Applied General Topology

Appl. Gen. Topol. 18, no. 1 (2017), 143-152

doi:10.4995/agt.2017.6663

Digital shy maps

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Communicated by S. Romaguera

Abstract

We study properties of shy maps in digital topology.

2010 MSC: *54C99; 05C99.*

Keywords: *digital image; continuous multivalued function; shy map; isomorphism; Cartesian product; wedge.*

1. Introduction

Continuous functions between digital images were introduced in [15] and have been explored in many subsequent papers. A *shy* map is a continuous function with certain additional restrictions (see Definition 2.5). Shy maps were studied in [4, 6, 9]. In the current paper, we develop additional fundamental properties of shy maps.

In section 3, we examine relationships between shy maps and other types of functions between digital images, such as constant functions and isomorphisms. Theorem 3.2 characterizes shy maps in terms of properties of their multivalued inverse functions. This suggests studying relations between other types of continuous surjections $f : (X, \kappa) \to (Y, \lambda)$ between digital images and their multivalued inverse functions $f^{-1} : Y \multimap X$. In section 4, we show that composition, Cartesian products using the normal product adjacency, and wedges preserve shyness. In section 5, we develop several special properties of shy maps into $(\mathbb{Z}, c_1)$.

Received 28 September 2016 – Accepted 24 December 2016

2. Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, and let $\mathbb{Z}$ be the set of integers. A digital image is a pair (X, κ) where $X \subset \mathbb{Z}^n$ for some positive integer n, and κ is an adjacency relation for $\mathbb{Z}^n$. Thus, a digital image is a graph in which X is the set of vertices and edges correspond to κ-adjacent points of X.

Much of the exposition in this section is quoted or paraphrased from the papers referenced.

2.1. Digitally continuous functions.

We will assume familiarity with the topological theory of digital images. See, e.g., [2] for the standard definitions. All digital images X are assumed to carry their own adjacency relations (which may differ from one image to another). We write the image as (X, κ), where κ represents the adjacency relation, when we wish to emphasize or clarify the adjacency relation.

Among the commonly used adjacencies are the c_u-adjacencies. Let $x, y \in \mathbb{Z}^n$, $x \neq y$. Let u be an integer, $1 \leq u \leq n$. We say x and y are c_u-adjacent if

- There are at most u indices i for which $|x_i - y_i| = 1$.
- For all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

We often label a c_u-adjacency by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

A subset Y of a digital image (X, κ) is *κ-connected* [15], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a sequence $P = \{y_i\}_{i=0}^m \subset Y$ such that $a = y_0$, $b = y_m$, and y_i and y_{i+1} are κ-adjacent for $0 \leq i < m$. The set P is called a *path* from a to b. The following generalizes a definition of [15].

Definition 2.1 ([3])**.** Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is (κ, λ)-continuous if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y. $\square$

When the adjacency relations are understood, we will simply say that f is *continuous.*

Given positive integers u, v such that $u \leq v$ a digital interval is a set of the form

$$[u, v]_{\mathbb{Z}} = \{z \in \mathbb{Z} \,|\, u \leq z \leq v\}$$

treated as a digital image with the c_1-adjacency.

The term *path from a to b* is also used for a continuous function $p : ([0, m]_{\mathbb{Z}}, c_1) \to (Y, \kappa)$ such that $f(0) = a$ and $f(m) = b$. Context generally clarifies which meaning of *path* is appropriate.

Continuity can be reformulated in terms of adjacency of points:

Theorem 2.2 ([15, 3]). *A function $f : X \to Y$ between digital images is continuous if and only if, for any adjacent points $x, x' \in X$, the points $f(x)$ and $f(x')$ are equal or adjacent.* $\square$

Theorem 2.3 ([2, 3]). *Let $f : (A, \kappa) \to (B, \lambda)$ and $g : (B, \lambda) \to (C, \mu)$ be continuous functions between digital images. Then $g \circ f : (A, \kappa) \to (C, \mu)$ is continuous.* $\square$

Definition 2.4. A function $f : X \to Y$ is an *isomorphism* [5] (called a *homeomorphism* in [2]) if f is a continuous bijection and f^{-1} is continuous. $\square$

Definition 2.5 ([4]). Let $f : (X, \kappa) \to (Y, \lambda)$ be a continuous surjection of digital images. We say f is *shy* if

- for every $y \in Y$, $f^{-1}(y)$ is κ-connected; and
- for every λ-adjacent $y_0, y_1 \in Y$, $f^{-1}(\{y_0, y_1\})$ is κ-connected. $\square$

It is known [4] that shy maps induce surjections of fundamental groups. We also have the following.

