (4.11). The field equations are then explicitly given by

$$
\begin{aligned}
\dot{\mathcal{J}} &= 3\mathcal{J}'\mathcal{P} + 3\mathcal{J}\mathcal{P}' - c_{\mathcal{P}}\mathcal{J}''' - c_{\mathcal{J}}\mathcal{P}''' + a\left(3\mathcal{P}'\mathcal{P} - c_{\mathcal{P}}\mathcal{P}'''\right) , \\
\dot{\mathcal{P}} &= 3\mathcal{P}'\mathcal{P} - c_{\mathcal{P}}\mathcal{P}''' \, .
\end{aligned}
\tag{4.12}
$$

Note that $\mathcal{P}$ evolves according to the KdV equation,[2] while the remaining equation is linear in $\mathcal{J}$, with an inhomogeneous source term that is entirely determined by $\mathcal{P}$ and their spatial derivatives.

[2]This is a direct consequence of the choice of $\mu_{\mathcal{J}}$ and the fact that the off-diagonal terms in the Poisson structure $\mathcal{D}^{(2)}$ exclusively depend on $\mathcal{D}^{(\mathcal{P})}$. Besides, a common practice in the literature is rescaling the field and the coordinates so that the KdV equation does not depend on $c_{\mathcal{P}}$ (see, e.g., [72, 75]). However, as explained in section 4.2.1, for our purposes, and for the sake of simplicity, keeping $c_{\mathcal{P}}$ explicitly in the field equations turns out to be very useful and convenient.

The field equations in (4.12) can also be seen to arise from a bi-Hamiltonian system with the same BMS$_3$ and canonical Poisson structures given by (2.1) and (4.9), respectively. Indeed, they can be written as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(1)} \\ \mu_{\mathcal{P}}^{(1)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(2)} \\ \mu_{\mathcal{P}}^{(2)} \end{pmatrix} , \tag{4.13}$$

where the functions $\mu_{\mathcal{J}}^{(2)}$ and $\mu_{\mathcal{P}}^{(2)}$ are given by

$$\begin{pmatrix} \mu_{\mathcal{J}}^{(2)} \\ \mu_{\mathcal{P}}^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}'' \\ 3\mathcal{J}\mathcal{P} - c_{\mathcal{P}}\mathcal{J}'' - c_{\mathcal{J}}\mathcal{P}'' + a\left(\frac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}''\right) \end{pmatrix} , \tag{4.14}$$

which can be obtained from the functional derivatives of a different Hamiltonian, as in (2.6), that reads

$$H^{(2)} = \int d\phi \left[\frac{3}{2}\mathcal{P}^2\mathcal{J} - c_{\mathcal{P}}\mathcal{P}''\mathcal{J} + \frac{c_{\mathcal{J}}}{2}\mathcal{P}'^2 + a\left(\frac{1}{2}\mathcal{P}^3 + \frac{c_{\mathcal{P}}}{2}\mathcal{P}'^2\right)\right] , \tag{4.15}$$

being clearly conserved.

4.2.1 Symmetries

As explained in section 3.1, in order to find the remaining conserved quantities, it is necessary to find the general solution of the consistency conditions in (3.11) for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries of the field equations. In this case ($k = 1$), the consistency conditions in (3.11), with $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ given by eq. (4.10), explicitly reduce to

$$\begin{aligned} \dot{\varepsilon}_{\mathcal{J}} &= 3\mathcal{P}\varepsilon_{\mathcal{J}}' - c_{\mathcal{P}}\varepsilon_{\mathcal{J}}''' , \\ \dot{\varepsilon}_{\mathcal{P}} &= 3\mathcal{J}\varepsilon_{\mathcal{J}}' + 3\mathcal{P}\varepsilon_{\mathcal{P}}' - c_{\mathcal{P}}\varepsilon_{\mathcal{P}}''' - c_{\mathcal{J}}\varepsilon_{\mathcal{J}}''' + a\left(3\mathcal{P}\varepsilon_{\mathcal{J}}' - c_{\mathcal{P}}\varepsilon_{\mathcal{J}}'''\right) . \end{aligned} \tag{4.16}$$

Note that the equations in (4.16) are linear for the parameters $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$. However, finding their solution is not that simple because their coefficients depend on $\mathcal{J}$ and $\mathcal{P}$, who evolve according to the nonlinear field equations in (4.12). Nevertheless, if one assumes that the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ depend only on the dynamical fields and their spatial derivatives, but not explicitly on the coordinates t, ϕ, and takes into account that the parameters must correspond to the variation of a functional, the theorem in [75, 76] then guarantees that the general solution of the consistency conditions (4.16) can be formally found. In our case, the explicit solution turns out to be given by a linear combination of two independent arrays, $K^{(j)}$ and $\tilde{K}^{(j)}$, that stand for a suitable generalization of the Gelfand-Dikii polynomials. The solution can then be written as

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = \sum_{j=0}^{\infty} \left[\eta_j K^{(j)} + \tilde{\eta}_j \tilde{K}^{(j)} \right] , \tag{4.17}$$

where η_j and $\tilde{\eta}_j$ are arbitrary constants, and both generalized polynomials $K^{(j)}$ and $\tilde{K}^{(j)}$ fulfill the same recursive relationship, given by

$$\mathcal{D}^{(1)} K^{(i+1)} = \mathcal{D}^{(2)} K^{(i)} . \tag{4.18}$$

If the initial seeds of the independent arrays are chosen as

$$K^{(0)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} , \qquad \tilde{K}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{4.19}$$

the recursion relation (4.18) then implies that the remaining ones are given by

$$K^{(n)} = \begin{pmatrix} 0 \\ R^{(n)} \end{pmatrix} , \qquad \tilde{K}^{(n)} = \begin{pmatrix} R^{(n)} \\ T^{(n)} \end{pmatrix} , \tag{4.20}$$

where $R^{(n)}$ stand for the standard Gelfand-Dikii polynomials, while $T^{(n)}$ correspond to a different set of polynomials that fulfill the following recursion relationships

$$\partial_\phi R^{(n+1)} = \mathcal{D}^{(\mathcal{P})} R^{(n)} , \tag{4.21}$$

$$\partial_\phi T^{(n+1)} = \mathcal{D}^{(\mathcal{P})} T^{(n)} + \mathcal{D}^{(\mathcal{J})} R^{(n)} . \tag{4.22}$$

Remarkably, both sets of polynomials can be obtained from the variation of two independent functionals, $H^{(n)}_{\text{KdV}}[\mathcal{P}]$ and $\tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]$, so that

$$R^{(n)} = \frac{\delta H^{(n)}_{\text{KdV}}[\mathcal{P}]}{\delta \mathcal{P}} = \frac{\delta \tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]}{\delta \mathcal{J}} , \tag{4.23}$$

$$T^{(n)} = \frac{\delta \tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]}{\delta \mathcal{P}} , \tag{4.24}$$

where $H^{(n)}_{\text{KdV}}$ stands for n-th conserved quantity of the KdV equation, while $\tilde{H}^{(n)}[\mathcal{P},\mathcal{J}]$ depends linearly on $\mathcal{J}$ and it is given by

$$\tilde{H}^{(n)}[\mathcal{P},\mathcal{J}] = c_{\mathcal{J}} \frac{\partial H^{(n)}_{\text{KdV}}[\mathcal{P}]}{\partial c_{\mathcal{P}}} + \int d\phi \mathcal{J} \frac{\delta H^{(n)}_{\text{KdV}}[\mathcal{P}]}{\delta \mathcal{P}} . \tag{4.25}$$

Therefore, the generalized polynomials can also be compactly defined as

$$K^{(n)} = \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}} \\ \frac{\delta}{\delta \mathcal{P}} \end{pmatrix} H^{(n)}_{\text{KdV}} , \qquad \tilde{K}^{(n)} = \begin{pmatrix} \frac{\delta}{\delta \mathcal{J}} \\ \frac{\delta}{\delta \mathcal{P}} \end{pmatrix} \tilde{H}^{(n)} . \tag{4.26}$$

An explicit list of the first six polynomials $R^{(n)}$ and $T^{(n)}$, with their corresponding functionals $H^{(n)}_{\text{KdV}}$ and $\tilde{H}^{(n)}$ is given in appendix A.

4.2.2 Conserved charges

Since the general form of the parameters that describe the symmetries of the field equations has been explicitly found to be given by (4.17), by virtue of the fact that the generalized polynomials come from the functional derivatives of suitable functionals as in (4.26), the variation of the canonical generators in (3.12) reduces to

$$\delta Q[\eta,\tilde{\eta}] = -\sum_{j=0}^{\infty} \int d\phi \left[\eta_j \frac{\delta H^{(j)}_{\text{KdV}}}{\delta \mathcal{P}} \delta \mathcal{P} + \tilde{\eta}_j \left(\frac{\delta \tilde{H}^{(j)}}{\delta \mathcal{P}} \delta \mathcal{P} + \frac{\delta \tilde{H}^{(j)}}{\delta \mathcal{J}} \delta \mathcal{J} \right) \right] , \tag{4.27}$$

which then readily integrates as

$$Q\left[\eta,\tilde{\eta}\right]=-\sum_{j=0}^{\infty}\left(\eta_j H_{\mathrm{KdV}}^{(j)}+\tilde{\eta}_j\tilde{H}^{(j)}\right). \tag{4.28}$$

Therefore, we have explicitly found two infinite independent towers of conserved quantities, being spanned by $H_{\mathrm{KdV}}^{(j)}$ and $\tilde{H}^{(j)}$, which by virtue of the recursion relation in (4.18), turn out to be in involution for both Poisson structures $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$, i.e.,

$$\left\{Q[\eta,\tilde{\eta}],Q[\zeta,\tilde{\zeta}]\right\}_{(2)}=\left\{Q[\eta,\tilde{\eta}],Q[\zeta,\tilde{\zeta}]\right\}_{(1)}=0\,, \tag{4.29}$$

where the subscripts for the brackets in (4.29) stand for the corresponding Poisson structures.[3]

Note that the pair of Hamiltonians that yield the same field equations in (4.13), given by (4.11) and (4.15), can then be written as

$$H^{(1)}=\tilde{H}^{(1)}+aH_{\mathrm{KdV}}^{(1)}\,,\qquad H^{(2)}=\tilde{H}^{(2)}+aH_{\mathrm{KdV}}^{(2)}\,. \tag{4.30}$$

In sum, as pointed out in the introduction, eq. (4.28) turns out to be an explicit realization of the infinite set of commuting conserved charges that is constructed out from precise nonlinear combinations of the generators of the BMS_3 algebra and their spatial derivatives. As discussed in section 4.3, this provides the basis to extend this integrable system to an entire hierarchy.

4.2.3 Remarks on some additional symmetries

The existence of additional symmetries, enlarging the set spanned by the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (4.17), is unveiled by relaxing our hypotheses, so that the parameters of the symmetries are now allowed to depend not just on the dynamical fields and their spatial derivatives, but also explicitly on the coordinates t, ϕ. Hence, apart from the infinite set of symmetries spanned by (3.8), with $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ given by (4.17), the field equations (4.12) can also be seen to be invariant under Galilean and anisotropic scale transformations:

Galilean transformations. They are parametrized by a single constant velocity parameter v_0, so that the coordinates and the fields transform according to

$$\bar{\phi}=\phi-v_0t\,,\qquad\qquad \bar{t}=t\,, \tag{4.31}$$

and

$$\bar{\mathcal{P}}=\mathcal{P}+\frac{v_0}{3}\,,\qquad\qquad \bar{\mathcal{J}}=\mathcal{J}-\frac{a}{3}v_0\,, \tag{4.32}$$

respectively.

For simplicity, if one chooses the Poisson structure $\mathcal{D}^{(1)}$, the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ that correspond to the infinitesimal Galilean transformations are then given by

$$\begin{aligned}\varepsilon_{\mathcal{J}}&=v_0t\mathcal{P}+\frac{\phi}{3}\,,\\ \varepsilon_{\mathcal{P}}&=v_0t\mathcal{J}-\frac{a\phi}{3}\,,\end{aligned} \tag{4.33}$$

[3]For an explicit proof of the involution of the conserved charges see appendix B.

