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On properties of meromorphic solutions for difference Painlevé equations

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Abstract

In this paper, we mainly investigate properties of finite order transcendental meromorphic solutions of difference Painlevé equations. If f is a finite order transcendental meromorphic solution of difference Painlevé equations, then we get some estimates of the order and the exponent of convergence of poles of $\Delta f(z)$, where $\Delta f(z) = f(z+1) - f(z)$.

MSC: 30D35; 39B12

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1 Introduction and main results

Let f be a function transcendental and meromorphic in the plane. The forward difference is defined in the standard way by $\Delta f(z) = f(z+1) - f(z)$. In what follows, we assume that the reader is familiar with the basic notions of Nevanlinna value distribution theory (see [1–3]). In addition, we use the notations $\sigma(f)$ to denote the order of growth of the meromorphic function $f(z)$, and $\lambda(f)$ and $\lambda(\frac{1}{f})$ to denote, respectively, the exponents of convergence of zeros and poles of $f(z)$.

Painlevé and his colleagues [4] classified all equations of the Painlevé type of the form

$$\frac{d^2y}{dz^2} = F\left(z, y, \frac{dy}{dz}\right),$$

where F is rational in y and $\frac{dy}{dz}$ and (locally) analytic in z. The first two of these are P_{I} and P_{II}:

$$\frac{d^2y}{dz^2} = 6y^2 + z, \qquad \frac{d^2y}{dz^2} = 2y^2 + zy + \alpha,$$

where α is a constant. The differential Painlevé equations, discovered at the beginning of the last century, have been an important research subject in the field of mathematics and physics.

In the past 18 years, the discrete Painlevé equations became important research problems (see [5]). For example, discrete Painlevé I equations

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \gamma,$$

$$y_{n+1} + y_{n-1} = \frac{\alpha n + \beta}{y_n} + \frac{\gamma}{y_n^2},$$

and discrete Painlevé II equation

$$y_{n+1} + y_{n-1} = \frac{(\alpha n + \beta)y_n + \gamma}{1 - y_n^2},$$

where α, β and γ are constants, $n \in N$.

Some results on the existence of meromorphic solutions for certain difference equations were obtained by Shimomura [6] and Yanagihara [7] 30 years ago.

Recently, a number of papers (see [8–20]) focused on complex difference equations and difference analogues of Nevanlinna theory. As the difference analogues of Nevanlinna theory are being investigated, many results on the complex difference equations are rapidly obtained.

Ablowitz *et al.* [8] looked at a difference equation of the type

$$y(z+1) + y(z-1) = R(z, y), \tag{1}$$

where R is rational in both of its arguments, and showed the following theorem.

Theorem A (see [8]) *If the second-order difference equation*

$$y(z+1) + y(z-1) = \frac{a_0(z) + a_1(z)y + \cdots + a_p(z)y^p}{b_0(z) + b_1(z)y + \cdots + b_q(z)y^q},$$

where a_i and b_i are polynomials, admits a non-rational meromorphic solution of finite order, then $\max\{p, q\} \leq 2$.

Halburd and Korhonen [15–17] used value distribution theory and a reasoning related to the singularity confinement to single out the difference Painlevé I and II equations from difference equation (1). They obtained that if (1) has a finite order admissible meromorphic solution $f(z)$, then either f satisfies a difference Riccati equation, or (1) can be transformed by a linear change in f to some classical difference equations, which include difference Painlevé I equations

$$f(z+1) + f(z-1) = \frac{az + b}{f(z)} + c, \tag{2}$$

$$f(z+1) + f(z-1) = \frac{az + b}{f(z)} + \frac{c}{f(z)^2}, \tag{3}$$

$$f(z+1) + f(z) + f(z-1) = \frac{az + b}{f(z)} + c, \tag{4}$$

and difference Painlevé II equation

$$f(z+1) + f(z-1) = \frac{(az + b)f(z) + c}{1 - f(z)^2}, \tag{5}$$

where a, b and c are constants.

From above, we see that difference Painlevé I and II equations are the development of the differential and discrete Painlevé I and II equations. So they are an important class of difference equations.

Chen and Chen [10] investigated some properties of meromorphic solutions of difference Painlevé I equation and proved the following Theorem B.

Theorem B (see [10]) *Let a, b, c be constants such that $|a| + |b| \neq 0$. If $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé* I *equation* (4), *then*

(i) $\lambda(\frac{1}{f}) = \lambda(f) = \sigma(f)$;

(ii) *if $p(z)$ is a non-constant polynomial, then $f(z) - p(z)$ has infinitely many zeros and* $\lambda(f - p) = \sigma(f)$;

(iii) *if $a \neq 0$, then $f(z)$ has no Borel exceptional value*; *if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set* $E = \{z \mid 3z^2 - cz - b = 0\}$.

The main aims of this paper are to consider the properties of finite order transcendental meromorphic solutions of difference Painlevé I and II equations (2)-(5), and we obtain the following results.