Theorem 2.6 ([6, 9]). *Let $f : X \to Y$ be a continuous surjection of digital images. Then f is shy if and only if for every connected subset Y' of Y, $f^{-1}(Y')$ is connected.*

2.2. **Normal product adjacency.** The *normal product adjacency* has been used in many papers for Cartesian products of graphs.

Definition 2.7 ([1]). Let (X, κ) and (Y, λ) be digital images. The *normal product adjacency* $k_*(\kappa, \lambda)$ for $X \times Y$ is defined as follows. Two members (x_0, y_0) and (x_1, y_1) of $X \times Y$ are $k_*(\kappa, \lambda)$-adjacent if and only if one of the following is valid.

- $x_0 = x_1$, and y_0 and y_1 are λ-adjacent; or
- x_0 and x_1 are κ-adjacent, and $y_0 = y_1$; or
- x_0 and x_1 are κ-adjacent, and y_0 and y_1 are λ-adjacent. $\square$

We will use the following properties.

Proposition 2.8 ([13]). *The projection maps $p_1 : (X \times Y, k_*(\kappa, \lambda)) \to (X, \kappa)$ and $p_2 : (X \times Y, k_*(\kappa, \lambda)) \to (Y, \lambda)$ defined by $p_1(x, y) = x$, $p_2(x, y) = y$ are $(k_*(\kappa, \lambda), \kappa)$-continuous and $(k_*(\kappa, \lambda), \lambda)$-continuous, respectively.* $\square$

Proposition 2.9 ([8]). *In $\mathbb{Z}^{m+n}$, $k_*(c_m, c_n) = c_{m+n}$; i.e., given points $x, x' \in \mathbb{Z}^m$ and $y, y' \in \mathbb{Z}^n$, (x, y) and (x', y') are $k_*(c_m, c_n)$-adjacent in $\mathbb{Z}^{m+n}$ if and only if they are c_{m+n}-adjacent.* $\square$

2.3. **Digital multivalued functions.** To ameliorate limitations and anomalies that appear in the study of continuous functions between digital images, several authors have considered multivalued functions with various forms of continuity. Functions with *weak continuity* and *strong continuity* were introduced in [16] and studied further in [9]. *Connectivity preserving multivalued functions* were introduced in [14] and studied further in [9]. *Continuous multivalued functions* were introduced in [10, 11] and studied further in [12, 6, 9]. We use the following.

Definition 2.10 ([14]). A multivalued function $f : X \multimap Y$ is *connectivity preserving* if for every connected subset A of X, $f(A)$ is a connected subset of Y. $\square$

Definition 2.11. Let A and B be subsets of a digital image (X, κ). We say A and B are *κ-adjacent*, or *adjacent* for short, if there exist $a \in A$ and $b \in B$ such that either $a = b$ or a and b are κ-adjacent. $\square$

Definition 2.12 ([16]). A multivalued function $f : X \multimap Y$ has *weak continuity* if for every pair x, y of adjacent points in X, the sets $f(x)$ and $f(y)$ are adjacent in Y. $\square$

Proposition 2.13 ([9]). *Let $f : X \multimap Y$ be a multivalued function between digital images. Then f is connectivity preserving if and only if f has weak continuity and for all $x \in X$, $f(x)$ is a connected subset of Y.* $\square$

Theorem 2.14 ([9]). *A continuous surjection $f : X \to Y$ of digital images is shy if and only if $f^{-1} : Y \multimap X$ is a connectivity preserving multivalued function.* $\square$

3. Shy, constant, and isomorphism functions

Proposition 3.1. *A constant map between connected digital images is shy. I.e., if X is a connected digital image and $y \in \mathbb{Z}^n$, then the function $f : X \to \{y\}$ is shy.*

Proof. This is obvious. $\square$

Previous results give the following characterizations of shy maps.

