Chang et al. *Fixed Point Theory and Applications* (2015) 2015:208
DOI 10.1186/s13663-015-0458-3

Fixed Point Theory and Applications
a SpringerOpen Journal

RESEARCH Open Access

Split equality fixed point problem for quasi-pseudo-contractive mappings with applications

Shih-sen Chang[1*], Lin Wang[2] and Li-Juan Qin[3]

*Correspondence:
changss2013@163.com
[1]Center for General Education, China Medical University, Taichung, 40402, Taiwan
Full list of author information is available at the end of the article

Abstract
In this paper, we consider a split equality fixed point problem for quasi-pseudo-contractive mappings which includes split feasibility problem, split equality problem, split fixed point problem *etc.*, as special cases. A unified framework for the study of this kind of problems and operators is provided. The results presented in the paper extend and improve many recent results.

MSC: 49J40; 49J52; 47J20

Keywords: split equality fixed point problem; quasi-pseudo-contractive mapping; quasi-nonexpansive mapping; directed mapping; demicontractive mapping

1 Introduction

Let C and Q be nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively. The *split feasibility problem* (*SFP*) is formulated as:

$$\text{to find } x^* \in C \text{ such that } Ax^* \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. In 1994, Censor and Elfving [1] first introduced the SFP in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. It has been found that the SFP can also be used in various disciplines such as image restoration, computer tomography, and radiation therapy treatment planning [3–5]. The SFP in an infinite-dimensional real Hilbert space can be found in [2, 4, 6–10].

Recently, Moudafi [11–13] introduced the following *split equality feasibility problem* (SEFP):

$$\text{to find } x \in C, y \in Q \text{ such that } Ax = By, \tag{1.2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators. Obviously, if $B = I$ (identity mapping on H_2) and $H_3 = H_2$, then (1.2) reduces to (1.1). The kind of split equality feasibility problems (1.2) allows asymmetric and partial relations between the variables x and y. The interest is to cover many situations, such as decomposition methods for PDEs, applications in game theory and intensity-modulated radiation therapy.

In order to solve split equality feasibility problem (1.2), Moudafi [11] introduced the following simultaneous iterative method:

$$\begin{cases} x_{k+1} = P_C(x_k - \gamma A^*(Ax_k - By_k)), \\ y_{k+1} = P_Q(y_k + \beta B^*(Ax_{k+1} - By_k)), \end{cases} \tag{1.3}$$

and under suitable conditions he proved the weak convergence of the sequence $\{(x_n, y_n)\}$ to a solution of (1.2) in Hilbert spaces.

In order to avoid using the projection, recently, Moudafi [13] introduced and studied the following problem: Let $T : H_1 \to H_1$ and $S : H_2 \to H_2$ be nonlinear operators such that $\mathrm{Fix}(T) \neq \emptyset$ and $\mathrm{Fix}(S) \neq \emptyset$, where $\mathrm{Fix}(T)$ and $\mathrm{Fix}(S)$ denote the sets of fixed points of T and S, respectively. If $C = \mathrm{Fix}(T)$ and $Q = \mathrm{Fix}(S)$, then split equality problem (1.2) reduces to

$$\text{find } x \in \mathrm{Fix}(T) \text{ and } y \in \mathrm{Fix}(S) \text{ such that } Ax = By, \tag{1.4}$$

which is called a *split equality fixed point problem (in short, SEFPP)*.

Denote by Γ the solution set of split equality fixed point problem (1.4).

Recently Moudafi [13] proposed the following iterative algorithm for finding a solution of SEFPP (1.4):

$$\begin{cases} x_{n+1} = T(x_n - \gamma_n A^*(Ax_n - By_n)), \\ y_{n+1} = S(y_n + \beta_n B^*(Ax_{n+1} - By_n)). \end{cases} \tag{1.5}$$

He also studied the weak convergence of the sequences generated by scheme (1.5) under the condition that T and S are firmly quasi-nonexpansive mappings. Very recently, Che and Li [14] proposed the following iterative algorithm for finding a solution of SEFPP (1.4):

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tu_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n)Sv_n. \end{cases} \tag{1.6}$$

They also established the weak convergence of the scheme (1.6) under the condition that the operators T and S are quasi-nonexpansive mappings.

The purpose of this paper is two-fold. First, we will consider split equality fixed point problem (1.4) for the class of quasi-pseudo-contractive mappings which is more general than the classes of quasi-nonexpansive mappings, directed mappings, and demicontractive mappings. Second, we modify the iterative scheme (1.6) and propose the following iterative algorithms with weak convergence without using the projection:

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)((1 - \xi)I + \xi T((1 - \eta)I + \eta T))u_n, \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n)((1 - \xi)I + \xi S((1 - \eta)I + \eta S))v_n. \end{cases} \tag{1.7}$$

Our results provide a unified framework for the study of this kind of problems and this class of operators.

2 Preliminaries

In this section, we collect some definitions, notations, and conclusions, which will be needed in proving our main results.

Let H be a real Hilbert space, C be a nonempty closed convex subset of H, and $T : C \to C$ be a nonlinear mapping.