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which manifestly fulfill their consistency conditions in (4.16). Equivalently, the parameters in (4.33) can be expressed in terms of the generalized polynomials as in (4.17), where now the coefficients η_j and $\tilde{\eta}_j$ acquire an explicit dependence on the coordinates t, ϕ, so that

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = v_0 t \tilde{K}^{(1)} + \frac{v_0 \phi}{3} \left(\tilde{K}^{(0)} - a K^{(0)} \right) . \tag{4.34}$$

Therefore, since $\varepsilon_{\mathcal{J}} = v_0 \frac{\delta G}{\delta \mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = v_0 \frac{\delta G}{\delta \mathcal{P}}$, one readily obtains the conserved charge that corresponds to the generator of the Galilean transformations, which reads[4]

$$G = \int d\phi \left[t \mathcal{J} \mathcal{P} + \frac{\phi}{3} \left(\mathcal{J} - a \mathcal{P} \right) \right] . \tag{4.35}$$

Anisotropic scaling of Lifshitz type. This symmetry is defined through a constant parameter σ, and it is generated by the transformations

$$\bar{t} = \sigma^3 t \,, \qquad \bar{\phi} = \sigma \phi \,, \qquad \begin{pmatrix} \bar{\mathcal{J}} \\ \bar{\mathcal{P}} \end{pmatrix} = \sigma^{-2} \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} , \tag{4.36}$$

which correspond to anisotropic scaling of Lifshitz type with dynamical exponent $z = 3$ (for a deeper discussion on anisotropic scaling of Lifshitz type, see e.g. [1, 78–86]).

In this case, in terms of the Poisson structure $\mathcal{D}^{(2)}$, the parameters that span the infinitesimal anisotropic scaling transformations in (4.36) read as

$$\begin{aligned} \varepsilon_{\mathcal{J}} &= \lambda \left(3t\mathcal{P} + \phi \right) , \\ \varepsilon_{\mathcal{P}} &= 3\lambda t \left(\mathcal{J} + a \mathcal{P} \right) , \end{aligned} \tag{4.37}$$

which satisfy the consistency conditions in (4.16). These parameters can also be written as a linear combination of the generalized polynomials as in (4.17), where the coefficients depend on t and ϕ, so that

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = 3\lambda t \left(\tilde{K}^{(1)} + a K^{(1)} \right) + \lambda \phi \tilde{K}^{(0)} . \tag{4.38}$$

Hence, the form of the generator of the anisotropic scaling transformations D can be directly read from $\varepsilon_{\mathcal{J}} = \lambda \frac{\delta D}{\delta \mathcal{J}}$ and $\varepsilon_{\mathcal{P}} = \lambda \frac{\delta D}{\delta \mathcal{P}}$, with

$$D = \int d\phi \left[3t \left(\mathcal{J}\mathcal{P} + \frac{a}{2} \mathcal{P}^2 \right) + \phi \mathcal{J} \right] . \tag{4.39}$$

For later purposes, it is worth noting that both sets of conserved quantities, $H_{\text{KdV}}^{(n)}$ and $\tilde{H}^{(n)}$, scale under (4.36) according to

$$\bar{H}^{(n)} = \sigma^{-(2n+1)} H^{(n)} . \tag{4.40}$$

[4] We thank an anonymous referee for providing the explicit form of the generator of Galilean transformations.

4.2.4 Equivalence with the Hirota-Satsuma coupled KdV system of type ix

Here we show that the field equations of the bi-Hamiltonian integrable system with BMS$_3$ and canonical Poisson structures, described by (4.12), can be seen to be equivalent to a particular class of a generalization of the Hirota-Satsuma coupled KdV system [87]. Specifically, according to the classification in [88], the equivalence is shown to hold for the field equations of type ix, which have been shown to be integrable through a method that differs from the one we have used above. The equivalence can be seen as follows.

If one changes the dynamical fields and rescale time according to

$$u = \frac{1}{4c_{\mathcal{P}}}\mathcal{P}\,, \qquad v = \frac{1}{4}\left(\mathcal{J} + a\mathcal{P}\right)\,, \qquad \tau = -c_{\mathcal{P}}t\,, \tag{4.41}$$

the field equations in (4.12) read

$$\partial_\tau v = -12uv' - 12vu' + v''' + \gamma u'''\,, \tag{4.42}$$

$$\partial_\tau u = -12uu' + u'''\,, \tag{4.43}$$

with

$$\gamma \equiv c_{\mathcal{J}} + ac_{\mathcal{P}}\,. \tag{4.44}$$

The equations in (4.42), (4.43) then turn out to be precisely the ones of type ix in [88]. Thus, at the level of the field equations there are actually only two inequivalent cases. The generic one corresponds to $\gamma = 1$, since the field equation in (4.42) can always be brought to this form provided that v is rescaled as $v \to \gamma v$. The remaining case is described by $\gamma = 0$, which is also known in the literature as "perturbed KdV" (see e.g. [89–95]).

Note that for $\gamma = 0$ ($c_{\mathcal{J}} = -ac_{\mathcal{P}}$), configurations with $v = 0$ ($\mathcal{J} = -a\mathcal{P}$) are devoid of energy, since $H = H^{(0)}$ in (4.2) vanishes. Nonetheless, they are generically endowed with both towers of conserved charges $H^{(j)}_{\text{KdV}}$ and $\tilde{H}^{(j)}$.

The structures discussed in this subsection provide all what is needed in order to generalize the integrable system to a hierarchy of them that is labeled by a nonnegative integer k.

4.3 The hierarchy ($k \geq 0$)

The results obtained in section 4.2 ensure that a hierarchy of bi-Hamiltonian integrable systems with BMS$_3$ and canonical Poisson structures, given by $\mathcal{D}^{(2)}$ and $\mathcal{D}^{(1)}$, can be readily constructed out from choosing any of their Hamiltonians to be given by an arbitrary linear combination of the independent conserved charges $H^{(n)}_{\text{KdV}}$ and $\tilde{H}^{(n)}$ in (4.25). Hereafter we focus in the subclass of them that possesses well-defined scaling properties. In order to achieve this task, we recall that if one rescales the spacelike coordinate and the fields as in (4.36), then both sets of conserved charges scale according to (4.40). Therefore, the most general combination that possesses the suitable scaling properties that we look for is described by a Hamiltonian of the form

$$H^{(k)} = \tilde{H}^{(k)} + aH^{(k)}_{\text{KdV}}\,, \tag{4.45}$$

with a constant and any fixed value of the integer $k \geq 0$.

The field equations of the corresponding hierarchy of integrable systems then read

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(k)} \\ \mu_{\mathcal{P}}^{(k)} \end{pmatrix} , \tag{4.46}$$

with

$$\mu_{\mathcal{J}}^{(k)} = \frac{\delta H^{(k)}}{\delta \mathcal{J}} \quad \text{and} \quad \mu_{\mathcal{P}}^{(k)} = \frac{\delta H^{(k)}}{\delta \mathcal{P}} . \tag{4.47}$$

The hierarchy defined through (4.46) is clearly bi-Hamiltonian, since by virtue of the recursion relationship in (4.18), the field equations can also be expressed as

$$\begin{pmatrix} \dot{\mathcal{J}} \\ \dot{\mathcal{P}} \end{pmatrix} = \mathcal{D}^{(2)} \begin{pmatrix} \mu_{\mathcal{J}}^{(k)} \\ \mu_{\mathcal{P}}^{(k)} \end{pmatrix} = \mathcal{D}^{(1)} \begin{pmatrix} \mu_{\mathcal{J}}^{(k+1)} \\ \mu_{\mathcal{P}}^{(k+1)} \end{pmatrix} . \tag{4.48}$$

The field equations of the hierarchy can also be explicitly written in terms of the polynomials $T^{(k)}$ defined through (4.24) and the Gelfand-Dikii polynomials $R^{(k)}$, so that they read

$$\begin{aligned} \dot{\mathcal{J}} &= \mathcal{D}^{(\mathcal{P})} T^{(k)} + \left(\mathcal{D}^{(\mathcal{J})} + a\mathcal{D}^{(\mathcal{P})}\right) R^{(k)} , \\ \dot{\mathcal{P}} &= \mathcal{D}^{(\mathcal{P})} R^{(k)} . \end{aligned} \tag{4.49}$$

It is worth pointing out that for any representative of the hierarchy with $k > 1$, not only the field equations in (4.49), but also the consistency condition for the functions $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ that parametrize the symmetries in (3.11) become severely more complicated than the simplest cases of $k = 0, 1$ (see eqs. (4.3), (4.5), and (4.12), (4.16), respectively). However, and remarkably, when $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ are assumed to depend only on the dynamical fields and their spatial derivatives, but not explicitly on t, ϕ, by virtue of the theorem in [75, 76], the general solution of the consistency condition for the parameters in (3.11) for $k \geq 0$ turns out to be precisely given by the same expansion in terms of the generalized Gelfand-Dikii polynomials $K^{(j)}$ and $\tilde{K}^{(j)}$, as in (4.17) for $k = 1$. This is explicitly verified in appendix C. Therefore, as a consequence, the corresponding canonical generators turn out to be given by the two independent sets of conserved charges given by (4.28), which are in involution, i.e., the commuting charges fulfill (4.29) for both Poisson structures. Nevertheless, depending on the choice of Poisson structure, $\mathcal{D}^{(2)}$ or $\mathcal{D}^{(1)}$, the energy of the system in (3.15), now corresponds to the Hamiltonian, $H^{(k)}$ or $H^{(k+1)}$, defined through (4.45), respectively.

Furthermore, by construction, the field equations for any representative of the hierarchy turn out to be invariant under anisotropic scaling transformations given by

$$t \to \sigma^{z} t , \qquad \phi \to \sigma \phi , \qquad \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} \to \sigma^{-2} \begin{pmatrix} \mathcal{J} \\ \mathcal{P} \end{pmatrix} , \tag{4.50}$$

which is of Lifshitz type, and characterized by a dynamical exponent $z = 2k+1$. Isotropic scaling then only holds for $k = 0$.

In terms of the BMS_3 Poisson structure $\mathcal{D}^{(2)}$, the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ that correspond to the infinitesimal anisotropic scaling transformation in (4.50), are then given by

$$\begin{pmatrix} \varepsilon_{\mathcal{J}} \\ \varepsilon_{\mathcal{P}} \end{pmatrix} = \lambda z t \left(\tilde{K}^{(k)} + aK^{(k)}\right) + \lambda \phi \tilde{K}^{(0)} , \tag{4.51}$$

and fulfill the consistency conditions in (3.11). The corresponding conserved charge can then be obtained from $\varepsilon_{\mathcal{J}}=\lambda\frac{\delta D}{\delta\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}=\lambda\frac{\delta D}{\delta\mathcal{P}}$, with

$$D=ztH^{(k)}+\int d\phi\,\phi\mathcal{J}\,, \tag{4.52}$$

where $H^{(k)}$ stands for the Hamiltonian, given by eq. (4.45).

As an ending remark of this subsection, it is worth noting that the field equations of the so-called "perturbed KdV hierarchy" (see e.g., [89, 91]) are precisely recovered from (4.49) for the particular case of $c_{\mathcal{J}}=a=0$. In this case, for the entire hierarchy, according to (4.45), configurations with $\mathcal{J}=0$ do not carry energy, which goes by hand with the fact that the Hamiltonian corresponds to the energy of a perturbation described by $\mathcal{J}$. Indeed, for this class of configurations, the entire set of conserved charges $\tilde{H}^{(j)}$ in (4.25) also vanishes. Note that the remaining ones, given by $H^{(j)}_{\text{KdV}}$ generically remain nontrivial, which can be interpreted as the conserved charges associated to an arbitrary "background configuration" described by $\mathcal{P}=\mathcal{P}(t,\phi)$ that solves the field equations of the k-th representative of the KdV hierarchy.

4.4 Analytic solutions

Exact analytic solutions of the KdV equation, as well as for the k-th representative of the KdV hierarchy, have been thoroughly studied in the literature since long ago through different methods (see e.g., [72, 96]).

Here we show how to obtain an interesting wide class of analytic solutions of the field equations in (4.49) for an arbitrary value of the nonnegative integer k that labels our hierarchy of integrable systems with BMS_3 Poisson structure.

As a warming up exercise, let us begin considering the simplest case, described by choosing the field $\mathcal{P}$ to be constant, so that the field equation of the KdV hierarchy in (4.49) is trivially solved for an arbitrary value of k. In this case, the field equation for $\mathcal{J}(t,\phi)$ in (4.49) just reduces to a dispersive linear homogeneous equation with constant coefficients, given by

$$\dot{\mathcal{J}}=\sum_{m=0}^{k}\alpha_{k,m}(-c_{\mathcal{P}})^{k-m}\mathcal{P}^{m}\partial_{\phi}^{2k-2m+1}\mathcal{J}\,, \tag{4.53}$$

with $\alpha_{k,m}\equiv\frac{(2k+1)!!}{m!(2k-2m+1)!!}$, which can be easily solved for an arbitrary member of the hierarchy. Indeed, expanding in Fourier modes according to[5]

$$\mathcal{J}(t,\phi)=\frac{1}{2\pi}\sum_{n=0}^{\infty}\mathcal{J}_{n}e^{-i\left(\omega_{k,n}t+n\phi\right)}\,, \tag{4.54}$$

one finds that the corresponding dispersion relation is given by

$$\omega_{k,n}=\sum_{m=0}^{k}\alpha_{k,m}{c_{\mathcal{P}}}^{k-m}\mathcal{P}^{m}n^{2(k-m)+1}\,. \tag{4.55}$$

[5] $\mathcal{J}(t,\phi)$ is real provided that the modes fulfill $(\mathcal{J}_n)^*=\mathcal{J}_{-n}$.