Theorem 3.2. *Let $f : X \to Y$ be a continuous surjection between digital images. The following are equivalent.*

1. *f is a shy map.*
2. *For every connected $Y' \subset Y$, $f^{-1}(Y')$ is a connected subset of X.*
3. *$f^{-1} : Y \multimap X$ is a connectivity preserving multi-valued function.*
4. *$f^{-1} : Y \multimap X$ is a multi-valued function with weak continuity such that for all $y \in Y$, $f^{-1}(y)$ is a connected subset of X.*

Proof. The equivalence of the first three statements follows from Theorem 2.6 and Theorem 2.14. The equivalence of the third and fourth statements follows from Proposition 2.13. $\square$

Theorem 3.3. *Let $f : (X, \kappa) \to (Y, \lambda)$ be a continuous surjection.*

- *If f is an isomorphism then f is shy.*
- *If f is shy and one-to-one, then f is an isomorphism.*

Proof. Since a single point is connected, the first assertion follows easily from Definitions 2.4 and 2.5.

Suppose f is shy and one-to-one. Shyness implies that f is a surjection, hence a bijection; and, from Definitions 2.1 and 2.5, that f^{-1} is continuous. Therefore, f is an isomorphism. $\square$

4. Operations that preserve shyness

We show that composition preserves shyness.

Theorem 4.1. *Let $f : A \to B$ and $g : B \to C$ be shy maps. Then $g \circ f : A \to C$ is shy.*

Proof. Let C' be a connected subset of C. By Theorem 2.6, $g^{-1}(C')$ is a connected subset of B. Therefore, by Theorem 2.6, $(g \circ f)^{-1}(C') = f^{-1}(g^{-1}(C'))$ is a connected subset of A. The assertion follows from Theorem 2.6. □

The following generalizes Proposition 4.1 of [8].

Proposition 4.2. *Let $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$. Then f and g are continuous if and only if the function $f \times g : (A \times B, k_*(\alpha, \beta)) \to (C \times D, k_*(\gamma, \delta))$ defined by $(f \times g)(a, b) = (f(a), g(b))$ is continuous.*

Proof. Suppose f and g are continuous. Let (a, b) and (a', b') be $k_*(\alpha, \beta)$-adjacent. Then a and a' are equal or α-adjacent, so $f(a)$ and $f(a')$ are equal or γ-adjacent; and b and b' are equal or β-adjacent, so $g(b)$ and $g(b')$ are equal or δ-adjacent. It follows from Definition 2.7 that $(f \times g)(a, b)$ and $(f \times g)(a', b')$ are equal or $k_*(\gamma, \delta)$-adjacent. Therefore, $f \times g$ is continuous.

Conversely, suppose $f \times g$ is continuous. From Proposition 2.8, we know the projection maps $p_1 : (X \times Y, k_*(\kappa, \lambda)) \to (X, \kappa)$ and $p_2 : (X \times Y, k_*(\kappa, \lambda)) \to (Y, \lambda)$ defined by $p_1(x, y) = x$, $p_2(x, y) = y$, are continuous. It follows from Theorem 2.3 that $f = p_1 \circ (f \times g)$ and $g = p_2 \circ (f \times g)$ are continuous. □

Proposition 4.3. *Let $f : A \to C$ and $g : B \to D$ be functions. Then the function $f \times g : A \times B \to C \times D$ defined by $(f \times g)(a, b) = (f(a), g(b))$ is a surjection if and only if f and g are surjections.*

Proof. Let $(c, d) \in C \times D$. If f and g are surjections, there are $a \in A$ and $b \in B$ such that $f(a) = c$ and $g(b) = d$. Therefore, $(f \times g)(a, b) = (c, d)$. Thus, $f \times g$ is a surjection.