Theorem 1.1 *Let a, b, c be constants with $|a| + |b| \neq 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé* I *equation* (2). *Then*

(i) *if $a \neq 0$ and $p(z)$ is a polynomial, then $f(z) - p(z)$ has infinitely many zeros and* $\lambda(f - p) = \sigma(f)$; *if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set* $E = \{z \mid 2z^2 - cz - b = 0\}$;

(ii) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f)$.

Theorem 1.2 *Let a, b, c be constants with $|a| + |b| + |c| \neq 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé* II *equation* (5). *Then*

(i) *if $a \neq 0$ and $p(z)$ is a nonzero polynomial, then $f(z) - p(z)$ has infinitely many zeros and $\lambda(f - p) = \sigma(f)$*; *if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set* $E = \{z \mid 2z^3 + (b - 2)z + c = 0\}$; *if $c \neq 0$, then $\lambda(f) = \sigma(f)$*;

(ii) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f)$.

Remark 1.1 By Theorem 1.2, we conclude that if $ac \neq 0$ and $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé II equation (5), then $f(z)$ has no Borel exceptional value.

Theorem 1.3 *Let a, b, c be constants with $|a| + |b| + |c| \neq 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé* I *equation* (3). *Then*

(i) $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f)$;

(ii) *if $a = 0$, then Borel exceptional values of $f(z)$ can only come from the set* $E = \{z \mid 2z^3 - bz - c = 0\}$.

Theorem 1.4 *Let a, b, c be constants with $|a| + |b| \neq 0$. Suppose that $f(z)$ is a finite order transcendental meromorphic solution of the difference Painlevé* I *equation* (4). *Then* $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f)$.

Remark 1.2 The following examples show that $\lambda(\Delta f) = \sigma(f)$ may not hold in above several theorems.

Example 1.1 The meromorphic function $f(z) = \tan(\frac{\pi}{2}z)$ satisfies the equation

$$f(z+1) + f(z-1) = \frac{-2}{f(z)},$$

where $a = c = 0$, $b = -2$ satisfying $|a| + |b| = 2 \neq 0$. We obtain that

$$\Delta f = -\cot\left(\frac{\pi}{2}z\right) - \tan\left(\frac{\pi}{2}z\right) = -\frac{2}{\sin(\pi z)}.$$

Thus, $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f) = 1$ and $\lambda(\Delta f) = 0 \neq \sigma(f)$. The solution $f(z) = \tan(\frac{\pi}{2}z)$ has two Borel exceptional values i and $-i$ satisfying the equation $2z^2 - cz - b = 2z^2 + 2 = 0$.

Example 1.2 The meromorphic function $f(z) = \tan(\frac{\pi}{4}z)$ satisfies the equation

$$f(z+1) + f(z-1) = \frac{4f(z)}{1 - f(z)^2},$$

where $a = c = 0$, $b = 4$ and $|a| + |b| + |c| = 4$. We have

$$\Delta f = \frac{1 + \tan(\frac{\pi}{4}z)}{1 - \tan(\frac{\pi}{4}z)} - \tan\left(\frac{\pi}{4}z\right) = \frac{1 + \tan^2(\frac{\pi}{4}z)}{1 - \tan(\frac{\pi}{4}z)}.$$

Thus, $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f) = 1$ and $\lambda(\Delta f) = 0 \neq \sigma(f)$. The solution $f(z) = \tan(\frac{\pi}{4}z)$ has two Borel exceptional values $\pm i$ satisfying $2z^3 + (b-2)z + c = 2z^3 + 2z = 0$.

Example 1.3 The meromorphic function $f(z) = \frac{1}{e^{2\pi iz}+z+1}$ satisfies the equation

$$f(z+1) + f(z-1) = \frac{2f(z)}{1 - f(z)^2},$$

where $a = c = 0$, $b = 2$ and $|a| + |b| + |c| = 2$. We have

$$\Delta f = \frac{1}{e^{2\pi iz} + z + 2} - \frac{1}{e^{2\pi iz} + z + 1} = \frac{-1}{(e^{2\pi iz} + z + 2)(e^{2\pi iz} + z + 1)}.$$

Thus, $\lambda(\frac{1}{f}) = \lambda(\frac{1}{\Delta f}) = \sigma(\Delta f) = \sigma(f) = 1$ and $\lambda(\Delta f) = 0$. The solution $f(z) = \frac{1}{e^{2\pi iz}+z+1}$ has a Borel exceptional value 0 satisfying the equation $2z^3 + (b-2)z + c = 2z^3 = 0$.

2 The proof of Theorem 1.1

We need the following lemmas to prove Theorem 1.1.

Lemma 2.1 (see [13, 18]) *Let $w(z)$ be a non-constant finite order meromorphic solution of*

$$P(z,w)=0,$$

where $P(z,w)$ is a difference polynomial in $w(z)$. If $P(z,a)\not\equiv 0$ for a meromorphic function a satisfying $\lim_{r\to\infty}\frac{T(r,a)}{T(r,w)}=0$, then

$$m\left(r,\frac{1}{w-a}\right)=S(r,w)$$

outside of a possible exceptional set of finite logarithmic measure.