In the case of nontrivial solutions $\mathcal{P} = \mathcal{P}(t,\phi)$ of the k-th KdV equation, it is also possible to find generic analytic solutions for $\mathcal{J}(t,\phi)$. Remarkably, in spite of the fact that $\mathcal{J}(t,\phi)$ obeys a linear differential equation, their exact solutions are able to be nondispersive. This effect occurs because the coefficients of the linear equation for $\mathcal{J}$ in (4.49) are determined by nontrivial solutions of the k-th KdV equation, and it persists even in presence of a source term (with $a \neq 0$ or $c_{\mathcal{J}} \neq 0$). This is explicitly discussed in what follows.

4.4.1 Generic analytic solutions

Let us assume that $\mathcal{P} = \mathcal{P}(t,\phi)$ corresponds to an arbitrary generic solution of the field equations of the k-th representative of the KdV hierarchy, described by the second line of (4.49). Note that, since the central extension $c_{\mathcal{P}}$ has not been scaled away, nontrivial solutions $\mathcal{P}(t,\phi)$ explicitly depend on $c_{\mathcal{P}}$.

In order to find the form of $\mathcal{J} = \mathcal{J}(t,\phi)$ one has to solve the remaining equation in (4.49), which turns out to be linear in $\mathcal{J}$, and possesses an inhomogeneous term that is completely specified by $\mathcal{P}$ and their spatial derivatives. Hence, the generic solution for $\mathcal{J}$ is given by the sum of the particular and the homogeneous solutions, i.e.,

$$\mathcal{J} = \mathcal{J}_h + \mathcal{J}_p\,. \tag{4.56}$$

Noteworthy, as it is shown in appendix D, for an arbitrary value of the label k of the hierarchy, a particular solution $\mathcal{J} = \mathcal{J}_p(t,\phi)$ can be analytically expressed in a very compact way, so that it reads[6]

$$\mathcal{J}_p = c_{\mathcal{J}}\frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}}\,. \tag{4.57}$$

Besides, a generic solution of the homogeneous equation can be found by virtue of the symmetries in (3.8), being spanned by the subset of parameters given by (4.17) that preserve the form of $\mathcal{P}(t,\phi)$, i.e., the ones for which $\delta\mathcal{P} = 0$. The suitable subset of symmetries we look for then becomes generated by an arbitrary combination of the generalized polynomials $K^{(j)}$, excluding $\tilde{K}^{(j)}$; and hence the parameters are given by $\varepsilon_{\mathcal{J}}$ and $\varepsilon_{\mathcal{P}}$ in (4.17) with $\tilde{\eta}_j = 0$. Thus, according to (3.8), the homogeneous solution is given by

$$\mathcal{J}_h = \delta_{\eta_j}\mathcal{J} = \sum_{j=0}^{\infty}\eta_j \mathcal{D}^{(\mathcal{P})}R^{(j)}\,. \tag{4.58}$$

Therefore, by virtue of the recursive relation of the Gelfand-Dikii polynomials in (4.21), the generic solution for $\mathcal{J}$ acquires the form

$$\mathcal{J} = \sum_{j=0}^{\infty}\eta_j\partial_\phi R^{(j+1)} + c_{\mathcal{J}}\frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}}\,. \tag{4.59}$$

As a cross-check, in appendix D it is explicitly shown that eq. (4.59) solves the field equation for $\mathcal{J}$ in (4.49).

[6]Note that this particular solution becomes trivial ($\mathcal{J}_p = 0$) in the case of "perturbed KdV" described by $c_{\mathcal{J}} = a = 0$.

In sum, the generic solution for $\mathcal{J}$ in (4.59) has been generated through acting on the particular solution $\mathcal{J}_p$ with the symmetries that are spanned by the corresponding canonical generators in (4.28) with $\tilde{\eta}_j = 0$. Hence, since the generators are in involution, the full set of conserved charges for the solution characterized by the fields $\mathcal{P}$ and $\mathcal{J}_p$ must coincide with the ones for the fields $\mathcal{P}$ with $\mathcal{J}$ given by (4.59). In other words, the homogeneous part of the solution $\mathcal{J}_h$ does not contribute to the conserved charges.

The conserved charges are then described by the corresponding ones for KdV, given by $H^{(n)}_{\text{KdV}}$, together with the independent set $\tilde{H}^{(n)}$ in (4.25). Once the latter set is evaluated in the generic solution (4.59), it can be compactly written in terms of the total derivative of $H^{(n)}_{\text{KdV}}$ with respect to $c_{\mathcal{P}}$, so that it reads (see appendix D)

$$\tilde{H}^{(n)} = c_{\mathcal{J}} \frac{dH^{(n)}_{\text{KdV}}}{dc_{\mathcal{P}}} \,. \tag{4.60}$$

Therefore, for the BMS$_3$ Poisson structure $\mathcal{D}^{(2)}$, the energy of our generic solution for the k-th representative of the hierarchy is given by the Hamiltonian in (4.45), which reduces to

$$H^{(k)} = c_{\mathcal{J}} \frac{dH^{(k)}_{\text{KdV}}}{dc_{\mathcal{P}}} + aH^{(k)}_{\text{KdV}} \,. \tag{4.61}$$

It is worth pointing out that the conserved charges of the solution can be expressed exclusively in terms of $\mathcal{P}$ and their spatial derivatives.

Note that the generic class of solutions presented here is mapped into itself under the anisotropic scaling of Lifshitz type given in (4.50), where the arbitrary constants η_j of the homogeneous solution (4.58) transform as $\bar{\eta}_j = \sigma^{2j-1}\eta_j$.

Additionally, as pointed in section 4.2.3, in the case of $k = 1$ the field equations are also invariant under Galilean transformations. Hence, in this case, by virtue of the Galilean boost spanned by (4.31), our solution in (4.59) acquires nontrivial zero modes once expressed in the moving frame.

In the next subsection, we explicitly describe a couple of simple and interesting particular examples of analytic solutions in the case of $k = 1$.

4.4.2 Particular cases for $k = 1$

Single KdV soliton on the real line. The integrable system described in section 4.2 can be extended to $\mathbb{R}^2$ provided that the angular coordinate is unwrapped ($-\infty < \phi < \infty$), so that our previous analysis still holds once the fall-off of the fields is assumed to be fast enough so as to get rid of boundary terms.

In our conventions, the well-known single soliton solution of the KdV equation in (4.12) reads

$$\mathcal{P} = -v\, \text{sech}^2(x) \,, \tag{4.62}$$

where $x = \sqrt{\frac{v}{4c_{\mathcal{P}}}}(\phi - vt)$, and v stands for the integration constant that parametrizes the velocity and amplitude of the soliton.

An analytic solution for the remaining field equation in (4.12) can then be constructed out from (4.59), with $\mathcal{P}$ given by (4.62). For simplicity we consider that the integration

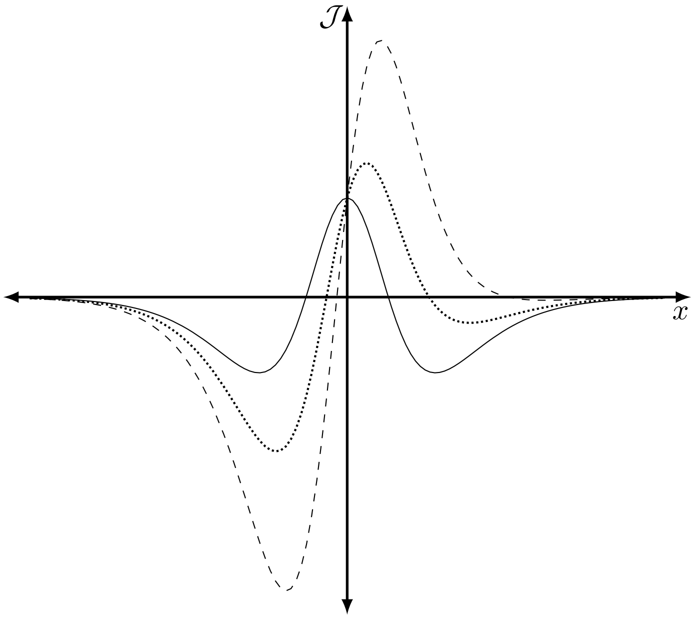

Figure 1. The form of $\mathcal{J}$ in (4.63) is plotted for generic fixed values of $c_{\mathcal{P}}$, $c_{\mathcal{J}}$, a, v and for different values of η_1. Note that when the integration constant η_1 vanishes, $\mathcal{J}$ becomes an even function of x (solid line).

constants in (4.59) are chosen as $\eta_j = \eta_1 \delta_{j,1}$, with η_1 arbitrary, so that the solution for $\mathcal{J}$ becomes explicitly given by

$$\mathcal{J} = \left(\eta_1 \frac{v^{3/2}}{\sqrt{c_{\mathcal{P}}}} - \frac{vx(ac_{\mathcal{P}} + c_{\mathcal{J}})}{c_{\mathcal{P}}} \right) \tanh(x) \operatorname{sech}^2(x) + av \operatorname{sech}^2(x) \, . \tag{4.63}$$

Therefore, although $\mathcal{J}$ obeys a linear differential equation, the solution in (4.63) clearly maintains its shape as it evolves in time. The profile of $\mathcal{J}$ in (4.63) is depicted in figure 1.

Cnoidal wave on S^1. In the case of periodic boundary conditions ($-\pi \leq \phi < \pi$) an analytic solution for the KdV equation in (4.12) of solitonic type is known as a "cnoidal wave", since it is described in terms the Jacobi elliptic cosine (cn) (see e.g. [96]). The solution is given by

$$\mathcal{P} = 4c_{\mathcal{P}} \left[A - \alpha \operatorname{cn}^2(y, m) \right] \, , \tag{4.64}$$

with $y = \sqrt{\frac{\alpha}{m}}(\phi - c_{\mathcal{P}} vt)$, and the velocity parameter is related to the remaining integration constants as

$$v = 4\alpha \left(2 - \frac{1}{m} \right) - 12A \, . \tag{4.65}$$

The wavelength of the solution is given by $2\sqrt{\frac{m}{\alpha}} K(m)$, where K stands for a complete elliptic integral of the first kind. Accordingly, the elliptic parameter m can take values within the range $0 < m < 1$, satisfying $2\sqrt{\frac{m}{\alpha}} K(m) = \frac{2\pi}{n}$ with $n \in \mathbb{N}$.

As in the previous example, we then construct the analytic solution for $\mathcal{J}$ from the generic one in (4.59), with $\mathcal{P}$ given by the cnoidal wave in (4.64). For the sake of simplicity,

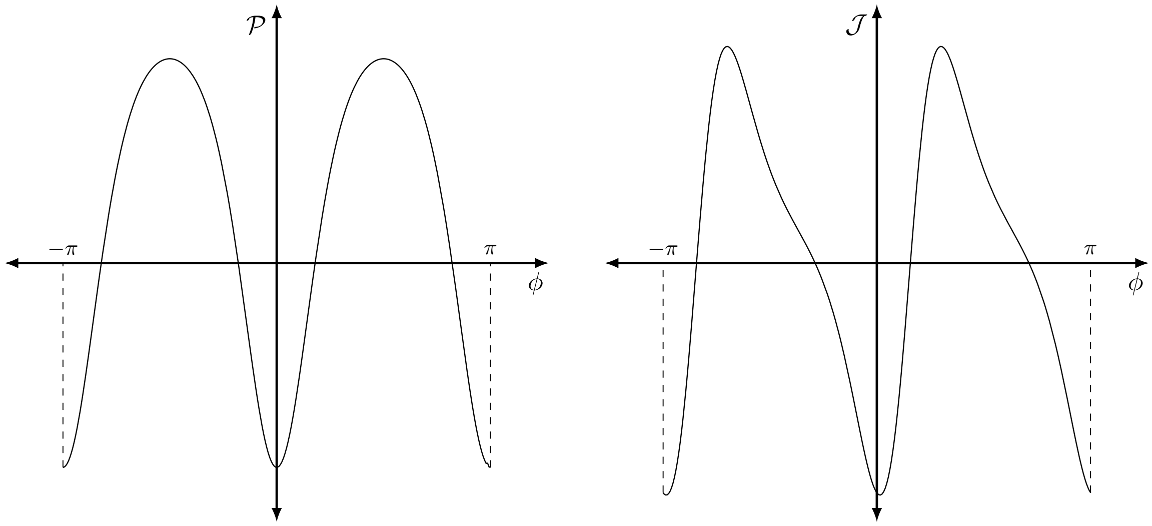

Figure 2. Profiles of $\mathcal{P}$ (cnoidal wave in (4.64)) and $\mathcal{J}$ in (4.66) for a fixed generic value of A, α, m and t.

we again choose the integration constants to be given by $\eta_j = \eta_1 \delta_{j,1}$, and hence the searched for analytic solution becomes

$$\mathcal{J} = 4c_{\mathcal{J}} \left[A - \alpha \mathrm{cn}^2(y,m)\right] + \left[8\alpha c_{\mathcal{P}} \sqrt{\frac{\alpha}{m}} (\eta_1 - (a c_{\mathcal{P}} + c_{\mathcal{J}}) v t)\right] \mathrm{cn}(y,m)\, \mathrm{sn}(y,m)\, \mathrm{dn}(y,m)\,, \tag{4.66}$$

where sn and dn stand for the elliptic sine and the delta amplitude, respectively.