Conversely, if $f \times g$ is a surjection, it follows easily that f and g are surjections. □

Cartesian products preserve shyness with respect to the normal product adjacency, as shown in the following.

Theorem 4.4. *Let $f : (A, \alpha) \to (C, \gamma)$ and $g : (B, \beta) \to (D, \delta)$ be continuous surjections. Then f and g are shy maps if and only if the function $f \times g : (A \times B, k_*(\alpha, \beta)) \to (C \times D, k_*(\gamma, \delta))$ is a shy map.*

Proof. Suppose f and g are shy. Then they are surjections. By Propositions 4.2 and 4.3, $f \times g$ is a continuous surjection.

Let $(c, d) \in C \times D$ and let $(a, b), (a', b') \in (f \times g)^{-1}(c, d)$. Since $f^{-1}(c)$ is connected, there is a path P in $f^{-1}(c)$ from a to a'. Therefore, $P \times \{b\}$ is a path in $f^{-1}(c) \times \{b\} \subset (f \times g)^{-1}(c, d)$ from (a, b) to (a', b). Since $g^{-1}(d)$ is connected, there is a path Q in $g^{-1}(d)$ from b to b'. Therefore, $\{a'\} \times Q$ is a path in $\{a'\} \times g^{-1}(d) \subset (f \times g)^{-1}(c, d)$ from (a', b) to (a', b'). Thus,

$(P\times\{b\})\cup(\{a'\}\times Q)$ is a path in $(f\times g)^{-1}(c,d)$ from (a,b) to (a',b'). Therefore, $(f\times g)^{-1}(c,d)$ is connected.

Let (c,d) and (c',d') be $k_*(\gamma,\delta)$-adjacent in $C\times D$. Then c and c' are equal or γ-adjacent, and d and d' are equal or δ-adjacent. Let $\{(a,b),(a',b')\}\subset (f\times g)^{-1}(\{(c,d),(c',d')\})$. Since f is shy, $f^{-1}(\{c,c'\})$ is connected, so there is a path P in $f^{-1}(\{c,c'\})$ from a to a'. Similarly, there is a path Q in $g^{-1}(\{d,d'\})$ from b' to b. Thus, $(P\times\{b\})\cup(\{a'\}\times Q)$ is a path in

$$(f^{-1}(\{c,c'\})\times\{b\})\cup(\{a'\}\times g^{-1}(\{d,d'\})\subset(f\times g)^{-1}(\{(c,d),(c',d')\})$$

from (a,b) to (a',b'). Therefore, $(f\times g)^{-1}(\{(c,d),(c',d')\})$ is connected.

It follows from Definition 2.5 that $f\times g$ is a shy map.

Conversely, suppose $f\times g$ is a shy map. It follows from Propositions 4.2 and 4.3 that f and g are continuous surjections.

Let U be a connected subset of C. Then for $d\in D$, $U\times\{d\}$ is a connected subset of $(C\times D,k_*(\gamma,\delta))$. The shyness of $f\times g$ implies

$$f^{-1}(U)\times g^{-1}(\{d\})=(f\times g)^{-1}(U\times\{d\})$$

is connected in $(A\times B,k_*(\alpha,\beta))$. From Proposition 2.8, it follows that

$$f^{-1}(U)=p_1(f^{-1}(U)\times g^{-1}(\{d\}))$$

is connected in A. Thus, f is shy. A similar argument shows g is shy. □

Corollary 4.5. *Let $A\subset\mathbb{Z}^m$, $B\subset\mathbb{Z}^n$, $C\subset\mathbb{Z}^u$, $D\subset\mathbb{Z}^v$. Suppose $f:(A,c_m)\to(C,c_u)$ and $g:(B,c_n)\to(D,c_v)$ are functions. Then the function $f\times g:(A\times B,c_{m+n})\to(C\times D,c_{u+v})$ is a shy map if and only if f and g are shy maps.*

