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A reconsideration of Jensen's inequality and its applications

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Abstract

A finite form of Jensen's inequality for a continuous *convex* function from a topological abelian semigroup to another topological ordered abelian semigroup is obtained under some assumption. As an application, a refinement of a mean inequality is also obtained.

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1 Introduction

This is the inheritance of the idea of [1, Theorem 1] which gives a new interpretation of Jensen's inequality by φ-mean. The finite form of Jensen's inequality proved by Jensen in 1906 asserts that if $t_1, \ldots, t_n$ are positive numbers with $\sum_{i=1}^n t_i = 1$ and f is a continuous convex (resp. concave) function on a real interval I, then

$$f\left(\sum_{i=1}^n t_i x_i\right) \le \sum_{i=1}^n t_i f(x_i) \quad \left(\text{resp.}\, f\left(\sum_{i=1}^n t_i x_i\right) \ge \sum_{i=1}^n t_i f(x_i)\right)$$

holds for all $x_1, \ldots, x_n \in I$.

We first introduce a concept called '$(*, \circ)$-convex (or concave)' for a continuous function from a topological abelian semigroup $(I, *)$ to another topological ordered abelian semigroup $(J, \circ)$ and give an interesting example of such a function (see Remark 1). Our purpose of this paper is to give a finite form of Jensen's inequality for such a function under some assumption (see Theorem 1). Also, as an application, we give a refinement of a mean inequality (see Theorem 2).

2 Terminology and main theorem

Let I be a topological space, and let $*$ be a topological abelian semigroup operation on I. For any $x \in I$ and $n \in \mathbf{N}$ with $n \ge 1$, define the nth power $x^{(n)_*}$ of x recursively by $x^{(1)_*} = x$ and $x^{(n+1)_*} = x^{(n)_*} * x$ for $n \ge 1$. We assume that

($\sharp_1$) any nth-power function $x \mapsto x^{(n)_*}$ is a bijection of I onto itself.

By the assumption ($\sharp_1$), for each $x \in I$ and $n \in \mathbf{N}$, there exists a unique element a of I such that $a^{(n)_*} = x$. Denote by $x^{(1/n)_*}$ such an element a. Moreover, we define

$$x^{(m/n)_*} = \left(x^{(1/n)_*}\right)^{(m)_*}$$

for each $m, n \in \mathbf{N}$. Then we can easily see that this definition is well defined, that is,

$$\frac{m}{n} = \frac{m'}{n'} \quad \Rightarrow \quad \left(x^{(1/n)_*}\right)^{(m)_*} = \left(x^{(1/n')_*}\right)^{(m')_*} \quad (\forall x \in I).$$

In this case, we can easily show that the following power laws:

$$x^{(p+q)_*} = x^{(p)_*} * x^{(q)_*}, \qquad x^{(pq)_*} = \left(x^{(p)_*}\right)^{(q)_*} \quad \text{and} \quad (x * y)^{(p)_*} = x^{(p)_*} * y^{(p)_*} \tag{1}$$

for all $p, q \in \mathbf{Q}_+$ and $x, y \in I$. Here $\mathbf{Q}_+$ denotes the set of all positive rational numbers. Moreover, we assume that

($\sharp_2$) for each $x \in I$, the function $p \mapsto x^{(p)_*}$ is continuous on $\mathbf{Q}_+$ and it has a continuous extension to $\mathbf{R}_+$, say $t \mapsto x^{(t)_*}$.

Here $\mathbf{R}_+$ denotes the set of all positive real numbers. Therefore power laws (1) hold for all $p, q \in \mathbf{R}_+$. Denote by $\mathcal{A}_+(I)$ the set of all topological abelian semigroup operations on I satisfying the assumptions ($\sharp_1$) and ($\sharp_2$). Our assumption ($\sharp_1$) leads to the following important concept called 'mean'. For each $x, y \in I$, put

$$M_*(x, y) = (x * y)^{(1/2)_*}.$$

We call $M_*(x, y)$ the mean of x and y with respect to the operation $*$.