Note that the profile of $\mathcal{J}$ preserves its form as it evolves in time only in the case of $\gamma = a c_{\mathcal{P}} + c_{\mathcal{J}} = 0$ (perturbed KdV), otherwise the amplitude grows linearly with time. Nonetheless, the energy in (4.61) as well as the remaining conserved charges given by $H^{(n)}_{\mathrm{KdV}}$, and $\tilde{H}^{(n)}$ in (4.25), turn out to be finite regardless the value of γ.

The profiles of $\mathcal{P}$ and $\mathcal{J}$ are sketched in figure 2.

As an ending remark of this section, it is worth pointing out that in the special case of $c_{\mathcal{J}} = a = 0$, the field equation for $\mathcal{J}$ becomes devoid of a source. The particular solutions, given by (4.63) for the real line, and by (4.66) in the case of S^1, in this case turn out to be described by odd analytic functions that maintain its shape as they propagate with the same velocity as their corresponding KdV solitons described by $\mathcal{P}$ in (4.62) and (4.64), respectively. Note that, according to eq. (4.61), the total energy of this sort of soliton-antisoliton bound states for $\mathcal{J}$ vanishes, and the conserved charges $\tilde{H}^{(n)}$ in (4.25) also do. Nevertheless, the conserved charges $H^{(n)}_{\mathrm{KdV}}$ remain being nontrivial.

5 Geometrization of the hierarchy: the dynamics of locally flat spacetimes in 3D

In this section, we show that the entire structure of the class of integrable systems with BMS_3 Poisson structure described above can be fully geometrized, in the sense that the dynamics turns out to be equivalently understood through the evolution of spacelike surfaces embedded in locally flat spacetimes in three dimensions.

For the sake of simplicity, here we focus in the case of BMS$_3$ Poisson structures with $c_{\mathcal{J}} = 0$. Thus, following the lines of [1], it is possible to unveil a deep link between the class of integrable systems aforementioned and General Relativity in three spacetime dimensions. Concretely, here we show that the Einstein-Hilbert action without cosmological constant in 3D can be endowed with an appropriate set of boundary conditions, so that in the reduced phase space, the Einstein equations in vacuum, which imply the vanishing of the Riemann tensor, precisely reduce to the ones of the dynamical systems with BMS$_3$ Poisson structure in (2.7). As a consequence, it is possible to establish a one-to-one map between any solution of this kind of integrable systems and certain specific locally flat metric in three spacetime dimensions. Furthermore, the symmetries of the integrable systems can be seen to naturally emerge from diffeomorphisms that preserve the asymptotic form of the spacetime metric. Hence, and remarkably, the symmetries manifestly become Noetherian in our geometric framework. Therefore, the infinite set of conserved charges for the integrable system is transparently recovered from the corresponding surface integrals in the canonical approach. This can be seen as follows.

The Einstein-Hilbert action in three spacetime dimensions

$$I = \frac{1}{16\pi G} \int d^3x \sqrt{-g} R \,, \tag{5.1}$$

can be equivalently expressed as a Chern-Simons action for the $isl(2,\mathbb{R})$ algebra [97, 98]. Thus, up to boundary terms, the action (5.1) can be written as

$$I = \frac{1}{2} \int d^3x \left\langle AdA + \frac{2}{3} A^3 \right\rangle \,, \tag{5.2}$$

where $\langle \cdots \rangle$ stands for the invariant bilinear form defined in eq. (3.2) with $c_{\mathcal{J}} = 0$ and $c_{\mathcal{P}} = 1/\left(8\pi G\right)$. The components of the $isl(2,\mathbb{R})$-valued gauge field are then identified with the dualized spin connection and the dreibein according to

$$A = \omega^a J_a + e^a P_a \,. \tag{5.3}$$

In order to describe the asymptotic structure of the fields, as explained in [99–101] it is useful to choose the gauge so that the connection reads

$$A = b^{-1} a b + b^{-1} db \,, \tag{5.4}$$

where the radial dependence is completely captured by the group element $b = b\left(r\right)$, which as shown in [66] can be conveniently chosen as $b = e^{rP_2}$. One of the advantages of this gauge choice is that the remaining analysis can be performed in terms of the auxiliary connection

$$a = a_t dt + a_\phi d\phi \,, \tag{5.5}$$

that only depends on t, ϕ. Here we propose that the asymptotic form of the auxiliary connection for the gravitational field in (5.5) is precisely given by the two-dimensional locally flat gauge field that describes the field equations of the dynamical system with BMS$_3$ Poisson structure in section 3. Their components a_ϕ and a_t are then described by eqs. (3.3) and (3.4), respectively. Therefore, from (5.2), the field equations imply that the connection A is flat ($F = dA + A^2 = 0$), which by virtue of (5.3) amounts to deal with three-dimensional manifolds with vanishing curvature and torsion; whereas eq. (5.4) means that the field strength of the auxiliary gauge field also vanishes as in (3.6). Hence, for our boundary conditions, the Einstein equations in vacuum precisely reduce to the ones of a dynamical system with BMS_3 Poisson structure in (2.5).

Besides, the asymptotic symmetries, being defined as the diffeomorphisms that preserve the asymptotic form of the spacetime metric, turn out to be equivalent to the set of gauge transformations $\delta A = d\tilde{\lambda} + [A, \tilde{\lambda}]$ that maintain the asymptotic form of the gauge field in (5.4). For our boundary conditions, one then finds that the asymptotic symmetries are spanned by a Lie-algebra-valued parameter of the form $\tilde{\lambda} = b^{-1}\lambda b$, where $\lambda = \Lambda\left(\varepsilon_{\mathcal{J}}, \varepsilon_{\mathcal{P}}\right)$ is exactly given as in eq. (3.7), including the consistency condition for the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (3.11). Furthermore, the transformation law of the dynamical fields $\mathcal{J}$, $\mathcal{P}$ precisely agrees with the transformations in eq. (3.8).

Since the asymptotic symmetries are Noetherian, the global conserved charges can be readily obtained using the canonical approach [102]. Indeed, their variation is explicitly given by surface integrals defined at the boundary of the spatial section,[7] which read

$$\delta Q[\tilde{\lambda}] = -\int d\phi \left\langle \tilde{\lambda}\delta A_\phi \right\rangle = -\int d\phi \left\langle \lambda \delta a_\phi \right\rangle = -\int d\phi \left(\varepsilon_{\mathcal{J}}\delta\mathcal{J} + \varepsilon_{\mathcal{P}}\delta\mathcal{P}\right) , \tag{5.6}$$

and precisely coincide with the variation of the conserved charges introduced in eq. (3.12), in the context of integrable systems.

In particular, if the Lagrange multipliers $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ are kept fixed at the boundary according to (4.47), so that they correspond to the variation of the functional $H^{(k)}$ defined in (4.45), the Einstein equations reduce to the ones of the hierarchy of integrable systems discussed in section 4.3. Therefore, in this case, the variation of the surface integrals in (5.6) integrates precisely as in (4.28). Note that the energy of a gravitational configuration that fulfills these boundary conditions is then given by the Hamiltonian of the corresponding integrable system as in (3.15), i.e., $E = Q[\partial_t] = H^{(k)}$.

In the simplest case of $k = 0$, described in section 4.1, we recover the set of boundary conditions proposed in [66][8] (see also [65]) which contain the boundary conditions in [6] for a particular choice of Lagrange multipliers at the boundary. Note that in this case, the BMS_3 algebra is realized as the asymptotic symmetry algebra. For the remaining cases ($k \geq 1$), the new class of boundary conditions is such that the asymptotic symmetry algebra is infinite-dimensional, abelian and devoid of central charges, which is equivalent to the fact that the conserved charges of the hierarchy are in involution (see eq. (4.29)).

It is worth pointing out that, in the metric formalism, our particular choices for the Lagrange multipliers $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ correspond to fixing the lapse and shift functions in the

[7]It is worth noting that, by virtue of the gauge choice in (5.4), since the radial coordinate has been explicitly gauged away, the analysis can be carried out for a boundary that is located at any fixed value of the radial coordinate, and hence, not necessarily at null infinity.

[8]Strictly speaking, we are dealing with the boundary conditions in [66] provided that the higher spin fields and their corresponding chemical potentials are turned-off.

ADM decomposition of the metric, as suitable local functionals of the dynamical variables at the boundary. Furthermore, and remarkably, the class of locally flat spacetimes described by this new set of asymptotic conditions inherits the anisotropic Lifshitz scaling of the corresponding integrable systems by virtue of the boundary conditions. This effect then becomes the flat analogue of the one found in [1] for the case of locally AdS_3 spacetimes, where isotropy is recovered in the particular case of the Brown-Henneaux boundary conditions [103].

It is also amusing to verify that trivial solutions of the class of integrable systems described above, like the ones just given by configurations with $\mathcal{J}$ and $\mathcal{P}$ constants, actually correspond to the geometries of the flat cosmological spacetimes in [104]. Hence, from a gravitational point of view, these configurations become certainly non-trivial, not only in a geometric sense, but also in the physical one since they carry Hawking temperature and entropy associated to the cosmological horizon. It would then be worth to explore additional simple configurations for the integrable systems that could naturally become non-trivial in the gravitational framework and vice versa.

The results of this section are also related to some interesting issues that will be discussed in a forthcoming work, including the precise details of the asymptotic structure in the metric formulation, as well as the microscopic entropy of flat cosmological spacetimes once they are embedded within the new set of boundary conditions. An intriguing link with the recent results in refs. [43, 105] concerning "soft hair" in the sense of [44–46] for asymptotically flat spacetimes in 3D, can also be established once the hierarchy of integrable systems with BMS_3 Poisson structure presented here is suitably extended to the case of $z = 0$ in a form akin to the one performed in [1] for the KdV hierarchy. It can also be shown that the geometrization of the class of integrable systems discussed here, but in the generic case with $c_{\mathcal{J}} \neq 0$, can be performed through a suitable extension of the analysis in [8, 69] once suitable parity odd terms in the action are included (see also [106]).

6 Extensions of our results

As pointed out in the introduction, the BMS_3 algebra admits some extensions which, according to our results, might be expected to be linked to new classes of integrable systems. In particular, an interesting nonlinear extension of BMS_3 that includes additional generators of spin $s > 2$ was found in [64] (in full agreement with the algebra simultaneously found in [63] for $s = 3$). This kind of extensions can be regarded as "flat W-algebras", since they can be recovered from a suitable Inönü-Wigner contraction of two copies of certain classical W-algebras (for a review about W-algebras, see e.g. [107]). Noteworthy, preliminary results [108] show that new hierarchies of integrable systems whose Poisson structures correspond to flat W-algebras indeed exist. In fact, this family of integrable systems turns out to be bi-Hamiltonian, and furthermore, following the lines of section 5, they can also be geometrized in terms of higher spin gravity without cosmological constant in three spacetime dimensions endowed with a suitable set of boundary conditions. Therefore, the symmetries of this novel class of integrable systems can be seen as combinations of diffeomorphisms and higher spin gauge transformations that preserve the asymptotic

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form of the three-dimensional configurations. As a consequence, in the three-dimensional geometric setup, the infinite set of conserved charges of the integrable systems emerge as the canonical generators that correspond to the asymptotic symmetries, being described by suitable surface integrals at the boundary, which turn out to be in involution.

Further interesting links between certain well-known classes of integrable systems and higher spin gravity on AdS_3 have been explored in [1, 109–112].