Lemma 2.2 (see [11, 16, 17]) *Let f be a non-constant finite order meromorphic function. Then*

$$N(r+1,f)=N(r,f)+S(r,f)$$

outside of a possible exceptional set of finite logarithmic measure.

Remark 2.1 In [12], Chiang and Feng proved that let f be a meromorphic function with exponent of convergence of poles $\lambda(\frac{1}{f})=\lambda<\infty$, $\eta\neq 0$ be fixed, then for each $\varepsilon>0$,

$$N\bigl(r,f(z+\eta)\bigr)=N(r,f)+O\bigl(r^{\lambda-1+\varepsilon}\bigr)+O(\log r).$$

Lemma 2.3 (see [18]) *Let $f(z)$ be a transcendental meromorphic solution of finite order σ of a difference equation of the form*

$$H(z,f)P(z,f)=Q(z,f),$$

where $H(z,f)$ is a difference product of total degree n in $f(z)$ and its shifts, and where $P(z,f)$, $Q(z,f)$ are difference polynomials such that the total degree of $Q(z,f)$ is $\le n$. Then, for each $\varepsilon>0$,

$$m\bigl(r,P(z,f)\bigr)=O\bigl(r^{\sigma-1+\varepsilon}\bigr)+S(r,f)$$

possibly outside of an exceptional set of finite logarithmic measure.

Lemma 2.4 (Valiron-Mohon'ko, see [2]) *Let $f(z)$ be a meromorphic function. Then, for all irreducible rational functions in f,*

$$R\bigl(z,f(z)\bigr)=\frac{a_0(z)+a_1(z)f(z)+\cdots+a_m(z)f(z)^m}{b_0(z)+b_1(z)f(z)+\cdots+b_n(z)f(z)^n}$$

with meromorphic coefficients $a_i(z)$ $(i=0,1,\ldots,m)$, $b_j(z)$ $(j=0,1,\ldots,n)$, the characteristic function of $R(z,f(z))$ satisfies

$$T\bigl(r,R\bigl(z,f(z)\bigr)\bigr)=dT(r,f)+O\bigl(\Psi(r)\bigr),$$

where $d=\deg_f R=\max\{m,n\}$ and $\Psi(r)=\max_{i,j}\{T(r,a_i),T(r,b_j)\}$.

Lemma 2.5 (see [12]) *Let $f(z)$ be a meromorphic function with order $\sigma = \sigma(f) < \infty$, and let η be a fixed nonzero complex number, then for each $\varepsilon > 0$, we have*

$$T\bigl(r, f(z+\eta)\bigr) = T(r,f) + O\bigl(r^{\sigma-1+\varepsilon}\bigr) + O(\log r).$$

Lemma 2.6 (see [21]) *Let $g : (0,+\infty) \to R$, $h : (0,+\infty) \to R$ be non-decreasing functions. If* (i) *$g(r) \leq h(r)$ outside of an exceptional set of finite linear measure, or* (ii) *$g(r) \leq h(r)$, $r \notin H \cup (0,1]$, where $H \subset (1,\infty)$ is a set of finite logarithmic measure, then for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.*

Proof of Theorem 1.1 Suppose that $f(z)$ is a transcendental meromorphic solution of finite order $\sigma(f)$ of equation (2).

(i) Let $a \neq 0$. Suppose that $p(z)$ is a polynomial. Set $g(z) = f(z) - p(z)$. Substituting $f(z) = g(z) + p(z)$ into (2), we obtain that

$$g(z+1) + p(z+1) + g(z-1) + p(z-1) = \frac{az+b}{g(z)+p(z)} + c. \tag{6}$$

It follows from (6) that

$$\begin{aligned} P(z,g) :=& \bigl(g(z+1) + p(z+1) + g(z-1) + p(z-1)\bigr) \\ &\times \bigl(g(z)+p(z)\bigr) - (az+b) - c\bigl(g(z)+p(z)\bigr) = 0. \end{aligned} \tag{7}$$

By (7), we have

$$P(z,0) = \bigl(p(z+1) + p(z-1)\bigr)p(z) - (az+b) - cp(z).$$

If $p(z) \equiv 0$, then $P(z,0) = -(az+b) \not\equiv 0$. If $p(z) \equiv \alpha \in C\backslash\{0\}$, then

$$P(z,0) = 2\alpha^2 - (az+b) - c\alpha \not\equiv 0$$

since $a \neq 0$. Suppose that $p(z)$ is a non-constant polynomial. Set $p(z) = d_k z^k + \cdots + d_0$, where $d_k, \ldots, d_0$ are constants, $d_k \neq 0$ and $k \geq 1$. Then we obtain that

$$\begin{aligned} P(z,0) &= \bigl(p(z+1) + p(z-1)\bigr)p(z) - (az+b) - cp(z) \\ &= 2d_k^2 z^{2k} + \cdots \not\equiv 0. \end{aligned}$$

Thus, by Lemma 2.1, we see that

$$m\left(r, \frac{1}{g}\right) = S(r,g)$$

outside of a possible exceptional set of finite logarithmic measure. Thus,

$$N\left(r, \frac{1}{f(z)-p(z)}\right) = N\left(r, \frac{1}{g}\right) = T(r,g) + S(r,g) = T(r,f) + S(r,f) \tag{8}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by (8) and Lemma 2.6, we have $\lambda(f-p)=\sigma(f)$.