Moreover, let J be a topological ordered space with relation $\leq$, and denote by $\mathcal{A}_+^0(J, \leq)$ the set of all operations $\circ \in \mathcal{A}_+(J)$ such that

($\flat_1$) $a \leq b \Leftrightarrow a \circ c \leq b \circ c$ $(a, b, c \in J)$

and

($\flat_2$) $a \leq b \Rightarrow a^{(t)_\circ} \leq b^{(t)_\circ}$ $(a, b \in J,\ t \in \mathbf{R}_+)$.

Let $C(I, J)$ be the set of all continuous functions from I to J. Take $* \in \mathcal{A}_+(I)$, $\circ \in \mathcal{A}_+^0(J, \leq)$ and $f \in C(I, J)$ arbitrarily. If f satisfies

$$f\left(M_*(x, y)\right) \leq M_\circ\left(f(x), f(y)\right) \quad \left(\text{resp.}\ f\left(M_*(x, y)\right) \geq M_\circ\left(f(x), f(y)\right)\right)$$

for all $x, y \in I$, then we say that f is said to be $(*, \circ)$-convex (resp. concave).

The following theorem states a finite form of Jensen's inequality for a $(*, \circ)$-convex (or concave) function.

Theorem 1 *Let $* \in \mathcal{A}_+(I)$ and $\circ \in \mathcal{A}_+^0(J, \leq)$. If $f \in C(I, J)$ is $(*, \circ)$-convex, then*

$$f\left(x_1^{(t_1)_*} * \cdots * x_n^{(t_n)_*}\right) \leq f(x_1)^{(t_1)_\circ} \circ \cdots \circ f(x_n)^{(t_n)_\circ}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$.

If f is $(*, \circ)$-concave, then the inequality above is reversed.

Remark 1 Let $\mathbf{R}_+^2$ be the product space of $\mathbf{R}_+$ with ordinary topology. Let $*$ be the operation on $\mathbf{R}_+^2$ defined by

$$(a,b) * (c,d) = (ac, ad + bc)$$

for each $(a,b),(c,d) \in \mathbf{R}_+^2$. Then $*$ is a topological abelian semigroup operation on $\mathbf{R}_+^2$ (*cf.* [2, p.157-160]). In fact, $(\mathbf{R}_+^2, *)$ is topologically isomorphic to an abelian subsemigroup of the semigroup of all 2×2 matrices with usual product under the following mapping:

$$(a,b) \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.$$

Note that

$$(a,b)^{(n)_*} = \left(a^n, na^{n-1}b\right) \quad \text{and} \quad (a,b)^{(1/n)_*} = \left(a^{1/n}, \frac{b}{na^{1-1/n}}\right)$$

for all $n \in \mathbf{N}$. Then a simple calculation implies that

$$(a,b)^{(m/n)_*} = \left(a^{m/n}, \frac{m}{n} a^{\frac{m}{n}-1} b\right)$$

for each $m, n \in \mathbf{N}$. Therefore we see that $* \in \mathcal{A}_+(\mathbf{R}_+^2)$. In this case, we obtain from a simple calculation that

$$M_*\big((a,b),(c,d)\big) = \left(\sqrt{ac}, \frac{ad+bc}{2\sqrt{ac}}\right)$$

for each $(a,b),(c,d) \in \mathbf{R}_+^2$.

Now let $\cdot$ be the ordinary multiplication on $\mathbf{R}_+$. Since $\mathbf{R}_+$ becomes a topological ordered space with the ordinary topology and the ordinary order $\leq$, we have that $\cdot \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and $M_\cdot(x,y) = \sqrt{xy}$ for each $x, y \in \mathbf{R}_+$. Let α and β be real numbers and put

$$f_{\alpha,\beta}(a,b) = a^\alpha b^\beta$$

for each $(a,b) \in \mathbf{R}_+^2$. Then $f_{\alpha,\beta}$ is a continuous function from $\mathbf{R}_+^2$ to $\mathbf{R}_+$ such that

$$f_{\alpha,\beta}\big(M_*\big((a,b),(c,d)\big)\big) = (\sqrt{ac})^{\alpha-\beta}\left(\frac{ad+bc}{2}\right)^\beta$$

and

$$M_\cdot\big(f_{\alpha,\beta}(a,b), f_{\alpha,\beta}(c,d)\big) = (\sqrt{ac})^\alpha(\sqrt{bd})^\beta$$

for each $(a,b),(c,d) \in \mathbf{R}_+^2$. Therefore we can easily see that if $\beta \leq 0$ (resp. $\beta \geq 0$), then $f_{\alpha,\beta}$ is $(*,\cdot)$-convex (resp. $(*,\cdot)$-concave).