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A List of conserved quantities and polynomials

Here we provide an explicit list of the first six conserved charges $H^{(n)}_{\text{KdV}}$ and $\tilde{H}^{(n)}$, and their associated polynomials $R^{(n)}$ and $T^{(n)}$.

The conserved quantities $H^{(n)}_{\text{KdV}}$ correspond to the ones of the KdV equation in (4.12), which in our conventions, generically depend on the central extension $c_{\mathcal{P}}$. Therefore, they read

$$
\begin{aligned}
H^{(0)}_{\text{KdV}} &= \int d\phi \mathcal{P}\,, \\
H^{(1)}_{\text{KdV}} &= \int d\phi \frac{\mathcal{P}^2}{2}\,, \\
H^{(2)}_{\text{KdV}} &= \int d\phi \left[\frac{1}{2} c_{\mathcal{P}} \mathcal{P}'^2 + \frac{1}{2}\mathcal{P}^3\right]\,, \\
H^{(3)}_{\text{KdV}} &= \int d\phi \left[\frac{1}{2} c_{\mathcal{P}}^2 \mathcal{P}''^2 + \frac{5}{2} c_{\mathcal{P}} \mathcal{P}\mathcal{P}'^2 + \frac{5}{8}\mathcal{P}^4\right]\,, \\
H^{(4)}_{\text{KdV}} &= \int d\phi \left[\frac{1}{2} c_{\mathcal{P}}^3 \left(\mathcal{P}^{(3)}\right)^2 + \frac{7}{2} c_{\mathcal{P}}^2 \mathcal{P}\mathcal{P}''^2 + \frac{35}{4} c_{\mathcal{P}} \mathcal{P}^2\mathcal{P}'^2 + \frac{7}{8}\mathcal{P}^5\right]\,, \\
H^{(5)}_{\text{KdV}} &= \int d\phi \left[\frac{1}{2} c_{\mathcal{P}}^4 \left(\mathcal{P}^{(4)}\right)^2 + \frac{9}{2} c_{\mathcal{P}}^3 \left(\mathcal{P}^{(3)}\right)^2 \mathcal{P} + \frac{63}{4} c_{\mathcal{P}}^2 \mathcal{P}^2\mathcal{P}''^2 - 5 c_{\mathcal{P}}^3 \mathcal{P}''^3 \right. \\
&\qquad \left. + \frac{105}{4} c_{\mathcal{P}} \mathcal{P}^3\mathcal{P}'^2 - \frac{35}{8} c_{\mathcal{P}}^2 \mathcal{P}'^4 + \frac{21}{16}\mathcal{P}^6\right]\,.
\end{aligned}
$$

The remaining conserved charges $\tilde{H}^{(n)}$, can then be readily obtained from (4.25), which are given by

$$\begin{aligned}
\tilde{H}^{(0)} &= \int d\phi \mathcal{J}\,, \\
\tilde{H}^{(1)} &= \int d\phi \mathcal{J}\mathcal{P}\,, \\
\tilde{H}^{(2)} &= \int d\phi \Big[\mathcal{J}\Big(\tfrac{3}{2}\mathcal{P}^2 - c_{\mathcal{P}}\mathcal{P}''\Big) + \tfrac{1}{2}c_{\mathcal{J}}\mathcal{P}'^2\Big]\,, \\
\tilde{H}^{(3)} &= \int d\phi \Big[\mathcal{J}\Big(c_{\mathcal{P}}^2\mathcal{P}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{P}'' - \tfrac{5}{2}c_{\mathcal{P}}\mathcal{P}'^2 + \tfrac{5}{2}\mathcal{P}^3\Big) + c_{\mathcal{J}}\Big(c_{\mathcal{P}}\mathcal{P}''^2 + \tfrac{5}{2}\mathcal{P}\mathcal{P}'^2\Big)\Big]\,, \\
\tilde{H}^{(4)} &= \int d\phi \Big[\mathcal{J}\Big(- c_{\mathcal{P}}^3\mathcal{P}^{(6)} + 7c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P} - \tfrac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'' + \tfrac{21}{2}c_{\mathcal{P}}^2\mathcal{P}''^2 - \tfrac{35}{2}c_{\mathcal{P}}\mathcal{P}\mathcal{P}'^2 \\
&\qquad + 14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}' + \tfrac{35}{8}\mathcal{P}^4\Big) + c_{\mathcal{J}}\Big(\tfrac{3}{2}c_{\mathcal{P}}^2(\mathcal{P}^{(3)})^2 + 7c_{\mathcal{P}}\mathcal{P}\mathcal{P}''^2 + \tfrac{35}{4}\mathcal{P}^2\mathcal{P}'^2\Big)\Big]\,, \\
\tilde{H}^{(5)} &= \int d\phi \Big[\mathcal{J}\Big(c_{\mathcal{P}}^4\mathcal{P}^{(8)} - 9c_{\mathcal{P}}^3\mathcal{P}^{(6)}\mathcal{P} - \tfrac{69}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2 + \tfrac{189}{2}c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}''^2 \\
&\qquad - 27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{P}'' + \tfrac{63}{2}c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}^2 - \tfrac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{P}'' \\
&\qquad + 126c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}\mathcal{P}' + \tfrac{231}{2}c_{\mathcal{P}}^2\mathcal{P}'^2\mathcal{P}'' - \tfrac{315}{4}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'^2 + \tfrac{63\mathcal{P}^5}{8}\Big) \\
&\qquad + c_{\mathcal{J}}\Big(2c_{\mathcal{P}}^3(\mathcal{P}^{(4)})^2 + \tfrac{27}{2}c_{\mathcal{P}}^2\mathcal{P}(\mathcal{P}^{(3)})^2 - 15c_{\mathcal{P}}^2\mathcal{P}''^3 + \tfrac{63}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}''^2 \\
&\qquad - \tfrac{35}{4}c_{\mathcal{P}}\mathcal{P}'^4 + \tfrac{105}{4}\mathcal{P}^3\mathcal{P}'^2\Big)\Big]\,.
\end{aligned}$$

Hence, according to (4.23), the Gelfand-Dikii polynomials read

$$\begin{aligned}
R^{(0)} &= 1\,, \quad R^{(1)} = \mathcal{P}\,, \quad R^{(2)} = -c_{\mathcal{P}}\mathcal{P}'' + \frac{3}{2}\mathcal{P}^2\,, \\
R^{(3)} &= c_{\mathcal{P}}^2\mathcal{P}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{P}'' - \frac{5}{2}c_{\mathcal{P}}\mathcal{P}'^2 + \frac{5}{2}\mathcal{P}^3\,, \\
R^{(4)} &= -c_{\mathcal{P}}^3\mathcal{P}^{(6)} + 7c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P} - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'' + \frac{21}{2}c_{\mathcal{P}}^2\mathcal{P}''^2 - \frac{35}{2}c_{\mathcal{P}}\mathcal{P}\mathcal{P}'^2 \\
&\quad + 14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}' + \frac{35}{8}\mathcal{P}^4\,, \\
R^{(5)} &= c_{\mathcal{P}}^4\mathcal{P}^{(8)} - 9c_{\mathcal{P}}^3\mathcal{P}^{(6)}\mathcal{P} + \frac{63}{2}c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}^2 - \frac{69}{2}c_{\mathcal{P}}^3(\mathcal{P}^{(3)})^2 - \frac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{P}'' \\
&\quad + \frac{189}{2}c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}''^2 - \frac{315}{4}c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}'^2 - 27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{P}' - 57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{P}'' \\
&\quad + 126c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{P}\mathcal{P}' + \frac{231}{2}c_{\mathcal{P}}^2\mathcal{P}'^2\mathcal{P}'' + \frac{63}{8}\mathcal{P}^5\,,
\end{aligned}$$

while the polynomials $T^{(n)}$ can be obtained from (4.24), so that they are given by

$$\begin{aligned}
T^{(0)} &= 0\,, \quad T^{(1)} = \mathcal{J}\,, \quad T^{(2)} = -c_{\mathcal{P}}\mathcal{J}'' - c_{\mathcal{J}}\mathcal{P}'' + 3\mathcal{J}\mathcal{P}\,, \\
T^{(3)} &= c_{\mathcal{P}}^2\mathcal{J}^{(4)} - 5c_{\mathcal{P}}\mathcal{P}\mathcal{J}'' - 5c_{\mathcal{P}}\mathcal{J}\mathcal{P}'' - 5c_{\mathcal{P}}\mathcal{J}'\mathcal{P}' + \frac{15}{2}\mathcal{J}\mathcal{P}^2 \\
&\quad + c_{\mathcal{J}}\left(2c_{\mathcal{P}}\mathcal{P}^{(4)} - 5\mathcal{P}\mathcal{P}'' - \tfrac{5}{2}\mathcal{P}'^2\right)\,,
\end{aligned}$$

$$
\begin{aligned}
T^{(4)} =& -c_{\mathcal{P}}^3\mathcal{J}^{(6)}+7c_{\mathcal{P}}^2\mathcal{J}^{(4)}\mathcal{P}+14c_{\mathcal{P}}^2\mathcal{J}^{(3)}\mathcal{P}'+21c_{\mathcal{P}}^2\mathcal{J}''\mathcal{P}''-\frac{35}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{J}''+14c_{\mathcal{P}}^2\mathcal{P}^{(3)}\mathcal{J}' \\
& +7c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}^{(4)}-35c_{\mathcal{P}}\mathcal{P}\mathcal{J}'\mathcal{P}'-35c_{\mathcal{P}}\mathcal{J}\mathcal{P}\mathcal{P}''-\frac{35}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}'^2+\frac{35}{2}\mathcal{J}\mathcal{P}^3-c_{\mathcal{J}}\Big(3c_{\mathcal{P}}^2\mathcal{P}^{(6)} \\
& -14c_{\mathcal{P}}\mathcal{P}\mathcal{P}^{(4)}-21c_{\mathcal{P}}\mathcal{P}''^2+\frac{35}{2}\mathcal{P}^2\mathcal{P}''+\frac{35}{2}\mathcal{P}\mathcal{P}'^2-28c_{\mathcal{P}}\mathcal{P}^{(3)}\mathcal{P}'\Big), \\
T^{(5)} =& c_{\mathcal{P}}^4\mathcal{J}^{(8)}-9c_{\mathcal{P}}^3\mathcal{J}^{(6)}\mathcal{P}-27c_{\mathcal{P}}^3\mathcal{J}^{(5)}\mathcal{P}'-57c_{\mathcal{P}}^3\mathcal{J}^{(4)}\mathcal{P}''+\frac{63}{2}c_{\mathcal{P}}^2\mathcal{J}^{(4)}\mathcal{P}^2-69c_{\mathcal{P}}^3\mathcal{J}^{(3)}\mathcal{P}^{(3)} \\
& +126c_{\mathcal{P}}^2\mathcal{J}^{(3)}\mathcal{P}\mathcal{P}'-57c_{\mathcal{P}}^3\mathcal{P}^{(4)}\mathcal{J}''+189c_{\mathcal{P}}^2\mathcal{P}\mathcal{J}''\mathcal{P}''+\frac{231}{2}c_{\mathcal{P}}^2\mathcal{J}''\mathcal{P}'^2-\frac{105}{2}c_{\mathcal{P}}\mathcal{P}^3\mathcal{J}'' \\
& -27c_{\mathcal{P}}^3\mathcal{P}^{(5)}\mathcal{J}'+126c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}^{(3)}\mathcal{J}'-\frac{315}{2}c_{\mathcal{P}}\mathcal{P}^2\mathcal{J}'\mathcal{P}'+231c_{\mathcal{P}}^2\mathcal{J}'\mathcal{P}'\mathcal{P}''-9c_{\mathcal{P}}^3\mathcal{J}\mathcal{P}^{(6)} \\
& +63c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}\mathcal{P}^{(4)}+\frac{189}{2}c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}''^2-\frac{315}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}^2\mathcal{P}''-\frac{315}{2}c_{\mathcal{P}}\mathcal{J}\mathcal{P}\mathcal{P}'^2+126c_{\mathcal{P}}^2\mathcal{J}\mathcal{P}^{(3)}\mathcal{P}' \\
& +\frac{315}{8}\mathcal{J}\mathcal{P}^4+c_{\mathcal{J}}\Big(4c_{\mathcal{P}}^3\mathcal{P}^{(8)}-27c_{\mathcal{P}}^2\mathcal{P}\mathcal{P}^{(6)}+63c_{\mathcal{P}}\mathcal{P}^2\mathcal{P}^{(4)}-\frac{207}{2}c_{\mathcal{P}}^2(\mathcal{P}^{(3)})^2+189c_{\mathcal{P}}\mathcal{P}\mathcal{P}''^2 \\
& -81c_{\mathcal{P}}^2\mathcal{P}^{(5)}\mathcal{P}'-171c_{\mathcal{P}}^2\mathcal{P}^{(4)}\mathcal{P}''+252c_{\mathcal{P}}\mathcal{P}\mathcal{P}^{(3)}\mathcal{P}'+231c_{\mathcal{P}}\mathcal{P}'^2\mathcal{P}''-\frac{105}{2}\mathcal{P}^3\mathcal{P}''-\frac{315}{4}\mathcal{P}^2\mathcal{P}'^2\Big).
\end{aligned}
$$

B Involution of the conserved quantities

In order to prove that our set of conserved charges, $\bar{H}^{(m)} = \left(H_{\text{KdV}}^{(n)};\tilde{H}^{(n)}\right)$, is abelian in both Poisson brackets, one can follow the lines of the proof of the same statement in the case of the pure KdV equation that can be found in standard textbooks (see e.g., [72, 75]).