Definition 2.1 $T : C \to C$ is said to be:

(i) Nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ $\forall x, y \in C$.

(ii) Quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - x^*\| \leq \|x - x^*\| \quad \forall x \in C \text{ and } x^* \in \mathrm{Fix}(T).$$

(iii) Firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C.$$

(iv) Firmly quasi-nonexpansive if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|(I - T)x\|^2 \quad \forall x \in C \text{ and } x^* \in \mathrm{Fix}(T).$$

(v) Strictly pseudo-contractive if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C.$$

(vi) Directed if $\mathrm{Fix}(T) \neq \emptyset$ and $\langle Tx - x^*, Tx - x\rangle \leq 0$ $\forall x \in C$ and $x^* \in \mathrm{Fix}(T)$.

(vii) Demicontractive if $\mathrm{Fix}(T) \neq \emptyset$ and there exists $k \in [0, 1)$ such that

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 + k\|Tx - x\|^2 \quad \forall x \in C \text{ and } x^* \in \mathrm{Fix}(T).$$

Remark 2.2 As pointed out by Bauschke and Combettes [15], $T : C \to C$ is directed if and only if

$$\|Tx - x^*\|^2 \leq \|x - x^*\|^2 - \|Tx - x\|^2 \quad \forall x \in C \text{ and } x^* \in F(T).$$

That is to say that the class of directed mappings coincides with that of firmly quasi-nonexpansive mappings.

Remark 2.3 From the above definitions, we note that the class of demicontractive mappings is fundamental; it includes many kinds of nonlinear mappings such as the directed mappings, the quasi-nonexpansive mappings, and the strictly pseudo-contractive mappings with fixed points as special cases.

Definition 2.4 An operator $T : C \to C$ is said to be *pseudo-contractive* if

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 \quad \forall x, y \in C.$$

The interest of pseudo-contractive operators lies in their connection with monotone mappings, namely, T is a pseudo-contraction if and only if $I - T$ is a monotone mapping. It is well known that T is pseudo-contractive if and only if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2 \quad \forall x, y \in C.$$

Definition 2.5 An operator $T : C \to C$ is said to be *quasi-pseudo-contractive* if $\mathrm{Fix}(T) \neq \emptyset$ and

$$\|Tx - x^*\|^2 \le \|x - x^*\|^2 + \|Tx - x\|^2 \quad \forall x \in C \text{ and } x^* \in F(T).$$

It is obvious that the class of quasi-pseudo-contractive mappings includes the class of demicontractive mappings.

Definition 2.6 (1) A mapping $T : C \to C$ is said to be *demiclosed at 0* if, for any sequence $\{x_n\} \subset C$ which converges weakly to x and with $\|x_n - T(x_n)\| \to 0$, $T(x) = x$.

(2) A mapping $T : H \to H$ is said to be semi-compact if, for any bounded sequence $\{x_n\} \subset H$ with $\|x_n - Tx_n\| \to 0$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $\{x_{n_i}\}$ converges strongly to some point $x \in H$.

Lemma 2.7 *Let H be a real Hilbert space. For any $x, y \in H$, the following conclusions hold*:

$$\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \quad t \in [0;1]; \tag{2.1}$$

$$\|x + y\|^2 \le \|x\|^2 + 2\langle y, x + y\rangle. \tag{2.2}$$

Recall that a Banach space X is said to satisfy Opial's condition, if for any sequence $\{x_n\}$ in X which converges weakly to x^*,

$$\limsup_{n\to\infty} \|x_n - x^*\| < \limsup_{n\to\infty} \|x_n - y\| \quad \forall y \in X \text{ with } y \neq x^*.$$

It is well known that every Hilbert space satisfies the Opial condition.

Lemma 2.8 *Let $\{a_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - \gamma_n)a_n + \delta_n \quad \forall n \ge 1,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(2) $\limsup_{n\to\infty} \frac{\delta_n}{\gamma_n} \le 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9 *Let H be a real Hilbert space and $T : H \to H$ be a L-Lipschitzian mapping with $L \ge 1$. Denote*

$$K := (1 - \xi)I + \xi T\bigl((1 - \eta)I + \eta T\bigr). \tag{2.3}$$

If $0 < \xi < \eta < \frac{1}{1+\sqrt{1+L^2}}$, *then the following conclusions hold*:

(1) $\mathrm{Fix}(T) = \mathrm{Fix}(T((1-\eta)I + \eta T)) = \mathrm{Fix}(K)$.
(2) *If T is demiclosed at* 0, *then K is also demiclosed at* 0.
(3) *In addition, if $T : H \to H$ is quasi-pseudo-contractive, then the mapping K is quasi-nonexpansive, that is,*

$$\|Kx - u^*\| \le \|x - u^*\| \quad \forall x \in H \text{ and } u^* \in \mathrm{Fix}(T) = \mathrm{Fix}(K).$$

Proof (1) If $x^* \in \mathrm{Fix}(T)$, it is obvious that $x^* \in \mathrm{Fix}(T((1-\eta)I + \eta T))$.