Remark 2 Let E be a nontrivial real interval with the ordinary topology and the ordinary order $\leq$. In this case, Craigen and Pales [3] showed that if $\circ$ is a continuous cancellative semigroup operation on E, then there exists a continuous order-preserving bijection φ of E onto another (necessarily unbounded) real interval such that

$$a \circ b = \varphi^{-1}\big(\varphi(a) + \varphi(b)\big)$$

for all $a, b \in E$ (*cf.* Aczel [4, 5]). Therefore we can easily see that if $\varphi(E) = \mathbf{R}_+$, then all continuous cancellative semigroup operations on E are in $\mathcal{A}_+^0(E, \leq)$.

Remark 3 It is clear that a direct product of $\mathbf{R}_+$ admits a semigroup operation in $\mathcal{A}_+^0$ that is given as the product of semigroup operations on each $\mathbf{R}_+$. However, the semigroup operation on $\mathbf{R}_+^2$ described in Remark 1 does not satisfy properties ($\flat_1$) and ($\flat_2$). So, it would be of interest to give an example of an ordered abelian semigroup with a semigroup operation in $\mathcal{A}_+^0$, which is not isomorphic, as a topological semigroup, to the direct product of the topological semigroup $\mathbf{R}_+$.

3 Lemmas and proof of Theorem 1

Throughout this section, let I and J be as in Section 2 and suppose that $* \in \mathcal{A}_+(I)$, $\circ \in \mathcal{A}_+^0(J, \leq)$ and that $f \in C(I, J)$ is $(*, \circ)$-convex.

Remark 4 If $f \in C(I, J)$ is $(*, \circ)$-concave, then all inequalities in this section are reversed.

Lemma 1 *The inequality*

$$f\big(x_1^{(1/2)*} * \cdots * x_n^{(1/2^n)*}\big) \leq f(x_1)^{(1/2)\circ} \circ \cdots \circ f(x_{n-1})^{(1/2^{n-1})\circ} \circ \big[f\big(x_n^{(1/2)*}\big)\big]^{(1/2^{n-1})\circ}$$

holds for all $n \in \mathbf{N}$ *and* $x_1, \ldots, x_n \in I$.

Proof Let $n \in \mathbf{N}$ and $x_1, \ldots, x_n \in I$. Since f is $(*, \circ)$-convex, it follows that

$$\begin{aligned}
&f\big(x_1^{(1/2)*} * x_2^{(1/2^2)*} * \cdots * x_n^{(1/2^n)*}\big) \\
&\quad = f\big(\big(x_1 * x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)^{(1/2)*}\big) \\
&\quad \leq \big(f(x_1) \circ f\big(x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)\big)^{(1/2)\circ} \\
&\quad = f(x_1)^{(1/2)\circ} \circ \big[f\big(x_2^{(1/2)*} * \cdots * x_n^{(1/2^{n-1})*}\big)\big]^{(1/2)\circ} \\
&\quad = f(x_1)^{(1/2)\circ} \circ \big[f\big(\big(x_2 * x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)^{(1/2)*}\big)\big]^{(1/2)\circ} \\
&\quad \leq f(x_1)^{(1/2)\circ} \circ \big[\big(f(x_2) \circ f\big(x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)\big)^{(1/2)\circ}\big]^{(1/2)\circ} \\
&\quad = f(x_1)^{(1/2)\circ} \circ f(x_2)^{(1/2^2)\circ} \circ \big[f\big(x_3^{(1/2)*} * \cdots * x_n^{(1/2^{n-2})*}\big)\big]^{(1/2^2)\circ} \\
&\quad \cdots \\
&\quad \leq f(x_1)^{(1/2)\circ} \circ f(x_2)^{(1/2^2)\circ} \circ \cdots \circ f(x_{n-1})^{(1/2^{n-1})\circ} \circ \big[f\big(x_n^{(1/2)*}\big)\big]^{(1/2^{n-1})\circ}.
\end{aligned}$$

Therefore we obtain the desired inequality. □

Let p and q be two nonnegative rational numbers. We define $x^{(p)*} * y^{(q)*} = y^{(q)*}$ if $p = 0$, $q > 0$ and $x^{(p)*} * y^{(q)*} = x^{(p)*}$ if $q = 0$, $p > 0$. We also define in the same way as for $\circ$. Applying these notations, we show the following lemma.