Without loss of generality, let us assume that $m > n$. Thus, the Poisson bracket associated to the BMS$_3$ operator $\mathcal{D}^{(2)}$ of two conserved charges, which reads

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = \int d\phi \begin{pmatrix} \frac{\delta\bar{H}^{(m)}}{\delta\mathcal{J}} & \frac{\delta\bar{H}^{(m)}}{\delta\mathcal{P}} \end{pmatrix} \mathcal{D}^{(2)} \begin{pmatrix} \frac{\delta\bar{H}^{(n)}}{\delta\mathcal{J}} \\ \frac{\delta\bar{H}^{(n)}}{\delta\mathcal{P}} \end{pmatrix}, \tag{B.1}
$$

by virtue of the recursion relation in (4.18), can be written in terms of the Poisson bracket associated to the "canonical" operator $\mathcal{D}^{(1)}$, according to

$$
\begin{aligned}
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} &= \int d\phi \begin{pmatrix} \frac{\delta\bar{H}^{(m)}}{\delta\mathcal{J}} & \frac{\delta\bar{H}^{(m)}}{\delta\mathcal{P}} \end{pmatrix} \mathcal{D}^{(1)} \begin{pmatrix} \frac{\delta\bar{H}^{(n+1)}}{\delta\mathcal{J}} \\ \frac{\delta\bar{H}^{(n+1)}}{\delta\mathcal{P}} \end{pmatrix} \\
&= \{\bar{H}^{(m)},\bar{H}^{(n+1)}\}_{(1)} \\
&= -\{\bar{H}^{(n+1)},\bar{H}^{(m)}\}_{(1)}\,.
\end{aligned} \tag{B.2}
$$

Analogously, making use of the recursion relationship again, one finds that

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = -\{\bar{H}^{(n+1)},\bar{H}^{(m)}\}_{(1)} = \{\bar{H}^{(m-1)},\bar{H}^{(n+1)}\}_{(2)}\,. \tag{B.3}
$$

Therefore, once the procedure is applied $m-n$ times, one obtains

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = \{\bar{H}^{(n)},\bar{H}^{(m)}\}_{(2)}\,, \tag{B.4}
$$

which implies that the conserved charges are involution in both Poisson brackets, i.e.,

$$
\{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(2)} = \{\bar{H}^{(m)},\bar{H}^{(n)}\}_{(1)} = 0\,. \tag{B.5}
$$

C Verification of the solution in (4.17) for the consistency condition of the symmetry parameters

Here we explicitly verify that eq. (4.17) solves the consistency condition for the parameters $\varepsilon_{\mathcal{J}}$, $\varepsilon_{\mathcal{P}}$ in (3.11) for an arbitrary representative of the hierarchy, being characterized by the Hamiltonian $H^{(k)}$ given by (4.45). Since the equation is linear for the parameters, it is then enough proving that

$$\varepsilon_{\mathcal{J}}(\phi)=\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\phi)}\,,\qquad \varepsilon_{\mathcal{P}}(\phi)=\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\phi)}\,,\tag{C.1}$$

fulfills, for an arbitrary member of the set of conserved charges $\bar{H}^{(j)}=\left(H^{(j)}_{\text{KdV}};\tilde{H}^{(j)}\right)$.

Note that the consistency condition for the parameters in (3.11) can also be written as

$$\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=-\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\left(\mathcal{D}^{(2)}\begin{pmatrix}\mu_{\mathcal{J}}(\varphi)\\ \mu_{\mathcal{P}}(\varphi)\end{pmatrix}\right)^{T}\right]\begin{pmatrix}\varepsilon_{\mathcal{J}}(\varphi)\\ \varepsilon_{\mathcal{P}}(\varphi)\end{pmatrix},\tag{C.2}$$

which by virtue of the definition of $\mu_{\mathcal{J}}$ and $\mu_{\mathcal{P}}$ in (4.47), it reads

$$\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=-\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\left(\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\right)^{T}\right]\begin{pmatrix}\varepsilon_{\mathcal{J}}(\varphi)\\ \varepsilon_{\mathcal{P}}(\varphi)\end{pmatrix}.\tag{C.3}$$

Therefore, once (C.1) is evaluated on (C.3), the consistency condition reduces to

$$\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=-\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\left(\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\right)^{T}\right]\begin{pmatrix}\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}.\tag{C.4}$$

Besides, taking the time derivative of (C.1), by virtue of the field equations in (4.46) one can readily show that

$$\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\begin{pmatrix}\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\varphi)} & \frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\right]\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}.\tag{C.5}$$

Then, integrating by parts, the latter equation reads

$$\begin{aligned}\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=&\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\int d\varphi\begin{pmatrix}\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\varphi)} & \frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\\ &-\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\left(\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\right)^{T}\right]\begin{pmatrix}\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix},\end{aligned}\tag{C.6}$$

which reduces to

$$\begin{pmatrix}\dot{\varepsilon}_{\mathcal{J}}(\phi)\\ \dot{\varepsilon}_{\mathcal{P}}(\phi)\end{pmatrix}=-\int d\varphi\left[\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\left(\mathcal{D}^{(2)}\begin{pmatrix}\frac{\delta H^{(k)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta H^{(k)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}\right)^{T}\right]\begin{pmatrix}\frac{\delta\bar{H}^{(j)}}{\delta\mathcal{J}(\varphi)}\\ \frac{\delta\bar{H}^{(j)}}{\delta\mathcal{P}(\varphi)}\end{pmatrix}+\begin{pmatrix}\frac{\delta}{\delta\mathcal{J}(\phi)}\\ \frac{\delta}{\delta\mathcal{P}(\phi)}\end{pmatrix}\{\bar{H}^{(j)},H^{(k)}\},\tag{C.7}$$

where we have made use of the definition of the Poisson brackets in (2.3). Therefore, since the conserved charges commute with the Hamiltonian ($\{\bar{H}^{(j)}, H^{(k)}\} = 0$), the second term at the r.h.s. of (C.7) vanishes; and consequently, the consistency condition evaluated on the parameters (C.1), given by (C.4), has been shown to be fulfilled.

D Generic solution for the field equations of the hierarchy

Here we show that the generic solution in (4.59) solves the field equations (4.49) for an arbitrary value of the label of the hierarchy k, provided that $\mathcal{P}$ stands for an arbitrary generic solution for the field equations of the k-th representative of the KdV hierarchy.

In sum, we want to prove that

$$\mathcal{J} = \sum_{j=0}^{\infty} \eta_j \partial_\phi R^{(j+1)} + c_{\mathcal{J}} \frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} + at\dot{\mathcal{P}}\,, \tag{D.1}$$

is a solution of

$$\dot{\mathcal{J}} = \mathcal{D}^{(\mathcal{P})} T^{(k)} + \left(\mathcal{D}^{(\mathcal{J})} + a\mathcal{D}^{(\mathcal{P})}\right) R^{(k)}\,, \tag{D.2}$$

provided that $\mathcal{P}$ solves

$$\dot{\mathcal{P}} = \mathcal{D}^{(\mathcal{P})} R^{(k)}\,, \tag{D.3}$$

where the Gelfand-Dikii polynomials $R^{(k)}$, and the polynomials $T^{(k)}$ are defined through (4.23) and (4.24), with $\tilde{H}^{(k)}$ given by (4.25), i.e.,

$$R^{(k)} = \frac{\delta H^{(k)}_{\mathrm{KdV}}[\mathcal{P}]}{\delta \mathcal{P}}\,, \tag{D.4}$$

$$T^{(k)} = c_{\mathcal{J}} \frac{\partial R^{(k)}}{\partial c_{\mathcal{P}}} + \int d\varphi \mathcal{J}(\varphi) \frac{\delta R^{(k)}(\varphi)}{\delta \mathcal{P}}\,. \tag{D.5}$$

Thus, once (D.5) is evaluated on (D.1), it reduces to

$$T^{(k)} = \sum_{j=0}^{\infty} \eta_j \int d\varphi \partial_\varphi R^{(j+1)}(\varphi) \frac{\delta R^{(k)}}{\delta \mathcal{P}(\varphi)} + c_{\mathcal{J}} \frac{dR^{(k)}}{dc_{\mathcal{P}}} + at\partial_t R^{(k)}\,, \tag{D.6}$$

where $\frac{dR^{(k)}}{dc_{\mathcal{P}}}$ stands for the total derivative of $R^{(k)}$ with respect to the central charge $c_{\mathcal{P}}$, given by

$$\frac{dR^{(k)}}{dc_{\mathcal{P}}} = \frac{\partial R^{(k)}}{\partial c_{\mathcal{P}}} + \int d\varphi \frac{\partial \mathcal{P}(\varphi)}{\partial c_{\mathcal{P}}} \frac{\delta R^{(k)}}{\delta \mathcal{P}(\varphi)}\,. \tag{D.7}$$

Besides, the time derivative of (D.1) can be written as

$$\begin{aligned}\dot{\mathcal{J}} &= \sum_{j=0}^{\infty} \eta_j \partial_\phi \left[\int d\varphi \frac{\delta R^{(j+1)}}{\delta \mathcal{P}(\varphi)} \dot{\mathcal{P}}(\varphi)\right] + c_{\mathcal{J}} \frac{\partial \dot{\mathcal{P}}}{\partial c_{\mathcal{P}}} + a\partial_t (t\partial_t P) \\ &= a\mathcal{D}^{(\mathcal{P})} R^{(k)} + \partial_\phi \left[-\sum_{j=0}^{\infty} \eta_j \int d\varphi \frac{\delta \partial_\varphi R^{(j+1)}}{\delta \mathcal{P}(\varphi)} R^{(k+1)}(\varphi) + c_{\mathcal{J}} \frac{\partial R^{(k+1)}}{\partial c_{\mathcal{P}}} + at\partial_t R^{(k+1)}\right]\,, \end{aligned} \tag{D.8}$$

where we have made use of the k-th KdV equation in (D.3), as well as the recursion relation for the Gelfand-Dikii polynomials in (4.21). Note that by virtue of (D.4), the following identity holds

$$\int d\varphi \frac{\delta \partial_\varphi R^{(j+1)}}{\delta \mathcal{P}(\varphi)} R^{(k+1)}(\varphi) = \frac{\delta}{\delta \mathcal{P}} \int d\varphi \partial_\varphi R^{(j+1)}(\varphi) R^{(k+1)}(\varphi) - \int d\varphi \partial_\varphi R^{(j+1)}(\varphi) \frac{\delta R^{(k+1)}(\varphi)}{\delta \mathcal{P}}, \tag{D.9}$$

where the first term in the r.h.s. of (D.9) vanishes due to the fact that the conserved charges $H_{\rm KdV}^{(k)}$ are in involution, i.e., $\{H_{\rm KdV}^{(k+1)}, H_{\rm KdV}^{(j+1)}\}_{(1)} = 0$. Hence, eq. (D.9) implies that (D.8) reduces to

$$\begin{aligned} \dot{\mathcal{J}} &= a\mathcal{D}^{(\mathcal{P})} R^{(k)} + \partial_\phi \left[\sum_{j=0}^{\infty} \eta_j \int d\varphi \partial_\varphi R^{(j+1)}(\varphi) \frac{\delta R^{(k+1)}(\varphi)}{\delta \mathcal{P}} + c_{\mathcal{J}} \frac{\partial R^{(k+1)}}{\partial c_{\mathcal{P}}} + at\partial_t R^{(k+1)} \right] \\ &= a\mathcal{D}^{(\mathcal{P})} R^{(k)} + \partial_\phi T^{(k+1)} . \end{aligned} \tag{D.10}$$

Therefore, making use of the recursion relation for the polynomials $T^{(k)}$ in (4.22), one finally proves that eq. (D.10) reduces to the field equation in (D.2), which implies that $\mathcal{J}$ in (D.1) is indeed a solution.