Proof. By Proposition 2.9, in $\mathbb{Z}^{m+n}$, $k_*(c_m,c_n)=c_{m+n}$ and in $\mathbb{Z}^{u+v}$, $k_*(c_u,c_v)=c_{u+v}$. The assertion follows from Theorem 4.4. □

The digital image (C,κ) is the *wedge* of its subsets X and Y, denoted $C=X\wedge Y$, if $C=X\cup Y$ and $X\cap Y=\{x_0\}$ for some point x_0, such that if $x\in X$, $y\in Y$, and x and y are κ-adjacent, then $x_0\in\{x,y\}$. Let $f:A\to C$ and $g:B\to D$ be functions between digital images, with $A\cap B=\{x_0\}$, $C\cap D=\{y_0\}$, and $f(x_0)=y_0=g(x_0)$. We define $f\wedge g:A\wedge B\to C\wedge D$ by

$$(f\wedge g)(x)=\begin{cases} f(x) & \text{if } x\in A;\\ g(x) & \text{if } x\in B.\end{cases}$$

Since $f(x_0)=y_0=g(x_0)$, $f\wedge g$ is well defined. We have the following.

Theorem 4.6. *Let $f:A\to C$ and $g:B\to D$ be functions between digital images, with $A\cap B=\{x_0\}$, $C\cap D=\{y_0\}$, and $f(x_0)=y_0=g(x_0)$. Then f and g are shy maps if and only if $f\wedge g$ is a shy map.*

Proof. If f and g are shy, it is easy to see that $f\wedge g$ is continuous and surjective.

Let U be a connected subset of $C\wedge D$. Let $v_0,v_1\in f^{-1}(U)$. The connectedness of U implies there is a path P in U from $f(v_0)$ to $f(v_1)$, i.e., a continuous $p:[0,m]_{\mathbb{Z}}\to U$ such that $p(0)=f(v_0)$ and $p(m)=f(v_1)$, where $P=p([0,m]_{\mathbb{Z}})$.

- If $U \subset C$, then $(f \wedge g)^{-1}(P) = f^{-1}(P)$ is a connected subset of A.
- If $U \subset D$, then $(f \wedge g)^{-1}(P) = g^{-1}(P)$ is a connected subset of B.
- Otherwise, there are integers $0 = i_1 < i_2 < \cdots < i_n = m$ such that the sets $p([i_j, i_{j+1}]_{\mathbb{Z}})$ alternate between containment in C and containment in D, i.e., without loss of generality, $p([i_j, i_{j+1}]_{\mathbb{Z}}) \subset C \cap U$ for even j, and $p([i_j, i_{j+1}]_{\mathbb{Z}}) \subset D \cap U$ for odd j. Therefore,

$$(f \wedge g)^{-1}(P) = \bigcup_{\text{even } j} f^{-1}(p([i_j, i_{j+1}]_{\mathbb{Z}})) \cup \bigcup_{\text{odd } j} g^{-1}(p([i_j, i_{j+1}]_{\mathbb{Z}}))$$

is a union of connected sets, each containing x_0. Hence $(f \wedge g)^{-1}(P)$ is a connected subset of $(f \wedge g)^{-1}(U)$ containing $\{v_0, v_1\}$. Thus, $(f \wedge g)^{-1}(U)$ is connected.

In all cases, $(f \wedge g)^{-1}(U)$ is connected. We conclude from Theorem 2.6 that $f \wedge g$ is a shy map.

Conversely, suppose $f \wedge g$ is a shy map. Then both f and g are continuous surjections. If U is a connected subset of C and V is a connected subset of D, we have by Theorem 2.6 that $f^{-1}(U) = (f \wedge g)^{-1}(U)$ is a connected subset of A and $g^{-1}(V) = (f \wedge g)^{-1}(V)$ is a connected subset of B. From Theorem 2.6, it follows that f and g are shy maps. $\square$

5. Shy maps into $\mathbb{Z}$

Theorem 5.1. *Let X and Y be connected subsets of $(\mathbb{Z}, c_1)$, and let $f : X \to Y$ be a continuous surjection. Then f is shy if and only if f is either monotone non-decreasing or monotone non-increasing.*