If $a=0$ and $p(z)=\beta\notin E$, then we have

$$P(z,0)=2\beta^2-c\beta-b\not\equiv 0.$$

Using a similar method as above, we obtain $\lambda(f-\beta)=\sigma(f)$. Hence, the Borel exceptional values of $f(z)$ can only come from the set $E=\{z\mid 2z^2-cz-b=0\}$.

(ii) Set $z=w+1$. Substituting $z=w+1$ into (2), we obtain that

$$f(w+2)+f(w)=\frac{aw+a+b}{f(w+1)}+c.$$

That is,

$$f(w+1)\bigl(f(w+2)+f(w)\bigr)=(aw+a+b)+cf(w+1). \tag{9}$$

Substituting $f(w+1)=\Delta f(w)+f(w)$ and $f(w+2)=\Delta f(w+1)+\Delta f(w)+f(w)$ into (9), we have

$$\bigl(\Delta f(w)+f(w)\bigr)\bigl(\Delta f(w+1)+\Delta f(w)+2f(w)\bigr)=(aw+a+b)+c\bigl(\Delta f(w)+f(w)\bigr).$$

That is,

$$\begin{aligned}-2f(w)^2 = {}&\bigl(\Delta f(w+1)+3\Delta f(w)-c\bigr)f(w)\\&+\bigl(\Delta f(w+1)+\Delta f(w)-c\bigr)\Delta f(w)-(aw+a+b).\end{aligned} \tag{10}$$

Since $N(R,\Delta f(w+1))\le N(R+1,\Delta f(w))+o(N(R+1,\Delta f(w)))$, by Lemma 2.2 we see that there is a subset $E_1\subset(1,\infty)$ of finite logarithmic measure such that for $|w|=R\notin E_1\cup[0,1]$,

$$N\bigl(R,\Delta f(w+1)\bigr)\le N\bigl(R,\Delta f(w)\bigr)+S(R,f). \tag{11}$$

By (10) and (11), when $|w|=R\notin E_1\cup[0,1]$, we obtain that

$$\begin{aligned}2N\bigl(R,f(w)\bigr) &= N\bigl(R,\bigl(\Delta f(w+1)+3\Delta f(w)-c\bigr)f(w)\\&\qquad+\bigl(\Delta f(w+1)+\Delta f(w)-c\bigr)\Delta f(w)-(aw+a+b)\bigr)\\&\le N\bigl(R,f(w)\bigr)+7N\bigl(R,\Delta f(w)\bigr)+S(R,f)+O(\log R).\end{aligned}$$

That is,

$$N\bigl(R,f(w)\bigr)\le 7N\bigl(R,\Delta f(w)\bigr)+S(R,f) \tag{12}$$

for all $|w|=R\notin E_1\cup[0,1]$. By Lemma 2.6 and (12), for any $\beta_1>1$, there exists $R_0>0$ such that

$$N\bigl(R,f(w)\bigr)\le 7N\bigl(\beta_1R,\Delta f(w)\bigr)+S(R,f) \tag{13}$$

for all $R > R_0$. By (13), we have

$$\lambda\left(\frac{1}{\Delta f(w)}\right) \geq \lambda\left(\frac{1}{f(w)}\right). \tag{14}$$

By Remark 2.1 and (14), we have

$$\lambda\left(\frac{1}{\Delta f(w)}\right) = \lambda\left(\frac{1}{\Delta f(z-1)}\right) = \lambda\left(\frac{1}{\Delta f(z)}\right)$$

and

$$\lambda\left(\frac{1}{f(w)}\right) = \lambda\left(\frac{1}{f(z-1)}\right) = \lambda\left(\frac{1}{f(z)}\right).$$

Hence, we get

$$\lambda\left(\frac{1}{\Delta f(z)}\right) \geq \lambda\left(\frac{1}{f(z)}\right). \tag{15}$$

By (2), we obtain that

$$f(z)\big(f(z+1)+f(z-1)\big) = az + b + cf(z). \tag{16}$$

By (16) and Lemma 2.3, we see that for any given $\varepsilon > 0$, there is a subset $E_2 \subset (1,\infty)$ of finite logarithmic measure such that for $|z| = r \notin E_2 \cup [0,1]$,

$$m\big(r, f(z+1)+f(z-1)\big) = O\big(r^{\sigma(f)-1+\varepsilon}\big) + S(r,f). \tag{17}$$

By Lemma 2.2, we see that there is a subset $E_3 \subset (1,\infty)$ of finite logarithmic measure such that for $|z| = r \notin E_3 \cup [0,1]$,

$$N\big(r, f(z+1)+f(z-1)\big) \leq 2N(r+1,f) + S(r,f) = 2N(r,f) + S(r,f). \tag{18}$$