Lemma 2 *The inequality* $f(x^{(p)_*} * y^{(1-p)_*}) \leq f(x)^{(p)_\circ} \circ f(y)^{(1-p)_\circ}$ *holds for all* $x, y \in I$ *and* $p \in \mathbf{R}_+$ *with* $0 < p < 1$.

Proof Let $x, y \in I$ and $p \in \mathbf{R}_+$ with $0 < p < 1$. By the binary system, we have the expansion $p = \sum_{i=1}^{\infty} p_i/2^i$, where $p_i \in \{0,1\}$ $(i = 1, 2, \ldots)$. Since $p_i \in \{0,1\}$ $(i = 1, 2, \ldots)$, it follows that

$$\begin{aligned}
&\lim_{n\to\infty} \bigl[f\bigl(x^{(p_1)_*} * y^{(1-p_1)_*}\bigr)\bigr]^{(1/2)_\circ} \circ \cdots \circ \bigl[f\bigl(x^{(p_{n-1})_*} * y^{(1-p_{n-1})_*}\bigr)\bigr]^{(1/2^{n-1})_\circ} \\
&\quad = \lim_{n\to\infty} \bigl[f(x)^{(p_1/2)_\circ} \circ f(y)^{((1-p_1)/2)_\circ}\bigr] \circ \cdots \\
&\qquad \circ \bigl[f(x)^{(p_n/2^{n-1})_\circ} \circ f(y)^{((1-p_{n-1})/2^{n-1})_\circ}\bigr] \\
&\quad = \lim_{n\to\infty} f(x)^{(\sum_{i=1}^{n-1} p_i/2^i)_\circ} \circ f(y)^{(\sum_{i=1}^{n-1} (1-p_i)/2^i)_\circ} \\
&\quad = f(x)^{(p)_\circ} \circ f(y)^{(1-p)_\circ}.
\end{aligned}$$

So, putting

$$a_n = \bigl[f\bigl(x^{(p_1)_*} * y^{(1-p_1)_*}\bigr)\bigr]^{(1/2)_\circ} \circ \cdots \circ \bigl[f\bigl(x^{(p_{n-1})_*} * y^{(1-p_{n-1})_*}\bigr)\bigr]^{(1/2^{n-1})_\circ}$$

for each $n \in \mathbf{N}$, we have

$$\lim_{n\to\infty} a_n = f(x)^{(p)_\circ} \circ f(y)^{(1-p)_\circ}. \tag{2}$$

Put

$$\mathbf{N}_1 = \bigl\{n \in \mathbf{N} : f\bigl(\bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2)_*}\bigr) = f\bigl(x^{(1/2)_*}\bigr)\bigr\}$$

and

$$\mathbf{N}_0 = \bigl\{n \in \mathbf{N} : f\bigl(\bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2)_*}\bigr) = f\bigl(y^{(1/2)_*}\bigr)\bigr\}.$$

Since

$$f\bigl(\bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2)_*}\bigr) = \begin{cases} f(x^{(1/2)_*}) & (p_n = 1), \\ f(y^{(1/2)_*}) & (p_n = 0) \end{cases}$$

for each $n \in \mathbf{N}$, it follows that $\mathbf{N} = \mathbf{N}_1 \cup \mathbf{N}_0$. Then either $\mathbf{N}_1$ or $\mathbf{N}_0$ are infinite. Put

$$e_n = \bigl[f\bigl(\bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2)_*}\bigr)\bigr]^{(1/2^{n-1})_\circ}$$

for each $n \in \mathbf{N}$. If $\mathbf{N}_1$ is infinite, then we have

$$\lim_{n\in\mathbf{N}_1} e_n \circ f\bigl(x^{(1/2)_*}\bigr) = \lim_{n\in\mathbf{N}_1} \bigl(f\bigl(x^{(1/2)_*}\bigr)\bigr)^{(1+1/2^{n-1})_\circ} = f\bigl(x^{(1/2)_*}\bigr). \tag{3}$$