Conversely, if $x^* \in \mathrm{Fix}(T((1-\eta)I + \eta T))$, *i.e.*, $x^* = T((1-\eta)x^* + \eta Tx^*)$, letting $U = (1-\eta)I + \eta T$, then $TUx^* = x^*$. Put $Ux^* = y^*$. Then $Ty^* = x^*$. Now we prove that $x^* = y^*$. In fact, we have

$$\begin{aligned}\|x^* - y^*\| &= \|x^* - Ux^*\| = \|x^* - \big((1-\eta)I + \eta T\big)x^*\| \\ &= \eta\|x^* - Tx^*\| = \eta\|Ty^* - Tx^*\| \le L\eta\|y^* - x^*\|.\end{aligned}$$

Since $0 < L\eta < 1$, we have $x^* = y^*$, *i.e.*, $x^* \in \mathrm{Fix}(T)$. This shows that $\mathrm{Fix}(T) = \mathrm{Fix}(T((1-\eta)I + \eta T))$.

It is obvious that $x \in \mathrm{Fix}(K)$ if and only if $x \in \mathrm{Fix}(T((1-\eta)I + \eta T))$.

The conclusion (1) is proved.

(2) For any sequence $\{x_n\} \subset H$ satisfying $x_n \rightharpoonup x^*$ and $\|x_n - Kx_n\| \to 0$. Next we show that $x^* \in \mathrm{Fix}(K)$. From conclusion (1), we only need to prove that $x^* \in \mathrm{Fix}(T)$. In fact, since T is L-Lipschizian, we have

$$\begin{aligned}\|x_n - Tx_n\| &\le \|x_n - T\big((1-\eta)I + \eta T\big)x_n\| + \|T\big((1-\eta)I + \eta T\big)x_n - Tx_n\| \\ &\le \frac{1}{\xi}\|x_n - (1-\xi)x_n - \xi T\big((1-\eta)I + \eta T\big)x_n\| + L\eta\|x_n - Tx_n\| \\ &= \frac{1}{\xi}\|x_n - Kx_n\| + L\eta\|x_n - Tx_n\|.\end{aligned}$$

Simplifying it, we have

$$\|x_n - Tx_n\| \le \frac{1}{\xi(1-L\eta)}\|x_n - Kx_n\| \to 0. \tag{2.4}$$

Since T is demiclosed at 0, we have $x^* \in F(T) = F(K)$. The conclusion (2) is proved.

The conclusion (3) is obvious (see also [16]). □

3 Main results

Throughout this section, we assume that:

(1) H_1, H_2, and H_3 are three real Hilbert spaces. $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operators with their adjoints A^* and B^*, respectively;
(2) $T : H_1 \to H_1$ and $S : H_2 \to H_2$ are two L-Lipschitzian and quasi-pseudo-contractive mappings with $L \ge 1$, $\mathrm{Fix}(T) \ne \emptyset$, and $\mathrm{Fix}(S) \ne \emptyset$.

In the sequel, we denote the strong convergence and weak convergence of a sequence $\{x_n\}$ to a point $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively.

Our object is to solve the following split equality fixed point problem:

$$\text{to find } x^* \in \operatorname{Fix}(T), y^* \in F(S) \text{ such that } Ax^* = By^*. \tag{3.1}$$

In the sequel we use Γ to denote the set of solutions of (3.1), that is,

$$\Gamma = \{(x^*, y^*) \in \operatorname{Fix}(T) \times \operatorname{Fix}(S) \text{ such that } Ax^* = By^*\}, \tag{3.2}$$

and we assume that $\Gamma \neq \emptyset$.

Now, we present our algorithm for finding $(x^*, y^*) \in \Gamma$.

Algorithm 3.1 **Initialization**: Choose $\{\alpha_n\} \subset (0,1)$. Take arbitrary $x_0 \in H_1$, $y_0 \in H_2$.

Iterative steps: For a given current $x_n \in H_1$, $y_n \in H_2$ compute

$$\begin{cases} \text{(a)} \quad u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ \text{(b)} \quad x_{n+1} = \alpha_n x_n + (1-\alpha_n)((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n, \\ \text{(c)} \quad v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ \text{(d)} \quad y_{n+1} = \alpha_n y_n + (1-\alpha_n)((1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S))v_n. \end{cases} \tag{3.3}$$

Theorem 3.2 *Let $H_1, H_2, H_3, A, B, S, T, \Gamma, \{x_n\}$ and $\{y_n\}$ be the same as above. If T and S are demiclosed at* 0 *and the following conditions are satisfied*:

(i) $\gamma_n \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ $\forall n \geq 1$;

(ii) $0 < a < \xi_n < \eta_n < b < \frac{1}{1+\sqrt{1+L^2}} \forall n \geq 1$.

Then the following conclusions hold:

(I) *the sequence* $(\{x_n, y_n\})$ *generated by* (3.3) *converges weakly to a solution of problem* (3.1);

(II) *In addition, if S, T are also semi-compact, then* $(\{x_n, y_n\})$ *converges strongly to a solution of problem* (3.1).

Proof First we prove the conclusion (I).