By Lemma 2.4 and (2), we see that

$$T\big(r, f(z+1)+f(z-1)\big) = T(r,f) + S(r,f) \tag{19}$$

since $|a| + |b| \neq 0$. By (17)-(19), when $|z| = r \notin E_3 \cup E_2 \cup [0,1]$, we have

$$\frac{1}{2}T\big(r,f(z)\big) \leq N\big(r,f(z)\big) + O\big(r^{\sigma(f)-1+\varepsilon}\big) + S(r,f). \tag{20}$$

By Lemma 2.6 and (20), for any $\beta_2 > 1$, there exists $r_0 > 0$ such that

$$\frac{1}{2}T\big(r,f(z)\big) \leq N\big(\beta_2 r,f(z)\big) + O\big(r^{\sigma(f)-1+\varepsilon}\big) + S(r,f)$$

for all $r > r_0$. Thus, we get

$$\lambda\left(\frac{1}{f(z)}\right) \geq \sigma\big(f(z)\big). \tag{21}$$

By (15) and (21), we have $\lambda(\frac{1}{\Delta f}) \geq \lambda(\frac{1}{f}) \geq \sigma(f)$. And we have $\sigma(\Delta f) \leq \sigma(f)$ from Lemma 2.5. Hence,

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{\Delta f}\right) = \sigma(\Delta f) = \sigma(f). \qquad \square$$

3 The proof of Theorem 1.2

Suppose that $f(z)$ is a transcendental meromorphic solution of finite order $\sigma(f)$ of equation (5).

(i) Suppose that $a \neq 0$ and $p(z)$ is a nonzero polynomial. Set $g_1(z) = f(z) - p(z)$. Substituting $f(z) = g_1(z) + p(z)$ into (5), we obtain that

$$g_1(z+1) + p(z+1) + g_1(z-1) + p(z-1) = \frac{(az+b)(g_1(z) + p(z)) + c}{1 - (g_1(z) + p(z))^2}. \tag{22}$$

It follows from (22) that

$$\begin{aligned} P(z,g_1) := {} & \bigl(g_1(z+1) + p(z+1) + g_1(z-1) + p(z-1)\bigr) \\ & \times \bigl(g_1(z) + p(z)\bigr)^2 + (az+b)\bigl(g_1(z) + p(z)\bigr) + c \\ & - \bigl(g_1(z+1) + p(z+1) + g_1(z-1) + p(z-1)\bigr) = 0. \end{aligned} \tag{23}$$

By (23), we have

$$\begin{aligned} P(z,0) = {} & \bigl(p(z+1) + p(z-1)\bigr)p(z)^2 + (az+b)p(z) \\ & + c - \bigl(p(z+1) + p(z-1)\bigr). \end{aligned}$$

If $p(z) \equiv \alpha \in C\backslash\{0\}$, then

$$P(z,0) = 2\alpha^3 + \alpha(az+b) + c - 2\alpha \not\equiv 0$$

since $a \neq 0$. Suppose that $p(z)$ is a non-constant polynomial. Set $p(z) = d_k z^k + \cdots + d_0$, where $d_k, \ldots, d_0$ are constants, $d_k \neq 0$ and $k \geq 1$. Then we obtain that

$$\begin{aligned} P(z,0) = {} & \bigl(p(z+1) + p(z-1)\bigr)p(z)^2 + (az+b)p(z) \\ & + c - \bigl(p(z+1) + p(z-1)\bigr) \\ = {} & 2d_k^3 z^{3k} + \cdots \not\equiv 0. \end{aligned}$$

Thus, by Lemma 2.1, we see that

$$m\left(r, \frac{1}{g_1}\right) = S(r,g_1)$$

outside of a possible exceptional set of finite logarithmic measure. Thus,

$$N\left(r, \frac{1}{f(z) - p(z)}\right) = N\left(r, \frac{1}{g_1}\right) = T(r,g_1) + S(r,g_1) = T(r,f) + S(r,f) \tag{24}$$

outside of a possible exceptional set of finite logarithmic measure. Hence, by (24) and Lemma 2.6, we have $\lambda(f-p)=\sigma(f)$.

If $a=0$ and $p(z)=\beta\notin E$, then we have

$$P(z,0)=2\beta^3+(b-2)\beta+c\not\equiv 0.$$

Using a similar method as above, we obtain $\lambda(f-\beta)=\sigma(f)$. Hence, the Borel exceptional values of $f(z)$ can only come from the set $E=\{z\mid 2z^3+(b-2)z+c=0\}$.

If $c\neq 0$, then by (5) we have

$$P_1(z,f):=f^2(z)\bigl(f(z+1)+f(z-1)\bigr)+(az+b)f(z)+c-f(z+1)-f(z-1).$$

Hence, we have

$$P_1(z,0)\equiv c\not\equiv 0.$$

Using a similar method as above, we obtain $\lambda(f)=\sigma(f)$.