Therefore it follows from (2) and (3) that

$$f(x)^{(p)_\circ} \circ f(y)^{(1-p)_\circ} \circ f\bigl(x^{(1/2)_*}\bigr) = \lim_{n\in\mathbf{N}_1} a_n \circ e_n \circ f\bigl(x^{(1/2)_*}\bigr). \tag{4}$$

Also we have from Lemma 1 that

$$\begin{aligned}
&f\bigl(x^{(\sum_{i=1}^n p_i/2^i)_*} * y^{(\sum_{i=1}^n (1-p_i)/2^i)_*}\bigr) \\
&\quad = f\bigl(\bigl(x^{(p_1)_*} * y^{(1-p_1)_*}\bigr)^{(1/2)_*} * \cdots * \bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2^n)_*}\bigr) \\
&\quad \le \bigl[f\bigl(x^{(p_1)_*} * y^{(1-p_1)_*}\bigr)\bigr]^{(1/2)_\circ} \circ \cdots \circ \bigl[f\bigl(x^{(p_{n-1})_*} * y^{(1-p_{n-1})_*}\bigr)\bigr]^{(1/2^{n-1})_\circ} \\
&\qquad \circ \bigl[f\bigl(\bigl(x^{(p_n)_*} * y^{(1-p_n)_*}\bigr)^{(1/2)_*}\bigr)\bigr]^{(1/2^{n-1})_\circ} \\
&\quad = a_n \circ e_n
\end{aligned}$$

for all $n \in \mathbf{N}_1$. Then it follows from ($\flat_1$) that

$$f\bigl(x^{(\sum_{i=1}^n p_i/2^i)_*} * y^{(\sum_{i=1}^n (1-p_i)/2^i)_*}\bigr) \circ f\bigl(x^{(1/2)_*}\bigr) \le a_n \circ e_n \circ f\bigl(x^{(1/2)_*}\bigr) \tag{5}$$

for all $n \in \mathbf{N}_1$. Letting $n \in \mathbf{N}_1 \to \infty$ in (5), we obtain from (4) that

$$f\bigl(x^{(p)_*} * y^{(1-p)_*}\bigr) \circ f\bigl(x^{(1/2)_*}\bigr) \le f(x)^{(p)_\circ} \circ f(y)^{(1-p)_\circ} \circ f\bigl(x^{(1/2)_*}\bigr). \tag{6}$$

Canceling $f(x^{(1/2)_*})$ in (6) by ($\flat_1$), we obtain the desired inequality. Similarly, the desired inequality is obtained in case that $\mathbf{N}_0$ is infinite. $\square$

Lemma 3 *The inequality*

$$f\bigl((x_1 * \cdots * x_n)^{(1/n)_*}\bigr) \le \bigl(f(x_1) \circ \cdots \circ f(x_n)\bigr)^{(1/n)_\circ}$$

holds for all $n \in \mathbf{N}$ *and* $x_1, \ldots, x_n \in I$.

Proof It is clear that the lemma holds for $n = 1$. Suppose the lemma holds for $n = k$. Let $x_1, \ldots, x_k, x_{k+1} \in I$ and put

$$a = \bigl(f(x_1) \circ \cdots \circ f(x_k)\bigr)^{(1/k)_\circ} \quad \text{and} \quad x = (x_1 * \cdots * x_k)^{(1/k)_*}.$$

Then $a \ge f(x)$ by hypothesis, and hence $a^{(k/k+1)_\circ} \ge f(x)^{(k/k+1)_\circ}$ by ($\flat_2$). It follows from ($\flat_1$) that

$$a^{(k/k+1)_\circ} \circ f(x_{k+1})^{(1/k+1)_\circ} \ge f(x)^{(k/k+1)_\circ} \circ f(x_{k+1})^{(1/k+1)_\circ}.$$