Consequently, making $\eta_j = 0$ in (D.1), one concludes that $\mathcal{J}_p$ in (4.57) provides a particular solution for the field equation (D.2).

The conserved charges associated to this exact solution are then given by $H_{\rm KdV}^{(n)}$ and $\tilde{H}^{(n)}$ defined in (4.25). Note that once $\tilde{H}^{(n)}$ is evaluated on the exact solution (D.1), the contribution due to the homogeneous part vanishes, because

$$\int d\phi \mathcal{J}_h \frac{\delta H_{\rm KdV}^{(n)}}{\delta \mathcal{P}} = \sum_{j=0}^{\infty} \eta_j \int d\phi \partial_\phi R^{(j+1)} \frac{\delta H_{\rm KdV}^{(n)}}{\delta \mathcal{P}} = \sum_{j=0}^{\infty} \eta_j \left\{ H_{\rm KdV}^{(n)}, H_{\rm KdV}^{(j+1)} \right\}_{(1)} = 0 . \tag{D.11}$$

Therefore, $\tilde{H}^{(n)}$ reduces to

$$\tilde{H}^{(n)} = c_{\mathcal{J}} \left(\frac{\partial H_{\rm KdV}^{(n)}}{\partial c_{\mathcal{P}}} + \int d\phi \frac{\partial \mathcal{P}}{\partial c_{\mathcal{P}}} \frac{\delta H_{\rm KdV}^{(n)}}{\delta \mathcal{P}} \right) + at\dot{H}_{\rm KdV}^{(n)} , \tag{D.12}$$

where the last term in (D.12) vanishes since $H_{\rm KdV}^{(n)}$ is conserved. Hence, (D.12) can be written in terms of the total derivative with respect to the central extension $c_{\mathcal{P}}$, according to

$$\tilde{H}^{(n)} = c_{\mathcal{J}} \frac{dH_{\rm KdV}^{(n)}}{dc_{\mathcal{P}}} . \tag{D.13}$$

References

[1] A. Pérez, D. Tempo and R. Troncoso, *Boundary conditions for General Relativity on AdS$_3$ and the KdV hierarchy*, *JHEP* **06** (2016) 103 [arXiv:1605.04490] [INSPIRE].

[2] P. Havas and J. Plebánski, *Conformal extensions of the galilei group and their relation to the schrödinger group*, *J. Math. Phys.* **19** (1978) 482.

[3] A. Bagchi and R. Gopakumar, *Galilean Conformal Algebras and AdS/CFT*, *JHEP* **07** (2009) 037 [arXiv:0902.1385] [INSPIRE].

[4] A. Bagchi, R. Gopakumar, I. Mandal and A. Miwa, *GCA in 2d*, *JHEP* **08** (2010) 004 [arXiv:0912.1090] [INSPIRE].

[5] A. Ashtekar, J. Bicak and B.G. Schmidt, *Asymptotic structure of symmetry reduced general relativity*, *Phys. Rev.* **D 55** (1997) 669 [gr-qc/9608042] [INSPIRE].

[6] G. Barnich and G. Compère, *Classical central extension for asymptotic symmetries at null infinity in three spacetime dimensions*, *Class. Quant. Grav.* **24** (2007) F15 [gr-qc/0610130] [INSPIRE].

[7] G. Barnich and C. Troessaert, *Aspects of the BMS/CFT correspondence*, *JHEP* **05** (2010) 062 [arXiv:1001.1541] [INSPIRE].

[8] G. Barnich, L. Donnay, J. Matulich and R. Troncoso, *Asymptotic symmetries and dynamics of three-dimensional flat supergravity*, *JHEP* **08** (2014) 071 [arXiv:1407.4275] [INSPIRE].

[9] G. Barnich and B. Oblak, *Notes on the BMS group in three dimensions: I. Induced representations*, *JHEP* **06** (2014) 129 [arXiv:1403.5803] [INSPIRE].

[10] G. Barnich and B. Oblak, *Notes on the BMS group in three dimensions: II. Coadjoint representation*, *JHEP* **03** (2015) 033 [arXiv:1502.00010] [INSPIRE].

[11] A. Bagchi, *Tensionless Strings and Galilean Conformal Algebra*, *JHEP* **05** (2013) 141 [arXiv:1303.0291] [INSPIRE].

[12] A. Bagchi, S. Chakrabortty and P. Parekh, *Tensionless Strings from Worldsheet Symmetries*, *JHEP* **01** (2016) 158 [arXiv:1507.04361] [INSPIRE].

[13] I. Mandal and A. Rayyan, *Super-GCA from $\mathcal{N} = (2,2)$ super-Virasoro*, *Phys. Lett.* **B 754** (2016) 195 [arXiv:1601.04723] [INSPIRE].

[14] E. Casali and P. Tourkine, *On the null origin of the ambitwistor string*, *JHEP* **11** (2016) 036 [arXiv:1606.05636] [INSPIRE].

[15] A. Bagchi, S. Chakrabortty and P. Parekh, *Tensionless Superstrings: View from the Worldsheet*, *JHEP* **10** (2016) 113 [arXiv:1606.09628] [INSPIRE].

[16] I. Mandal, *Addendum to "Super-GCA from $\mathcal{N} = (2,2)$ super-Virasoro": Super-GCA connection with tensionless strings*, *Phys. Lett.* **B 760** (2016) 832 [arXiv:1607.02439] [INSPIRE].

[17] E. Casali, Y. Herfray and P. Tourkine, *The complex null string, Galilean conformal algebra and scattering equations*, *JHEP* **10** (2017) 164 [arXiv:1707.09900] [INSPIRE].

[18] A. Bagchi, A. Banerjee, S. Chakrabortty and P. Parekh, *Inhomogeneous Tensionless Superstrings*, arXiv:1710.03482 [INSPIRE].

[19] D. Francia and A. Sagnotti, *On the geometry of higher spin gauge fields*, *Class. Quant. Grav.* **20** (2003) S473 [hep-th/0212185] [INSPIRE].

[20] D. Francia and A. Sagnotti, *Higher-spin geometry and string theory*, *J. Phys. Conf. Ser.* **33** (2006) 57 [hep-th/0601199] [INSPIRE].

[21] A. Sagnotti and M. Taronna, *String Lessons for Higher-Spin Interactions*, *Nucl. Phys.* **B 842** (2011) 299 [arXiv:1006.5242] [INSPIRE].

[22] D. Polyakov, *Interactions of Massless Higher Spin Fields From String Theory*, *Phys. Rev.* **D 82** (2010) 066005 [arXiv:0910.5338] [INSPIRE].

[23] D. Polyakov, *Gravitational Couplings of Higher Spins from String Theory*, *Int. J. Mod. Phys.* **A 25** (2010) 4623 [arXiv:1005.5512] [INSPIRE].

[24] X. Bekaert, N. Boulanger and P. Sundell, *How higher-spin gravity surpasses the spin two barrier: no-go theorems versus yes-go examples*, *Rev. Mod. Phys.* **84** (2012) 987 [arXiv:1007.0435] [INSPIRE].

[25] G. Barnich, A. Gomberoff and H.A. González, *Three-dimensional Bondi-Metzner-Sachs invariant two-dimensional field theories as the flat limit of Liouville theory*, *Phys. Rev.* **D 87** (2013) 124032 [arXiv:1210.0731] [INSPIRE].

[26] G. Barnich and H.A. González, *Dual dynamics of three dimensional asymptotically flat Einstein gravity at null infinity*, *JHEP* **05** (2013) 016 [arXiv:1303.1075] [INSPIRE].

[27] C. Batlle, V. Campello and J. Gomis, *Canonical realization of (2+1)-dimensional Bondi-Metzner-Sachs symmetry*, *Phys. Rev.* **D 96** (2017) 025004 [arXiv:1703.01833] [INSPIRE].

[28] A. Bagchi, *Correspondence between Asymptotically Flat Spacetimes and Nonrelativistic Conformal Field Theories*, *Phys. Rev. Lett.* **105** (2010) 171601 [arXiv:1006.3354] [INSPIRE].

[29] G. Barnich, A. Gomberoff and H.A. González, *The flat limit of three dimensional asymptotically anti-de Sitter spacetimes*, *Phys. Rev.* **D 86** (2012) 024020 [arXiv:1204.3288] [INSPIRE].

[30] H.A. González and M. Pino, *Asymptotically flat spacetimes in 3D bigravity*, *Phys. Rev.* **D 86** (2012) 084045 [arXiv:1207.0882] [INSPIRE].

[31] A. Bagchi, S. Detournay and D. Grumiller, *Flat-Space Chiral Gravity*, *Phys. Rev. Lett.* **109** (2012) 151301 [arXiv:1208.1658] [INSPIRE].

[32] G. Barnich, *Entropy of three-dimensional asymptotically flat cosmological solutions*, *JHEP* **10** (2012) 095 [arXiv:1208.4371] [INSPIRE].

[33] A. Bagchi, S. Detournay, R. Fareghbal and J. Simon, *Holography of 3D Flat Cosmological Horizons*, *Phys. Rev. Lett.* **110** (2013) 141302 [arXiv:1208.4372] [INSPIRE].

[34] C. Krishnan, A. Raju and S. Roy, *A Grassmann path from AdS_3 to flat space*, *JHEP* **03** (2014) 036 [arXiv:1312.2941] [INSPIRE].

[35] A. Bagchi, R. Basu, D. Grumiller and M. Riegler, *Entanglement entropy in Galilean conformal field theories and flat holography*, *Phys. Rev. Lett.* **114** (2015) 111602 [arXiv:1410.4089] [INSPIRE].

[36] A. Bagchi, D. Grumiller and W. Merbis, *Stress tensor correlators in three-dimensional gravity*, *Phys. Rev.* **D 93** (2016) 061502 [arXiv:1507.05620] [INSPIRE].

[37] G. Barnich, C. Troessaert, D. Tempo and R. Troncoso, *Asymptotically locally flat spacetimes and dynamical nonspherically-symmetric black holes in three dimensions*, *Phys. Rev.* **D 93** (2016) 084001 [arXiv:1512.05410] [INSPIRE].

[38] C. Troessaert, D. Tempo and R. Troncoso, *Asymptotically flat black holes and gravitational waves in three-dimensional massive gravity*, in *8th Aegean Summer School: Gravitational Waves: From Theory to Observations Rethymno, Crete, Greece, June 29 – July 4, 2015*, arXiv:1512.09046 [INSPIRE].

[39] A. Bagchi, M. Gary and Zodinmawia, *Bondi-Metzner-Sachs bootstrap*, *Phys. Rev.* **D 96** (2017) 025007 [arXiv:1612.01730] [INSPIRE].

[40] A. Bagchi, M. Gary and Zodinmawia, *The nuts and bolts of the BMS Bootstrap*, *Class. Quant. Grav.* **34** (2017) 174002 [arXiv:1705.05890] [INSPIRE].

[41] H. Afshar et al., *Soft Heisenberg hair on black holes in three dimensions*, *Phys. Rev.* **D 93** (2016) 101503 [arXiv:1603.04824] [INSPIRE].

[42] D. Grumiller, A. Pérez, S. Prohazka, D. Tempo and R. Troncoso, *Higher Spin Black Holes with Soft Hair*, *JHEP* **10** (2016) 119 [arXiv:1607.05360] [INSPIRE].

[43] H. Afshar, D. Grumiller, W. Merbis, A. Pérez, D. Tempo and R. Troncoso, *Soft hairy horizons in three spacetime dimensions*, *Phys. Rev.* **D 95** (2017) 106005 [arXiv:1611.09783] [INSPIRE].

[44] S.W. Hawking, M.J. Perry and A. Strominger, *Soft Hair on Black Holes*, *Phys. Rev. Lett.* **116** (2016) 231301 [arXiv:1601.00921] [INSPIRE].

[45] S.W. Hawking, M.J. Perry and A. Strominger, *Superrotation Charge and Supertranslation Hair on Black Holes*, *JHEP* **05** (2017) 161 [arXiv:1611.09175] [INSPIRE].

[46] A. Strominger, *Black Hole Information Revisited*, arXiv:1706.07143 [INSPIRE].

[47] S.W. Hawking, *The Information Paradox for Black Holes*, arXiv:1509.01147 [INSPIRE].

[48] L. Donnay, G. Giribet, H.A. González and M. Pino, *Supertranslations and Superrotations at the Black Hole Horizon*, *Phys. Rev. Lett.* **116** (2016) 091101 [arXiv:1511.08687] [INSPIRE].