(ii) Set $z=w+1$. Substituting $z=w+1$ into (5), we obtain that

$$f(w+2)+f(w)=\frac{(aw+a+b)f(w+1)+c}{1-f(w+1)^2}.$$

That is,

$$f(w+1)^2\bigl(f(w+2)+f(w)\bigr)=f(w+2)+f(w)-c-(aw+a+b)f(w+1). \tag{25}$$

Substituting $f(w+1)=\Delta f(w)+f(w)$ and $f(w+2)=\Delta f(w+1)+\Delta f(w)+f(w)$ into (25), we have

$$\begin{aligned}&\bigl(\Delta f(w)+f(w)\bigr)^2\bigl(\Delta f(w+1)+\Delta f(w)+2f(w)\bigr)\\&\quad=\Delta f(w+1)+\Delta f(w)+2f(w)-c-(aw+a+b)\bigl(\Delta f(w)+f(w)\bigr).\end{aligned}$$

Thus, we obtain that

$$-2f(w)^3=A(w)f(w)+B(w)\Delta f(w)+C(w), \tag{26}$$

where

$$\begin{aligned}A(w)&=\bigl(\Delta f(w+1)+5\Delta f(w)\bigr)f(w)\\&\quad+4\bigl(\Delta f(w)\bigr)^2+2\Delta f(w)\Delta f(w+1)+aw+a+b-2,\\B(w)&=\Delta f(w+1)\Delta f(w)+\bigl(\Delta f(w)\bigr)^2+aw+a+b-1,\\C(w)&=c-\Delta f(w+1).\end{aligned}$$

Since $N(R,\Delta f(w+1))\le N(R+1,\Delta f(w))+o(N(R+1,\Delta f(w)))$, by Lemma 2.2 we see that there is a subset $H_1\subset(1,\infty)$ having finite logarithmic measure such that for $|w|=R\notin$

$H_1 \cup [0,1]$,

$$N\bigl(R, \Delta f(w+1)\bigr) \leq N\bigl(R, \Delta f(w)\bigr) + S(R,f). \tag{27}$$

By (26) and (27), when $|w| = R \notin H_1 \cup [0,1]$, we obtain that

$$\begin{aligned} 3N\bigl(R,f(w)\bigr) &= N\bigl(R, A(w)f(w) + B(w)\Delta f(w) + C(w)\bigr) \\ &\leq 2N\bigl(R,f(w)\bigr) + 12N\bigl(R, \Delta f(w)\bigr) + S(R,f) + O(\log R). \end{aligned}$$

That is,

$$N\bigl(R,f(w)\bigr) \leq 12N\bigl(R, \Delta f(w)\bigr) + S(R,f) \tag{28}$$

for $|w| = R \notin H_1 \cup [0,1]$. By Lemma 2.6 and (28), for any $\beta_1 > 1$, there exists $R_0 > 0$ such that

$$N\bigl(R,f(w)\bigr) \leq 12N\bigl(\beta_1 R, \Delta f(w)\bigr) + S(R,f)$$

for all $R > R_0$. Thus, we have

$$\lambda\left(\frac{1}{\Delta f(w)}\right) \geq \lambda\left(\frac{1}{f(w)}\right). \tag{29}$$

By the same reasoning as in Theorem 1.1(ii), we get

$$\lambda\left(\frac{1}{\Delta f(z)}\right) \geq \lambda\left(\frac{1}{f(z)}\right). \tag{30}$$

By (5), we obtain that

$$f(z)^2\bigl(f(z+1) + f(z-1)\bigr) = f(z+1) + f(z-1) - (az+b)f(z) - c. \tag{31}$$

By (31) and Lemma 2.3, we see that for any given $\varepsilon > 0$, there is a subset $H_2 \subset (1,\infty)$ having finite logarithmic measure such that for $|z| = r \notin H_2 \cup [0,1]$,

$$m\bigl(r, f(z+1) + f(z-1)\bigr) = O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f). \tag{32}$$

By Lemma 2.2, we see that there is a subset $H_3 \subset (1,\infty)$ of finite logarithmic measure such that for $|z| = r \notin H_3 \cup [0,1]$,

$$N\bigl(r, f(z+1) + f(z-1)\bigr) \leq 2N(r+1,f) + S(r,f) = 2N(r,f) + S(r,f). \tag{33}$$

By Lemma 2.4 and (5), we see that

$$T\bigl(r, f(z+1) + f(z-1)\bigr) = 2T\bigl(r, f(z)\bigr) + S(r,f) \tag{34}$$

since $|a| + |b| + |c| \neq 0$. By (32)-(34), when $|z| = r \notin H_3 \cup H_2 \cup [0,1]$, we have

$$T\big(r,f(z)\big) \leq N\big(r,f(z)\big) + O\big(r^{\sigma(f)-1+\varepsilon}\big) + S(r,f). \tag{35}$$

By Lemma 2.6 and (35), for any $\beta_2 > 1$, there exists $r_1 > 0$ such that

$$T\big(r,f(z)\big) \leq N\big(\beta_2 r,f(z)\big) + O\big(r^{\sigma(f)-1+\varepsilon}\big) + S(r,f)$$

for all $r > r_1$. Thus, we get

$$\lambda\left(\frac{1}{f(z)}\right) \geq \sigma\big(f(z)\big). \tag{36}$$

By (30) and (36), we see that $\lambda(\frac{1}{\Delta f}) \geq \lambda(\frac{1}{f}) \geq \sigma(f)$. And we have $\sigma(\Delta f) \leq \sigma(f)$ from Lemma 2.5. Hence,

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{\Delta f}\right) = \sigma(\Delta f) = \sigma(f).$$

4 Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3 Suppose that $f(z)$ is a transcendental meromorphic solution of finite order $\sigma(f)$ of equation (3).