Therefore we have from Lemma 2 that

$$\begin{aligned}
\bigl(f(x_1) \circ \cdots \circ f(x_{k+1})\bigr)^{(1/k+1)_\circ} &= \bigl(a^{(k)_\circ} \circ f(x_{k+1})\bigr)^{(1/k+1)_\circ} \\
&= a^{(k/k+1)_\circ} \circ f(x_{k+1})^{(1/k+1)_\circ} \\
&\ge f(x)^{(k/k+1)_\circ} \circ f(x_{k+1})^{(1/k+1)_\circ} \\
&\ge f\bigl(x^{(k/k+1)_*} * x_{k+1}^{(1/k+1)_*}\bigr) \\
&= f\bigl((x_1 * \cdots * x_k * x_{k+1})^{(1/k+1)_*}\bigr).
\end{aligned}$$

In other words, the lemma holds for $n = k+1$. Then, by mathematical induction, the lemma holds for all $n \in \mathbf{N}$. □

Lemma 4 *The inequality*

$$f\left(x_1^{(p_1)_*} * \cdots * x_n^{(p_n)_*}\right) \leq f(x_1)^{(p_1)_\circ} \circ \cdots \circ f(x_n)^{(p_n)_\circ}$$

holds for all $x_1, \ldots, x_n \in I$, $n \in \mathbf{N}$ *and* $p_1, \ldots, p_n \in \mathbf{Q}_+$ *with* $p_1 + \cdots + p_n = 1$.

Proof This result follows directly from Lemma 3. □

We are now in a position to prove Theorem 1.

Proof Let $n \in \mathbf{N}$, $x_1, \ldots, x_n \in I$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$. For each $1 \leq i \leq n$, choose a sequence $\{p_{ik}\}_{k=1}^\infty$ in $\mathbf{Q}_+$ which converges to t_i. Put $q_k = p_{1k} + \cdots + p_{nk}$ for each $k \in \mathbf{N}$. Then we have from Lemma 4 that

$$f\left(x_1^{(p_{1k}/q_k)_*} * \cdots * x_n^{(p_{nk}/q_k)_*}\right) \leq f(x_1)^{(p_{1k}/q_k)_\circ} \circ \cdots \circ f(x_n)^{(p_{nk}/q_k)_\circ}. \tag{7}$$

Hence, after taking the limit with respect to k in (7), we obtain the desired inequality:

$$f\left(x_1^{(t_1)_*} * \cdots * x_n^{(t_n)_*}\right) \leq f(x_1)^{(t_1)_\circ} \circ \cdots \circ f(x_n)^{(t_n)_\circ}.$$

Of course, if f is $(*, \circ)$-concave, the above inequality is reversed, as stated in Remark 4. This completes the proof of Theorem 1. □

4 Applications

Let K be a topological ordered space with order $\leq$, and let $\circ$, $*$ and $\diamond$ be three operations in $\mathcal{A}_+^0(K, \leq)$ which have the following properties:

$$(a \circ b) * c = (a * c) \circ (b * c) \quad (\forall a, b, c \in K), \tag{8}$$

$$\exists e \in K : e * x = x \quad (\forall x \in K), \tag{9}$$

$$a \diamond b = a \circ b \circ (a * b) \quad (\forall a, b \in K). \tag{10}$$

In this case, it is clear that an element e in (9) is unique.

Lemma 5 *The equality* $x^{(t)_\diamond} \circ e = (x \circ e)^{(t)_*}$ *holds for each* $x \in K$ *and* $t \in \mathbf{R}_+$.

Proof Take $x \in K$ arbitrarily. By (8) and (9), we have $(a \diamond b) \circ e = (a \circ e) * (b \circ e)$ for all $a, b \in K$. By mathematical induction, we have

$$(x_1 \diamond \cdots \diamond x_k) \circ e = (x_1 \circ e) * \cdots * (x_k \circ e) \tag{11}$$