Proof. Suppose f is shy. If f is not monotone, then either

$$\text{for some } a, b, c \in X \text{ with } a < b < c,\ f(a) < f(b) \text{ and } f(b) > f(c), \tag{5.1}$$

or

$$\text{for some } a, b, c \in X \text{ with } a < b < c,\ f(a) > f(b) \text{ and } f(b) < f(c). \tag{5.2}$$

In case (5.1), the continuity of f implies there exist $s, t \in X$ such that $a \leq s < b < t \leq c$ and $f(s) = f(t) = f(b) - 1$. Thus, $s, t \in f^{-1}(f(b) - 1)$ and $b \notin f^{-1}(f(b) - 1)$, so $f^{-1}(f(b) - 1)$ is not c_1-connected, a contradiction of the shyness of f. Case (5.2) generates a similar contradiction. Thus, we obtain a contradiction by assuming that f is not monotone.

Suppose f is monotone. We may assume without loss of generality that f is non-decreasing. Let Y' be a connected subset of Y and let $x_0, x_1 \in f^{-1}(Y')$. We need to show there is a connected subset Y'' of $f^{-1}(Y')$ such that $\{x_0, x_1\} \subset Y''$.

- If $x_0 = x_1$ we can take $Y'' = \{x_0, x_1\}$.
- Otherwise, without loss of generality, $x_0 < x_1$. Since f is continuous and non-decreasing, $f([x_0, x_1]_{\mathbb{Z}}) = [f(x_0), f(x_1)]_{\mathbb{Z}}$ is a connected set containing $\{f(x_0), f(x_1)\} \subset Y'$, so we can take $Y'' = [x_0, x_1]_{\mathbb{Z}}$.

Thus,

$$[x_0, x_1]_{\mathbb{Z}} \subset f^{-1}([f(x_0), f(x_1)]_{\mathbb{Z}}) \subset f^{-1}(Y'),$$

so $[x_0, x_1]_{\mathbb{Z}}$ is a connected subset of $f^{-1}(Y')$ containing $\{x_0, x_1\}$. Since x_0 and x_1 were arbitrarily chosen, we must have that $f^{-1}(Y')$ is c_1-connected. Therefore, f^{-1} is a connectivity preserving multivalued function. It follows from Theorem 2.6 that f is shy. □

A *digital simple closed curve* is a set $S = \{x_i\}_{i=0}^m \subset (\mathbb{Z}^n, \kappa)$, for some $m, n \in \mathbb{N}$ with $m \geq 4$ and some adjacency κ, such that $i \neq j$ implies $x_i \neq x_j$, and x_i and x_j are κ-adjacent if and only if $j \in \{(i-1) \mod m, (i+1) \mod m\}$.

Theorem 5.2. *Let S be a digital simple closed curve. Let $f : S \to Y \subset (\mathbb{Z}, c_1)$ be a shy map. Then either $Y = \{z\}$ or $Y = \{z, z+1\}$ for some $z \in \mathbb{Z}$.*

Proof. If f is not a constant function, i.e., if $Y \neq \{z\}$, then

$$z_0 = \min\{f(x) \,|\, x \in S\} < \max\{f(x) \,|\, x \in S\} = z_1.$$

Let $x_i \in f^{-1}(z_i)$, $i \in \{0, 1\}$. There are two distinct digital arcs, A and B, connecting x_0 and x_1 in S. If $z_1 - z_0 > 1$, then the continuity of $f|_A$ and $f|_B$ implies there are points $a \in A$ and $b \in B$ such that $f(a) = f(b) = z_0 + 1$. Since a and b are in distinct components of $S \setminus (f^{-1}(\{z_0, z_1\})$, $f^{-1}(z_0 + 1)$ is disconnected. This is contrary to the assumption that f is shy. The assertion follows. □

Theorem 5.3. *Let (X, κ) be a connected digital image and let $r \in X$ be such that $X \setminus \{r\}$ is κ-disconnected. Let $f : (X, \kappa) \to Y \subset (\mathbb{Z}, c_1)$ be a shy map. Then there are at most 2 components of $X \setminus \{r\}$ on which f is not equal to the constant function with value $f(r)$.*