(i) Using the same methods as in the proofs of Theorems 1.1(ii) and 1.2(ii), we have

$$-2f(w)^3 = A(w)f(w) + B(w)\Delta f(w) + C(w), \tag{37}$$

where

$$\begin{aligned} A(w) &= \big(\Delta f(w+1) + 5\Delta f(w)\big)f(w) \\ &\quad + 4\big(\Delta f(w)\big)^2 + 2\Delta f(w)\Delta f(w+1) - (aw+a+b), \\ B(w) &= \big(\Delta f(w+1) + \Delta f(w)\big)\Delta f(w) - (aw+a+b), \\ C(w) &= -c. \end{aligned}$$

Set $|w| = R$. Since $N(R, \Delta f(w+1)) \leq N(R+1, \Delta f(w)) + o(N(R+1, \Delta f(w)))$, by Lemma 2.2 we see that

$$N\big(R, \Delta f(w+1)\big) \leq N\big(R, \Delta f(w)\big) + S(R,f) \tag{38}$$

outside of a possible exceptional set of finite logarithmic measure. By (37) and (38), we obtain that

$$\begin{aligned} 3N\big(R,f(w)\big) &= N\big(R,A(w)f(w) + B(w)\Delta f(w) + C(w)\big) \\ &\leq 2N\big(R,f(w)\big) + 10N\big(R, \Delta f(w)\big) \\ &\quad + S(R,f) + O(\log R) \end{aligned} \tag{39}$$

outside of a possible exceptional set of finite logarithmic measure. By (39) and Lemma 2.6, we have

$$\lambda\left(\frac{1}{\Delta f(w)}\right) \geq \lambda\left(\frac{1}{f(w)}\right). \tag{40}$$

By Remark 2.1 and (40), we obtain that

$$\lambda\left(\frac{1}{\Delta f(w)}\right) = \lambda\left(\frac{1}{\Delta f(z-1)}\right) = \lambda\left(\frac{1}{\Delta f(z)}\right)$$

and

$$\lambda\left(\frac{1}{f(w)}\right) = \lambda\left(\frac{1}{f(z-1)}\right) = \lambda\left(\frac{1}{f(z)}\right).$$

Hence, we get

$$\lambda\left(\frac{1}{\Delta f(z)}\right) \geq \lambda\left(\frac{1}{f(z)}\right). \tag{41}$$

By (3), we obtain that

$$f(z)^2\bigl(f(z+1)+f(z-1)\bigr) = (az+b)f(z)+c. \tag{42}$$

By (42) and Lemma 2.3, we see that

$$m\bigl(r,f(z+1)+f(z-1)\bigr) = O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f) \tag{43}$$

outside of a possible exceptional set of finite logarithmic measure. By Lemma 2.2, we get

$$N\bigl(r,f(z+1)+f(z-1)\bigr) \leq 2N\bigl(r+1,f(z)\bigr) + S(r,f) = 2N(r,f) + S(r,f) \tag{44}$$

outside of a possible exceptional set of finite logarithmic measure.

If $c \neq 0$, by Lemma 2.4 and (3), we see that

$$T\bigl(r,f(z+1)+f(z-1)\bigr) = 2T\bigl(r,f(z)\bigr) + S(r,f). \tag{45}$$

If $c = 0$, by the same reason, we have

$$T\bigl(r,f(z+1)+f(z-1)\bigr) = T\bigl(r,f(z)\bigr) + S(r,f) \tag{46}$$

since $|a| + |b| + |c| = |a| + |b| \neq 0$. By (43)-(46), we have

$$T\bigl(r,f(z)\bigr) \leq N\bigl(r,f(z)\bigr) + O\bigl(r^{\sigma(f)-1+\varepsilon}\bigr) + S(r,f) \tag{47}$$

outside of a possible exceptional set of finite logarithmic measure. By Lemma 2.6 and (47), we get

$$\lambda\left(\frac{1}{f(z)}\right) \geq \sigma\bigl(f(z)\bigr). \tag{48}$$

By (41) and (48), we have $\lambda(\frac{1}{\Delta f}) \geq \lambda(\frac{1}{f}) \geq \sigma(f)$. And we have $\sigma(\Delta f) \leq \sigma(f)$ from Lemma 2.5. Hence,

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{\Delta f}\right) = \sigma(\Delta f) = \sigma(f).$$

(ii) Let $a = 0$ and $p(z) = \beta \notin E$. Using the same methods as in the proof of Theorem 1.1(i), we have

$$P(z,0) = 2\beta^3 - b\beta - c \not\equiv 0.$$

Thus, we obtain $\lambda(f - \beta) = \sigma(f)$. Hence, the Borel exceptional values of $f(z)$ can only come from the set $E = \{z \mid 2z^3 - bz - c = 0\}$. □