for all $k \in \mathbf{N}$ and $x_1, \ldots, x_k \in K$. In particular, we have

$$x^{(k)_\diamond} \circ e = (x \circ e)^{(k)_*} \tag{12}$$

for all $k \in \mathbf{N}$. Then we have

$$\begin{aligned}
(x \circ e)^{(n/m)_*} &= \left((x \circ e)^{(n)_*}\right)^{(1/m)_*} \\
&= \left(x^{(n)_\diamond} \circ e\right)^{(1/m)_*} \quad \text{by (12)} \\
&= \left(\left(x^{(n/m)_\diamond}\right)^{(m)_\diamond} \circ e\right)^{(1/m)_*} \\
&= \left(\left(x^{(n/m)_\diamond} \circ e\right)^{(m)_*}\right)^{(1/m)_*} \quad \text{by (12)} \\
&= x^{(n/m)_\diamond} \circ e
\end{aligned}$$

for all $n, m \in \mathbf{N}$. Therefore $x^{(p)_\diamond} \circ e = (x \circ e)^{(p)_*}$ holds for all $p \in \mathbf{Q}_+$. Take $t \in \mathbf{R}_+$ arbitrarily, and choose a sequence $\{p_n\}$ in $\mathbf{Q}_+$ which converges to t. Then

$$x^{(t)_\diamond} \circ e = \lim_{n\to\infty} x^{(p_n)_\diamond} \circ e = \lim_{n\to\infty} (x \circ e)^{(p_n)_*} = (x \circ e)^{(t)_*}$$

holds and so the proof is complete. □

Lemma 6 *Suppose that $M_*(a,b) \le M_\circ(a,b)$ holds for all $a,b \in K$. Then $M_*(a,b) \le M_\diamond(a,b) \le M_\circ(a,b)$ holds for each $a,b \in K$.*

Proof Let $a, b \in K$. We first show that $M_*(a,b) \le M_\diamond(a,b)$. Since

$$\left((a * b)^{(1/2)_*}\right)^{(2)_\circ} \le a \circ b,$$

it follows from Lemma 5 that

$$\begin{aligned}
\left((a * b)^{(1/2)_*} \circ e\right)^{(2)_*} &= (a * b) \circ \left((a * b)^{(1/2)_*}\right)^{(2)_\circ} \circ e \\
&\le (a * b) \circ (a \circ b) \circ e \\
&= (a \diamond b) \circ e \\
&= \left((a \diamond b)^{(1/2)_\diamond}\right)^{(2)_\diamond} \circ e \\
&= \left((a \diamond b)^{(1/2)_\diamond} \circ e\right)^{(2)_*}.
\end{aligned}$$

Therefore we obtain from $(\flat_2)$ for $*$ that

$$(a * b)^{(1/2)_*} \circ e \le (a \diamond b)^{(1/2)_\diamond} \circ e.$$

Canceling e in the above inequality, we obtain the desired inequality.

We next show that $M_\diamond(a,b) \le M_\circ(a,b)$. Since

$$a * b \le \left((a \circ b)^{(1/2)_\circ}\right)^{(2)_*},$$

it follows from Lemma 5 that

$$\begin{aligned}
\left((a \diamond b)^{(1/2)_\diamond} \circ e\right)^{(2)_*} &= \left((a \diamond b)^{(1/2)_\diamond}\right)^{(2)_\diamond} \circ e \\
&= (a \diamond b) \circ e
\end{aligned}$$

$$\begin{aligned}&= (a * b) \circ a \circ b \circ e \\ &\leq \left((a \circ b)^{(1/2)_\circ}\right)^{(2)_*} \circ a \circ b \circ e \\ &= \left((a \circ b)^{(1/2)_\circ} \circ e\right)^{(2)_*}.\end{aligned}$$

Therefore we obtain from ($\flat_2$) for $*$ that

$$(a \diamond b)^{(1/2)_\diamond} \circ e \leq (a \circ b)^{(1/2)_\circ} \circ e.$$

Canceling e in the above inequality, we obtain the desired inequality and so the proof is complete. □

The following result is a refinement of the mean inequality.

Theorem 2 *Let K be a topological ordered space with order $\leq$ and $*, \circ, \diamond \in \mathcal{A}_+^0(K, \leq)$ satisfying* (8), (9) *and* (10). *If $M_*(x,y) \leq M_\circ(x,y)$ holds for all $x, y \in K$, then*

$$x_1^{(t_1)_*} * \cdots * x_n^{(t_n)_*} \leq x_1^{(t_1)_\diamond} \diamond \cdots \diamond x_n^{(t_n)_\diamond} \leq x_1^{(t_1)_\circ} \circ \cdots \circ x_n^{(t_n)_\circ}$$

holds for all $n \in \mathbf{N}$, $x_1, \ldots, x_n \in K$ and $t_1, \ldots, t_n \in \mathbf{R}_+$ with $t_1 + \cdots + t_n = 1$.