For any given $(p,q) \in \Gamma$, then $p \in \operatorname{Fix}(T)$, $q \in \operatorname{Fix}(S)$ and $Ap = Bq$. From (3.3)(a), we have

$$\begin{aligned} \|u_n - p\|^2 &= \|x_n - \gamma_n A^*(Ax_n - By_n) - p\|^2 \\ &= \|x_n - p\|^2 + \gamma_n^2 \|A^*(Ax_n - By_n)\|^2 - 2\gamma_n \langle x_n - p, A^*(Ax_n - By_n)\rangle \\ &\leq \|x_n - p\|^2 + \gamma_n^2 \|A\|^2 \|Ax_n - By_n\|^2 - 2\gamma_n \langle Ax_n - Ap, Ax_n - By_n\rangle. \end{aligned} \tag{3.4}$$

Similarly, from (3.3)(c), we have

$$\|v_n - q\|^2 \leq \|y_n - q\|^2 + \gamma_n^2 \|B\|^2 \|Ax_n - By_n\|^2 + 2\gamma_n \langle By_n - Bq, Ax_n - By_n\rangle. \tag{3.5}$$

Put

$$\begin{aligned} K &:= (1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T), \\ G &:= (1-\xi_n)I + \xi_n S((1-\eta_n)I + \eta_n S). \end{aligned}$$

By the assumptions of Theorem 3.2, condition (ii) and Lemma 2.9, we know that the mappings K and G have the following properties:

(1) Both K and G are quasi-nonexpansive;
(2) $\mathrm{Fix}(K) = \mathrm{Fix}(T)$ and $\mathrm{Fix}(G) = \mathrm{Fix}(S)$;
(3) K and G demiclosed at 0.

Hence from (3.3)(b) and (2.1) we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \bigl\|\alpha_n x_n + (1-\alpha_n)\bigl((1-\xi_n)I + \xi_n T\bigl((1-\eta_n)I + \eta_n T\bigr)\bigr)u_n - p)\bigr\|^2 \\ &= \bigl\|\alpha_n(x_n - p) + (1-\alpha_n)(Ku_n - p)\bigr\|^2 \\ &= \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|Ku_n - p\|^2 - \alpha_n(1-\alpha_n)\|Ku_n - x_n\|^2 \\ &\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\|u_n - p\|^2 - \alpha_n(1-\alpha_n)\|Ku_n - x_n\|^2. \end{aligned} \tag{3.6}$$

Similarly from (3.3)(c) and (2.1) we have

$$\|y_{n+1}-q\|^2 \le \alpha_n\|y_n - q\|^2 + (1-\alpha_n)\|v_n - q\|^2 - \alpha_n(1-\alpha_n)\|Gv_n - y_n\|^2. \tag{3.7}$$

Adding (3.6) and (3.7) and by virtue of (3.4) and (3.5), we have

$$\begin{aligned}&\|x_{n+1}-p\|^2 + \|y_{n+1}-q\|^2 \\ &\quad\le \alpha_n\|x_n-p\|^2 + \alpha_n\|y_n-q\|^2 + (1-\alpha_n)\|u_n-p\|^2 + (1-\alpha_n)\|v_n-q\|^2 \\ &\qquad - \alpha_n(1-\alpha_n)\|Ku_n-x_n\|^2 - \alpha_n(1-\alpha_n)\|Gv_n-y_n\|^2 \\ &\quad\le \alpha_n\|x_n-p\|^2 + (1-\alpha_n)\bigl\{\|x_n-p\|^2 + \gamma_n^2\|A\|^2\|Ax_n-By_n\|^2 \\ &\qquad - 2\gamma_n\langle Ax_n-Ap, Ax_n-By_n\rangle\bigr\} \\ &\qquad + \alpha_n\|y_n-q\|^2 + (1-\alpha_n)\bigl\{\|y_n-p\|^2 + \gamma_n^2\|B\|^2\|Ax_n-By_n\|^2 \\ &\qquad + 2\gamma_n\langle By_n-Bq, Ax_n-By_n\rangle\bigr\} \\ &\qquad - \alpha_n(1-\alpha_n)\|Ku_n-x_n\|^2 - \alpha_n(1-\alpha_n)\|Gv_n-y_n\|^2 \\ &\quad= \|x_n-p\|^2 + \|y_n-q\|^2 + \gamma_n^2(1-\alpha_n)\bigl\{\|A\|^2 + \|B\|^2\bigr\}\|Ax_n-By_n\|^2 \\ &\qquad - (1-\alpha_n)2\gamma_n\bigl\{\langle Ax_n-Ap, Ax_n-By_n\rangle - \langle By_n-Bq, Ax_n-By_n\rangle\bigr\} \\ &\qquad - \alpha_n(1-\alpha_n)\bigl\{\|Ku_n-x_n\|^2 + \|Gv_n-y_n\|^2\bigr\} \\ &\quad= \|x_n-p\|^2 + \|y_n-q\|^2 + \gamma_n^2(1-\alpha_n)\bigl\{\|A\|^2 + \|B\|^2\bigr\}\|Ax_n-By_n\|^2 \\ &\qquad - (1-\alpha_n)2\gamma_n\|Ax_n-By_n\|^2 - \alpha_n(1-\alpha_n)\bigl\{\|Ku_n-x_n\|^2 + \|Gv_n-y_n\|^2\bigr\} \\ &\qquad (\text{since } Ap = Bq) \\ &\quad= \|x_n-p\|^2 + \|y_n-q\|^2 - (1-\alpha_n)\gamma_n\bigl(2 - \gamma_n\bigl(\|A\|^2 + \|B\|^2\bigr)\bigr)\|Ax_n-By_n\|^2 \\ &\qquad - \alpha_n(1-\alpha_n)\bigl\{\|Ku_n-x_n\|^2 + \|Gv_n-y_n\|^2\bigr\}. \end{aligned} \tag{3.8}$$