Proof of Theorem 1.4 Suppose that $f(z)$ is a transcendental meromorphic solution of finite order $\sigma(f)$ of equation (4). Set $z = w + 1$. Using the same method as in the proof of Theorem 1.3(i), we have

$$\begin{aligned} -3f(w)^2 = {} & \bigl(\Delta f(w+1) + 5\Delta f(w) - c\bigr)f(w) \\ & + \bigl(\Delta f(w+1) + 2\Delta f(w) - c\bigr)\Delta f(w) - (aw + a + b). \end{aligned}$$

By the same reason as that in Theorem 1.3(i), when $|w| = R$, we have

$$2N\bigl(R, f(w)\bigr) \leq N\bigl(R, f(w)\bigr) + 5N\bigl(R, \Delta f(w)\bigr) + S(R,f) + O(\log R) \tag{49}$$

possibly outside of an exceptional set of finite logarithmic measure.

By Lemma 2.6 and (49), we obtain that $\lambda(\frac{1}{\Delta f(w)}) \geq \lambda(\frac{1}{f(w)})$. By the same method as above, we have $\lambda(\frac{1}{\Delta f(z)}) \geq \lambda(\frac{1}{f(z)})$. And we see that $\lambda(\frac{1}{f(z)}) = \sigma(f(z))$ from Theorem B and $\sigma(\Delta f) \leq \sigma(f)$ from Lemma 2.5. Hence, we have

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{\Delta f}\right) = \sigma(\Delta f) = \sigma(f).$$ □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
C-WP completed the main part of this article, C-WP and Z-XC corrected the main theorems. All authors read and approved the final manuscript.

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References

1. Hayman, WK: Meromorphic Functions. Clarendon, Oxford (1964)
2. Laine, I: Nevanlinna Theory and Complex Differential Equations. de Gruyter, Berlin (1993)
3. Yang, L: Value Distribution Theory. Science Press, Beijing (1993)
4. Painlevé, P: Mémoire sur les équations différentielles dont l'integrale générale est uniforme. Bull. Soc. Math. Fr. **28**, 201-261 (1900)
5. Fokas, AS: From continuous to discrete Painlevé equations. J. Math. Anal. Appl. **180**, 342-360 (1993)
6. Shimomura, S: Entire solutions of a polynomial difference equation. J. Fac. Sci., Univ. Tokyo, Sect. IA Math. **28**, 253-266 (1981)
7. Yanagihara, N: Meromorphic solutions of some difference equations. Funkc. Ekvacioj **23**, 309-326 (1980)
8. Ablowitz, M, Halburd, RG, Herbst, B: On the extension of Painlevé property to difference equations. Nonlinearity **13**, 889-905 (2000)
9. Bergweiler, W, Langley, JK: Zeros of differences of meromorphic functions. Math. Proc. Camb. Philos. Soc. **142**, 133-147 (2007)
10. Chen, MR, Chen, ZX: On properties of meromorphic solution of certain difference Painlevé I equation. Bull. Aust. Math. Soc. **85**, 463-475 (2012)
11. Chen, ZX: Value distribution of meromorphic solutions of certain difference Painlevé equations. J. Math. Anal. Appl. **364**, 556-566 (2010)
12. Chiang, YM, Feng, JF: On the Nevanlinna characteristic of $f(z+\eta)$ and difference equations in the complex plane. Ramanujan J. **16**, 105-129 (2008)
13. Halburd, RG, Korhonen, RJ: Difference analogue of the lemma on the logarithmic derivative with applications to difference equations. J. Math. Anal. Appl. **314**, 477-487 (2006)
14. Halburd, RG, Korhonen, RJ: Nevanlinna theory for the difference operator. Ann. Acad. Sci. Fenn., Math. **31**, 463-478 (2006)
15. Halburd, RG, Korhonen, RJ: Existence of finite-order meromorphic solutions as a detector of integrability in difference equations. Physica D **218**, 191-203 (2006)
16. Halburd, RG, Korhonen, RJ: Meromorphic solution of difference equation, integrability and the discrete Painlevé equations. J. Phys. A **40**, 1-38 (2007)
17. Halburd, RG, Korhonen, RJ: Finite-order meromorphic solutions and the discrete Painlevé equations. Proc. Lond. Math. Soc. **94**, 443-474 (2007)
18. Laine, I, Yang, CC: Clunie theorems for difference and q-difference polynomials. J. Lond. Math. Soc. **76**, 556-566 (2007)
19. Peng, CW, Chen, ZX: On a conjecture concerning some nonlinear difference equations. Bull. Malays. Math. Soc. **36**, 221-227 (2013)
20. Peng, CW, Chen, ZX: Properties of meromorphic solutions of some certain difference equations. Kodai Math. J. **37**, 97-119 (2014)
21. Gundersen, G: Finite order solutions of second order linear differential equations. Trans. Am. Math. Soc. **305**, 415-429 (1988)