Proof. Suppose A is a component of $X \setminus \{r\}$ on which f is not constant. Since X is connected and $X \setminus \{r\}$ is not, by continuity of f, there exists $a \in A$ such that $|f(a) - f(r)| = 1$. Suppose B is another component of $X \setminus \{r\}$ on which f is not constant. Similarly, there exists $b \in b$ such that $|f(b) - f(r)| = 1$. We must have $f(a) \neq f(b)$, since $f^{-1}(f(a))$ is connected and every path in X from a to b contains r. Therefore, we may assume $f(a) = f(r) - 1$, $f(b) = f(r) + 1$.

Suppose C is a component of $X \setminus \{r\}$ that is distinct from A and B. Suppose there exists $c \in C$ such that $f(c) \neq f(r)$. If $f(c) < f(r)$ then, by continuity of f, there exists $c' \in C$ such that $f(c') = f(r) - 1 = f(a)$. But this is impossible, since $f^{-1}(f(a))$ is connected and every path in X from a to c' contains r. Similarly, if we assume $f(c) > f(r)$ we get a contradiction. The assertion follows. □

Example 5.4. Let T be a tree. Let r be the root vertex of T. Let $\{v_i\}_{i=0}^m$ be the set of vertices adjacent to r. Let T_i be the subtree of T with vertices r, v_i, and the descendants of v_i in T (see Figure 1). Let $f : T \to Y \subset (\mathbb{Z}, c_1)$ be a shy function. Then f is constant on all but at most 2 of the T_i.

Proof. The assertion follows from Theorem 5.3. □

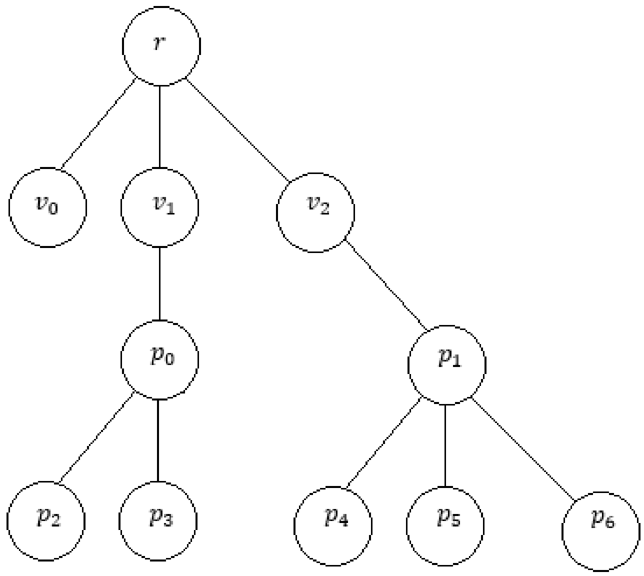

FIGURE 1. A tree T to illustrate Example 5.4. The vertex sets of T_0, T_1, and T_2 are, respectively, $\{r, v_0\}$, $\{r, v_1, p_0, p_2, p_3\}$, and $\{r, v_2, p_1, p_4, p_5, p_6\}$. A shy map from T to a subset of $(\mathbb{Z}, c_1)$ is non-constant on at most 2 of T_0, T_1, and T_2.

6. Further remarks

We have made several contributions to our knowledge of digital shy maps. In section 3, we studied the relations between shy maps and both constant functions and isomorphisms. In section 4, we showed that shyness is preserved by compositions, certain Cartesian products, and wedges. In section 5, we demonstrated several restrictions on shy maps onto subsets of $(\mathbb{Z}, c_1)$.

Additional results concerning shy maps, obtained after the initial submission of this paper, appear in [7].

This research did not receive any specific grant from funding agencies in the public, commercial, or not-for-profit sectors.

ACKNOWLEDGEMENTS. *The corrections and suggestions of the anonymous reviewers are gratefully acknowledged.*

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