Since $\gamma_n \in (0, \min\{\frac{1}{\|A\|^2}, \frac{1}{\|A\|^2}\})$, $\gamma_n\|A\|^2 < 1$ and $\gamma_n\|B\|^2 < 1$. So $0 < \gamma_n(\|A\|^2 + \|B\|^2) < 2$. This implies that $\gamma_n(2 - \gamma_n(\|A\|^2 + \|B\|^2)) > 0$.

Putting

$$X_n(p,q) = \|x_n - p\|^2 + \|y_n - q\|^2, \tag{3.9}$$

hence (3.8) can be written as

$$\begin{aligned} X_{n+1}(p,q) &\le X_n(p,q) - (1-\alpha_n)\gamma_n\big(2-\gamma_n\big(\|A\|^2+\|B\|^2\big)\big)\|Ax_n - By_n\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\big\{\|Ku_n - x_n\|^2 + \|Gv_n - y_n\|^2\big\} \\ &\le X_n(p,q). \end{aligned} \tag{3.10}$$

This implies that $\{X_n(p,q)\}$ is a non-increasing sequence, hence the limit $\lim_{n\to\infty} X_n(p,q)$ exists. Therefore the following limits exist:

$$\lim_{n\to\infty}\|x_n - p\| \quad \text{and} \quad \lim_{n\to\infty}\|y_n - q\| \quad \forall (p,q)\in\Gamma. \tag{3.11}$$

Rewritten (3.10) as

$$\begin{aligned} &(1-\alpha_n)\gamma_n\big(2-\gamma_n\big(\|A\|^2+\|B\|^2\big)\big)\|Ax_n - By_n\|^2 \\ &\quad + \alpha_n(1-\alpha_n)\big\{\|Ku_n - x_n\|^2 + \|Gv_n - y_n\|^2\big\} \le X_n(p,q) - X_{n+1}(p,q). \end{aligned} \tag{3.12}$$

Letting $n\to\infty$ and taking the limit in (3.12), we have

$$\|Ax_n - By_n\| \to 0; \qquad \|Ku_n - x_n\| \to 0; \qquad \|Gv_n - y_n\| \to 0. \tag{3.13}$$

From (3.13) and (3.3) we have

$$\begin{cases} \lim_{n\to\infty}\|u_n - x_n\| \to 0 \text{ and } \lim_{n\to\infty}\|v_n - y_n\| \to 0, \\ \lim_{n\to\infty}\|x_{n+1} - x_n\| \\ \quad = \lim_{n\to\infty}(1-\alpha_n)\|((1-\xi_n)I + \xi_n T((1-\eta_n)I + \eta_n T))u_n - x_n\| \\ \quad = \lim_{n\to\infty}(1-\alpha_n)\|Ku_n - x_n\| = 0, \\ \lim_{n\to\infty}\|y_{n+1} - y_n\| \\ \quad = \lim_{n\to\infty}(1-\beta_n)\|((1-\xi_n)S + \xi_n S((1-\eta_n)I + \eta_n S))y_n - y_n\| \\ \quad = \lim_{n\to\infty}(1-\alpha_n)\|Gv_n - y_n\| = 0. \end{cases} \tag{3.14}$$

This together with (3.13) shows that

$$\begin{cases} \|Ku_n - u_n\| \le \|Ku_n - x_n\| + \|x_n - u_n\| \to 0; \\ \|Gv_n - v_n\| \le \|Gv_n - y_n\| + \|y_n - v_n\| \to 0. \end{cases} \tag{3.15}$$

Since $\{x_n\}$ and $\{y_n\}$ are bounded sequences, there exist some weakly convergent subsequences, say $\{x_{n_i}\}\subset\{x_n\}$ and $\{y_{n_i}\}\subset\{y_n\}$ such that $x_{n_i}\rightharpoonup x^*$ and $y_{n_i}\rightharpoonup y^*$. Since every Hilbert space has the Opial property. The Opial property guarantees that the weakly subsequential limit of $\{(x_n,y_n)\}$ is unique. Therefore we have $x_n\rightharpoonup x^*$ and $y_n\rightharpoonup y^*$.

On the other hand, from (3.14), one gets $u_n\rightharpoonup x^*$ and $v_n\rightharpoonup y^*$. By (3.15) and the demi-closed property of K and G, we have $Kx^* = x^*$ and $Gy^* = y^*$. This implies that $x^*\in \mathrm{Fix}(T)$ and $y^*\in\mathrm{Fix}(S)$.