[49] H. Afshar, S. Detournay, D. Grumiller and B. Oblak, *Near-Horizon Geometry and Warped Conformal Symmetry*, *JHEP* **03** (2016) 187 [arXiv:1512.08233] [INSPIRE].

[50] L. Donnay, G. Giribet, H.A. González and M. Pino, *Extended Symmetries at the Black Hole Horizon*, *JHEP* **09** (2016) 100 [arXiv:1607.05703] [INSPIRE].

[51] S. Deser and J.H. Kay, *Topologically massive supergravity*, *Phys. Lett.* **B 120** (1983) 97 [INSPIRE].

[52] S. Deser, *Quantum theory of gravity: essays in honor of the 60th birthday of Bryce S DeWitt*, Adam Hilger Ltd., U.S.A, (1984).

[53] N. Marcus and J.H. Schwarz, *Three-Dimensional Supergravity Theories*, *Nucl. Phys.* **B 228** (1983) 145 [INSPIRE].

[54] G. Barnich, L. Donnay, J. Matulich and R. Troncoso, *Super-BMS$_3$ invariant boundary theory from three-dimensional flat supergravity*, *JHEP* **01** (2017) 029 [arXiv:1510.08824] [INSPIRE].

[55] A. Bagchi and I. Mandal, *Supersymmetric Extension of Galilean Conformal Algebras*, *Phys. Rev.* **D 80** (2009) 086011 [arXiv:0905.0580] [INSPIRE].

[56] I. Mandal, *Supersymmetric Extension of GCA in 2d*, *JHEP* **11** (2010) 018 [arXiv:1003.0209] [INSPIRE].

[57] N. Banerjee, D.P. Jatkar, S. Mukhi and T. Neogi, *Free-field realisations of the BMS$_3$ algebra and its extensions*, *JHEP* **06** (2016) 024 [arXiv:1512.06240] [INSPIRE].

[58] N. Banerjee, D.P. Jatkar, I. Lodato, S. Mukhi and T. Neogi, *Extended Supersymmetric BMS$_3$ algebras and Their Free Field Realisations*, *JHEP* **11** (2016) 059 [arXiv:1609.09210] [INSPIRE].

[59] I. Lodato and W. Merbis, *Super-BMS$_3$ algebras from $\mathcal{N} = 2$ flat supergravities*, *JHEP* **11** (2016) 150 [arXiv:1610.07506] [INSPIRE].

[60] N. Banerjee, I. Lodato and T. Neogi, $N = 4$ *Supersymmetric BMS3 algebras from asymptotic symmetry analysis*, Phys. Rev. **D 96** (2017) 066029 [arXiv:1706.02922] [INSPIRE].

[61] R. Basu, S. Detournay and M. Riegler, *Spectral Flow in 3D Flat Spacetimes*, JHEP **12** (2017) 134 [arXiv:1706.07438] [INSPIRE].

[62] O. Fuentealba, J. Matulich and R. Troncoso, *Asymptotic structure of* $\mathcal{N} = 2$ *supergravity in 3D: extended super-BMS*$_3$ *and nonlinear energy bounds*, JHEP **09** (2017) 030 [arXiv:1706.07542] [INSPIRE].

[63] H. Afshar, A. Bagchi, R. Fareghbal, D. Grumiller and J. Rosseel, *Spin-3 Gravity in Three-Dimensional Flat Space*, Phys. Rev. Lett. **111** (2013) 121603 [arXiv:1307.4768] [INSPIRE].

[64] H.A. González, J. Matulich, M. Pino and R. Troncoso, *Asymptotically flat spacetimes in three-dimensional higher spin gravity*, JHEP **09** (2013) 016 [arXiv:1307.5651] [INSPIRE].

[65] M. Gary, D. Grumiller, M. Riegler and J. Rosseel, *Flat space (higher spin) gravity with chemical potentials*, JHEP **01** (2015) 152 [arXiv:1411.3728] [INSPIRE].

[66] J. Matulich, A. Pérez, D. Tempo and R. Troncoso, *Higher spin extension of cosmological spacetimes in 3D: asymptotically flat behaviour with chemical potentials and thermodynamics*, JHEP **05** (2015) 025 [arXiv:1412.1464] [INSPIRE].

[67] H.A. González and M. Pino, *Boundary dynamics of asymptotically flat 3D gravity coupled to higher spin fields*, JHEP **05** (2014) 127 [arXiv:1403.4898] [INSPIRE].

[68] O. Fuentealba, J. Matulich and R. Troncoso, *Extension of the Poincaré group with half-integer spin generators: hypergravity and beyond*, JHEP **09** (2015) 003 [arXiv:1505.06173] [INSPIRE].

[69] O. Fuentealba, J. Matulich and R. Troncoso, *Asymptotically flat structure of hypergravity in three spacetime dimensions*, JHEP **10** (2015) 009 [arXiv:1508.04663] [INSPIRE].

[70] R. Caroca, P. Concha, E. Rodríguez and P. Salgado-ReboLledó, *Generalizing the* $\mathfrak{bms}_3$ *and 2D-conformal algebras by expanding the Virasoro algebra*, arXiv:1707.07209 [INSPIRE].

[71] P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal field theory*, Graduate texts in contemporary physics, Springer, New York, NY, U.S.A. (1997).

[72] A. Das, *Integrable Models*, World scientific lecture notes in physics, World Scientific, (1989).

[73] M. Dunajski, *Solitons, instantons, and twistors*, Oxford University Press, (2010).

[74] V.G. Drinfeld and V.V. Sokolov, *Lie algebras and equations of Korteweg-de Vries type*, J. Sov. Math. **30** (1984) 1975 [INSPIRE].

[75] P. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, U.S.A., (2000).

[76] I.M. Gel'fand and I. Ya. Dorfman, *Hamiltonian operators and algebraic structures related to them*, Func. Anal. Appl. **13** (1979) 248.

[77] R.F. Penna, BMS$_3$ *invariant fluid dynamics at null infinity*, Class. Quant. Grav. **35** (2018) 044002 [arXiv:1708.08470] [INSPIRE].

[78] M. Taylor, *Non-relativistic holography*, arXiv:0812.0530 [INSPIRE].

[79] G. Bertoldi, B.A. Burrington and A. Peet, *Black Holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent*, Phys. Rev. **D 80** (2009) 126003 [arXiv:0905.3183] [INSPIRE].

JHEP01(2018)148

[80] G. Bertoldi, B.A. Burrington and A.W. Peet, *Thermodynamics of black branes in asymptotically Lifshitz spacetimes*, Phys. Rev. **D 80** (2009) 126004 [arXiv:0907.4755] [INSPIRE].

[81] S.A. Hartnoll, *Lectures on holographic methods for condensed matter physics*, Class. Quant. Grav. **26** (2009) 224002 [arXiv:0903.3246] [INSPIRE].

[82] E. D'Hoker and P. Kraus, *Holographic Metamagnetism, Quantum Criticality and Crossover Behavior*, JHEP **05** (2010) 083 [arXiv:1003.1302] [INSPIRE].

[83] S.A. Hartnoll, *Horizons, holography and condensed matter*, arXiv:1106.4324 [INSPIRE].

[84] H.A. González, D. Tempo and R. Troncoso, *Field theories with anisotropic scaling in 2D, solitons and the microscopic entropy of asymptotically Lifshitz black holes*, JHEP **11** (2011) 066 [arXiv:1107.3647] [INSPIRE].

[85] S.A. Hartnoll, D.M. Ramirez and J.E. Santos, *Emergent scale invariance of disordered horizons*, JHEP **09** (2015) 160 [arXiv:1504.03324] [INSPIRE].

[86] M. Taylor, *Lifshitz holography*, Class. Quant. Grav. **33** (2016) 033001 [arXiv:1512.03554] [INSPIRE].

[87] R. Hirota and J. Satsuma, *Soliton Solutions of a Coupled Korteweg-de Vries Equation*, Phys. Lett. **A 85** (1981) 407 [INSPIRE].

[88] S.Y. Sakovich, *Coupled KdV Equations of Hirota-Satsuma Type*, J. Nonlinear Math. Phys. **6** (1999) 255.

[89] K.M. Tamizhmani and M. Lakshmanan, *Complete integrability of the Korteweg-De Vries equation under perturbation around its solution: Lie-Backlund symmetry approach*, J. Phys. **A 16** (1983) 3773 [INSPIRE].

[90] W.-X. Ma and B. Fuchssteiner, *Integrable theory of the perturbation equations*, Chaos Solitons Fractals **7** (1996) 1227 [solv-int/9604004] [INSPIRE].

[91] W. Ma and B. Fuchssteiner, *The bi-hamiltonian structure of the perturbation equations of the kdv hierarchy*, Phys. Lett. **A 213** (1996) 49.

[92] A. Kalkanli Karasu, *Painleve classification of coupled KdV systems*, J. Math. Phys. **38** (1997) 3616 [INSPIRE].

[93] W.-X. Ma, *Integrable Couplings of Soliton Equations by Perturbations I. A General Theory and Application to the KdV Hierarchy*, solv-int/9912004.

[94] L.C. Vega, A. Restuccia and A. Sotomayor, *Backlund transformation and solitonic solutions for a parametric coupled korteweg-de vries system*, J. Phys. Conf. Ser. **490** (2014) 012024.

[95] A. Restuccia and A. Sotomayor, *Duality relation among the Hamiltonian structures of a parametric coupled Korteweg-de Vries system*, Open Phys. **14** (2016) 95.

[96] P.G. Drazin and R.S. Johnson, *Solitons: An Introduction*, Cambridge University Press (1989).

[97] A. Achucarro and P.K. Townsend, *A Chern-Simons Action for Three-Dimensional anti-de Sitter Supergravity Theories*, Phys. Lett. **B 180** (1986) 89 [INSPIRE].

[98] E. Witten, *(2+1)-Dimensional Gravity as an Exactly Soluble System*, Nucl. Phys. **B 311** (1988) 46 [INSPIRE].

[99] O. Coussaert, M. Henneaux and P. van Driel, *The asymptotic dynamics of three-dimensional Einstein gravity with a negative cosmological constant*, Class. Quant. Grav. **12** (1995) 2961 [gr-qc/9506019] [INSPIRE].

[100] M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Chemical potentials in three-dimensional higher spin anti-de Sitter gravity*, *JHEP* **12** (2013) 048 [arXiv:1309.4362] [INSPIRE].

[101] C. Bunster, M. Henneaux, A. Pérez, D. Tempo and R. Troncoso, *Generalized Black Holes in Three-dimensional Spacetime*, *JHEP* **05** (2014) 031 [arXiv:1404.3305] [INSPIRE].

[102] T. Regge and C. Teitelboim, *Role of Surface Integrals in the Hamiltonian Formulation of General Relativity*, *Annals Phys.* **88** (1974) 286 [INSPIRE].

[103] J.D. Brown and M. Henneaux, *Central Charges in the Canonical Realization of Asymptotic Symmetries: An Example from Three-Dimensional Gravity*, *Commun. Math. Phys.* **104** (1986) 207 [INSPIRE].

[104] L. Cornalba and M.S. Costa, *A new cosmological scenario in string theory*, *Phys. Rev.* **D 66** (2002) 066001 [hep-th/0203031] [INSPIRE].

[105] D. Grumiller, A. Pérez, D. Tempo and R. Troncoso, *Log corrections to entropy of three dimensional black holes with soft hair*, *JHEP* **08** (2017) 107 [arXiv:1705.10605] [INSPIRE].

[106] A. Giacomini, R. Troncoso and S. Willison, *Three-dimensional supergravity reloaded*, *Class. Quant. Grav.* **24** (2007) 2845 [hep-th/0610077] [INSPIRE].

[107] P. Bouwknegt and K. Schoutens, *W symmetry in conformal field theory*, *Phys. Rept.* **223** (1993) 183 [hep-th/9210010] [INSPIRE].

[108] O. Fuentealba et al., work in progress.

[109] G. Compère and W. Song, *$\mathcal{W}$ symmetry and integrability of higher spin black holes*, *JHEP* **09** (2013) 144 [arXiv:1306.0014] [INSPIRE].

[110] M. Gutperle and Y. Li, *Higher Spin Lifshitz Theory and Integrable Systems*, *Phys. Rev.* **D 91** (2015) 046012 [arXiv:1412.7085] [INSPIRE].

[111] M. Beccaria, M. Gutperle, Y. Li and G. Macorini, *Higher spin Lifshitz theories and the Korteweg-de Vries hierarchy*, *Phys. Rev.* **D 92** (2015) 085005 [arXiv:1504.06555] [INSPIRE].

[112] M. Gutperle and Y. Li, *Higher Spin Chern-Simons Theory and the Super Boussinesq hierarchy*, arXiv:1709.02345 [INSPIRE].

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