Now we are left to show that $Ax^* = By^*$. In fact, since $Ax_n - By_n \rightharpoonup Ax^* - By^*$, by using the weakly lower semi-continuity of squared norm, we have

$$\big\|Ax^* - By^*\big\|^2 = \liminf_{n\to\infty}\|Ax_n - By_n\|^2 = \lim_{n\to\infty}\|Ax_n - By_n\|^2 = 0.$$

Thus $Ax^* = By^*$. This completes the proof of the conclusion (I).

Now we prove the conclusion (II). In fact, by virtue of (2.4), (3.13), and (3.14), we have

$$\begin{cases} \|x_n - Tx_n\| \leq \frac{1}{\xi_n(1-L\eta_n)} \|x_n - Kx_n\| \to 0; \\ \|y_n - Sy_n\| \leq \frac{1}{\xi_n(1-L\eta_n)} \|y_n - Gy_n\| \to 0. \end{cases} \tag{3.16}$$

Since S, T are semi-compact, it follows from (3.16) that there exist subsequences $\{x_{n_i}\} \subset \{x_n\}$ and $\{y_{n_j}\} \subset \{y_n\}$ such that $x_{n_i} \to x$ (some point in $F(T)$) and $y_{n_j} \to y$ (some point in $F(S)$). It follows from (3.11), $x_n \rightharpoonup x^*$, and $y_n \rightharpoonup y^*$ that $x_n \to x^*$ and $y_n \to y^*$ and $Ax^* = By^*$. □

4 Applications

4.1 Application to the split equality variational inequality problem

Throughout this section, we assume that H_1, H_2, and H_3 are three real Hilbert spaces. C and Q both are nonempty and closed convex subsets of H_1 and H_2, respectively and assume that $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear operator with its adjoint A^* and B^*, respectively.

Let $M : C \to H_1$ be a mapping. *The variational inequality problem for mapping M* is to find a point $x^* \in C$ such that

$$\langle Mx^*, z - x^*\rangle \geq 0 \quad \forall z \in C. \tag{4.1}$$

We will denote the solution set of (4.1) by $VI(M, C)$.

A mapping $M : C \to H_1$ is said to be *α-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y\rangle \geq \alpha \|Mx - My\|^2 \quad \forall x, y \in C. \tag{4.2}$$

It is easy to see that if M is α-inverse-strongly monotone, then M is $\frac{1}{\alpha}$-Lipschitzian.

Setting $h(x, y) = \langle Mx, y - x\rangle : C \times C \to \mathbb{R}$, it is easy to show that h is an *equilibrium function*, *i.e.*, it satisfies the following conditions, (A1)-(A4):

(A1) $h(x, x) = 0$, for all $x \in C$;

(A2) h is monotone, *i.e.*, $h(x, y) + h(y, x) \leq 0$ for all $x, y \in C$;

(A3) $\limsup_{t\downarrow 0} h(tz + (1 - t)x, y) \leq h(x, y)$ for all $x, y, z \in C$;

(A4) for each $x \in C$, $y \mapsto h(x, y)$ is convex and lower semi-continuous.

For given $\lambda > 0$ and $x \in H$, the *resolvent of the equilibrium function h* is the operator $R_{\lambda,h} : H \to C$ defined by

$$R_{\lambda,h}(x) := \left\{ z \in C : h(z, y) + \frac{1}{\lambda} \langle y - z, z - x\rangle \geq 0, \forall y \in C \right\}. \tag{4.3}$$

Proposition 4.1 [17] *The resolvent operator $R_{\lambda,h}$ of the equilibrium function h has the following properties* :

(1) $R_{\lambda,h}$ *is single-valued*;

(2) $\mathrm{Fix}(R_{\lambda,h}) = VI(M, C)$, *where $VI(M, C)$ is the solution set of variational inequality* (4.1) *which is a nonempty closed and convex subset of C*;

(3) $R_{\lambda,h}$ *is a firmly nonexpansive mapping. Therefore $R_{\lambda,h}$ is demiclosed at* 0.

Let $T : C \to H_1$ and $S : Q \to H_2$ be two α-inverse-strongly monotone mappings. The so-called *split equality variational inequality problem with respect to T and S* is to find $x^* \in C$ and $y^* \in Q$ such that

$$\begin{cases} \text{(a)} \quad \langle Tx^*, u - x^* \rangle \geq 0 \quad \forall u \in C, \\ \text{(b)} \quad \langle Sy^*, v - y^* \rangle \geq 0 \quad \forall v \in Q, \\ \text{(c)} \quad Ax^* = By^*. \end{cases} \tag{4.4}$$

In the sequel we use Ω to denote the solution set of split equality variational inequality problem (4.4), *i.e.*,

$$\Theta = \{(x^*, y^*) \in VI(T, C) \times VI(S, Q) : Ax^* = By^*\}, \tag{4.5}$$

where $VI(T, C)$ (resp. $VI(S, Q)$) is the solution set of variational inequality (4.4)(a) (resp. (4.4)(b)).

Denote by $f(x, y) = \langle Tx, y - x \rangle : C \times C \to \mathbb{R}$ and $g(u, v) = \langle Su, v - u \rangle : Q \times Q \to \mathbb{R}$. For given $\lambda > 0$, $x \in H_1$, and $u \in H_2$, let $R_{\lambda,f}(x)$ and $R_{\lambda,g}(u)$ be the resolvent operator of the equilibrium function f and g, respectively, which are defined by

$$R_{\lambda,f}(x) := \left\{ z \in C : f(z, y) + \frac{1}{\lambda} \langle y - z, z - x \rangle \geq 0, \forall y \in C \right\}$$

and

$$R_{\lambda,g}(u) := \left\{ z \in Q : g(z, v) + \frac{1}{\lambda} \langle v - z, z - u \rangle \geq 0, \forall v \in Q \right\}.$$

It follows from Proposition 4.1 that

$$\text{Fix}(R_{\lambda,f}) = VI(T, C) \neq \emptyset; \qquad \text{Fix}(R_{\lambda,g}) = VI(S, Q) \neq \emptyset, \tag{4.6}$$

and so $R_{\lambda,f}$ and $R_{\lambda,g}$ both are quasi-pseudo-contractive and 1-Lipschitzian. Therefore the split equality variational inequality problem with respect to T and S (4.4) is equivalent to the following split equality fixed point problem:

$$\text{to find } x^* \in \text{Fix}(R_{\lambda,f}), y^* \in \text{Fix}(R_{\lambda,g}) \text{ such that } Ax^* = By^*. \tag{4.7}$$

Since $R_{\lambda,f}$ and $R_{\lambda,g}$ are firmly nonexpansive with $\text{Fix}(R_{\lambda,f}) \neq \emptyset$ and $\text{Fix}(R_{\lambda,g}) \neq \emptyset$, the following theorem can be obtained from Theorem 3.2 immediately.

Theorem 4.2 *Let H_1, H_2, H_3, C, Q, A, B, T, S, $R_{\lambda,f}$, $R_{\lambda,g}$, Θ be the same as above and assume that $\Theta \neq \emptyset$. For given $x_0 \in C$, $y_0 \in Q$, let $(\{x_n\}, \{x_n\})$ be the sequence generated by*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = R_{\lambda,f}(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = R_{\lambda,g}(v_n). \end{cases} \tag{4.8}$$

If $\gamma_n \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ $\forall n \geq 1$, then the sequence $(\{x_n, y_n\})$ generated by (4.8) *converges weakly to a solution of split equality variational inequality problem* (4.4).

4.2 Application to the split equality convex minimization problem

Let C be a nonempty closed convex subset of H_1 and Q be a nonempty closed convex subset of H_2. Let $\phi : C \to \mathbb{R}$ and $\psi : Q \to \mathbb{R}$ be two proper and convex and lower semi-continuous functions and $A : H_1 \to H_3$ and $B : H_2 \to H_3$ be two bounded linear operator with its adjoint A^* and B^*, respectively.

The so-called *split equality convex minimization problem for ϕ and ψ* is to find $x^* \in C$, $y^* \in Q$ such that

$$\phi(x^*) = \min_{x\in C} \phi(x), \qquad \psi(y^*) = \min_{x\in Q} \psi(y), \quad \text{and} \quad Ax^* = By^*. \tag{4.9}$$

In the sequel, we denote by Ω the solution set of split equality convex minimization problem (4.9), *i.e.*,

$$\begin{aligned}\Omega = \Big\{&(p,q) \in C \times Q \text{ such that } \phi(x^*) = \min_{x\in C} \phi(x), \\ &\psi(y^*) = \min_{x\in Q} \psi(y) \text{ and } Ax^* = By^*\Big\}\end{aligned} \tag{4.10}$$

Let $j(x,y) := \phi(y) - \psi(x) : C \times C \to \mathbb{R}$ and $k(u,v) := \phi(v) - \psi(u) : Q \times Q \to \mathbb{R}$. It is easy to see that j and k both are equilibrium functions satisfying the conditions (A1)-(A4).

For given $\lambda > 0$, $x \in H_1$ and $u \in H_2$, we define the resolvent operators of j and k as follows:

$$R_{\lambda,j}(x) := \left\{z \in C : j(z,y) + \frac{1}{\lambda}\langle y - z, z - x\rangle \geq 0, \forall y \in C\right\}$$

and

$$R_{\lambda,k}(u) := \left\{z \in Q : k(z,v) + \frac{1}{\lambda}\langle v - z, z - u\rangle \geq 0, \forall v \in Q\right\}.$$

By the same argument as given in Section 4.1, we know that

$$\operatorname{Fix}(R_{\lambda,j}) = \left\{x^* \in C : \phi(x^*) = \min_{x\in C} \phi(x)\right\}, \qquad \operatorname{Fix}(R_{\lambda,k}) = \left\{y^* \in Q : \psi(y^*) = \min_{x\in Q} \psi(y)\right\}.$$

Therefore the split equality convex minimization problem for ϕ and ψ is equivalent to the following split equality fixed point problem:

$$\text{to find } x^* \in \operatorname{Fix}(R_{\lambda,j}), y^* \in \operatorname{Fix}(R_{\lambda,k}) \text{ such that } Ax^* = By^*. \tag{4.11}$$

Since $R_{\lambda,j}$ and $R_{\lambda,k}$ both are firmly nonexpansive with $\operatorname{Fix}(R_{\lambda,f}) \neq \emptyset$ and $\operatorname{Fix}(R_{\lambda,g}) \neq \emptyset$, the following theorem can be obtained from Theorem 3.2 immediately.

Theorem 4.3 *Let H_1, H_2, H_3, C, Q, A, B, ϕ, ψ, $R_{\lambda,j}$, $R_{\lambda,k}$, Ω be the same as above and assume that $\Omega \neq \emptyset$. For given $x_0 \in C$, $y_0 \in Q$, let $(\{x_n\}, \{x_n\})$ be the sequence generated by*

$$\begin{cases} u_n = x_n - \gamma_n A^*(Ax_n - By_n), \\ x_{n+1} = R_{\lambda,j}(u_n), \\ v_n = y_n + \gamma_n B^*(Ax_n - By_n), \\ y_{n+1} = R_{\lambda,k}(v_n). \end{cases} \tag{4.12}$$

If $\gamma_n \in (0, \min(\frac{1}{\|A\|^2}, \frac{1}{\|B\|^2}))$ $\forall n \geq 1$, *then the sequence* $(\{x_n, y_n\})$ *generated by* (4.12) *converges weakly to a solution of split equality convex minimization problem* (4.9).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

Author details
[1]Center for General Education, China Medical University, Taichung, 40402, Taiwan. [2]College of Statistics and Mathematics, Yunnan University of Finance and Economics, Kunming, Yunnan 650221, P.R. China. [3]Department of Mathematics, Kunming University, Kunming, Yunnan 650214, China.

Acknowledgements
This work was supported by the National Natural Science Foundation of China (Grant No. 11361070).

Received: 1 September 2015 Accepted: 5 November 2015 Published online: 17 November 2015

References
1. Censor, Y, Elfving, T: A multiprojection algorithm using Bregman projections in a product space. Numer. Algorithms **8**, 221-239 (1994)
2. Byrne, C: Iterative oblique projection onto convex subsets and the split feasibility problem. Inverse Probl. **18**, 441-453 (2002)
3. Censor, Y, Bortfeld, T, Martin, N, Trofimov, A: A unified approach for inversion problem in intensity-modulated radiation therapy. Phys. Med. Biol. **51**, 2353-2365 (2006)
4. Censor, Y, Elfving, T, Kopf, N, Bortfeld, T: The multiple-sets split feasibility problem and its applications. Inverse Probl. **21**, 2071-2084 (2005)
5. Censor, Y, Motova, A, Segal, A: Perturbed projections and subgradient projections for the multiple-sets split feasibility problem. J. Math. Anal. Appl. **327**, 1244-1256 (2007)
6. Xu, HK: A variable Krasnosel'skii-Mann algorithm and the multiple-sets split feasibility problem. Inverse Probl. **22**, 2021-2034 (2006)
7. Yang, Q: The relaxed *CQ* algorithm for solving the split feasibility problem. Inverse Probl. **20**, 1261-1266 (2004)
8. Zhao, J, Yang, Q: Several solution methods for the split feasibility problem. Inverse Probl. **21**, 1791-1799 (2005)
9. Chang, S-s, Agarwal, RP: Strong convergence theorems of general split equality problems for quasi-nonexpansive mappings. J. Inequal. Appl. **2014**, Article ID 367 (2014)
10. Chang, S-s, Wang, L, Tang, YK, Wang, G: Moudafi's open question and simultaneous iterative algorithm for general split equality variational inclusion problems and general split equality optimization problems. Fixed Point Theory Appl. **2014**, Article ID 215 (2014)
11. Moudafi, A: A relaxed alternating *CQ* algorithm for convex feasibility problems. Nonlinear Anal. **79**, 117-121 (2013)
12. Moudafi, A, Al-Shemas, E: Simultaneous iterative methods forsplit equality problem. Trans. Math. Program. Appl. **1**, 1-11 (2013)
13. Moudafi, A: Split monotone variational inclusions. J. Optim. Theory Appl. **150**, 275-283 (2011)
14. Che, H, Li, M: A simultaneous iterative method for split equality problems of two finite families of strictly pseudononspreading mappings without prior knowledge of operator norms. Fixed Point Theory Appl. **2015**, Article ID 1 (2015)
15. Bauschke, HH, Combettes, PL: A weak-to-strong convergence principle for Fejer-monotone methods in Hilbert spaces. Math. Oper. Res. **26**, 248-264 (2001)
16. Yao, Y, Liou, Y-C, Yao, J-C: Split common fixed point problem for two quasi-pseudo-contractive operators and its algorithm construction. Fixed Point Theory Appl. **2015**, Article ID 127 (2015)
17. Blum, E, Oettli, W: From optimization and variational inequalities to equilibrium problems. Math. Stud. **63**(1-4), 123-145 (1994)