Proof This follows immediately from Theorem 1 and Lemma 6. □

5 Examples

Throughout this section, let $\mathbf{R}_+$ be an ordinary topological ordered space.

Example 1 Put $x \circ_t y = (x^t + y^t)^{1/t}$ for each $x, y \in \mathbf{R}_+$ and $t \in \mathbf{R}\backslash\{0\}$. Then each $\circ_t$ is a topological abelian semigroup operation on $\mathbf{R}_+$ such that $x^{(n)_{\circ_t}} = n^{1/t}x$ and $x^{(1/n)_{\circ_t}} = (1/n)^{1/t}x$ for all $n \in \mathbf{N}$ and $x \in \mathbf{R}_+$. Also, since

$$x^{(n/m)_{\circ_t}} = \left(x^{(1/m)_{\circ_t}}\right)^{(n)_{\circ_t}} = n^{1/t}(1/m)^{1/t}x = (n/m)^{1/t}x$$

for each $m, n \in \mathbf{N}$, $t \in \mathbf{R}\backslash\{0\}$ and $x \in \mathbf{R}_+$, it follows that $\circ_t \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ for each $t \in \mathbf{R}\backslash\{0\}$. Let $x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n \in \mathbf{R}_+$ with $\alpha_1 + \cdots + \alpha_n = 1$. Since $x^{(\alpha)_{\circ_t}} = \alpha^{1/t}x$ for each $x, \alpha \in \mathbf{R}_+$ and $t \in \mathbf{R}\backslash\{0\}$, we have that

$$x_1^{(\alpha_1)_{\circ_t}} \circ_t \cdots \circ_t x_n^{(\alpha_n)_{\circ_t}} = \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t\right)^{1/t} \tag{13}$$

for all $t \in \mathbf{R}\backslash\{0\}$. Note that

$$M_{\circ_s}(x,y) = \left(\frac{x^s + y^s}{2}\right)^{1/s} \leq \left(\frac{x^t + y^t}{2}\right)^{1/t} = M_{\circ_t}(x,y)$$

for all $x, y \in \mathbf{R}_+$ and $s, t \in \mathbf{R}\backslash\{0\}$ with $s \leq t$. Therefore Theorem 1 implies the following well-known inequality:

$$\left(\alpha_1 x_1^s + \cdots + \alpha_n x_n^s\right)^{1/s} \leq \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t\right)^{1/t},$$

where $s, t \in \mathbf{R}\backslash\{0\}$ with $s \leq t$.

Example 2 Let t be a positive real number, and let $\circ_t$ be the operation on $\mathbf{R}_+$ defined in Example 1. Then we have $\circ_t \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$. Also, if $x * y = xy$ $(x, y \in \mathbf{R}_+)$, then $* \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and $x^{(\alpha)_*} = x^\alpha$ for all $x, \alpha \in \mathbf{R}_+$. Hence we have

$$M_*(x,y) = \sqrt{xy} \leq \left(\frac{x^t + y^t}{2}\right)^{1/t} = M_{\circ_t}(x,y)$$

for each $x, y \in \mathbf{R}_+$.

Let $n \in \mathbf{N}$ and $x_1, \dots, x_n, \alpha_1, \dots, \alpha_n \in \mathbf{R}_+$ with $\alpha_1 + \cdots + \alpha_n = 1$. Then it is clear that

$$x_1^{(\alpha_1)_*} * \cdots * x_n^{(\alpha_n)_*} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}. \tag{14}$$

If $x \diamond y = x \circ_t y \circ_t (x * y)$ $(x, y \in \mathbf{R}_+)$, then $\diamond \in \mathcal{A}_+^0(\mathbf{R}_+, \leq)$ and three operations $*$, $\circ$ and $\diamond$ on $\mathbf{R}_+$ satisfy (8), (9) and (10). If $x, \alpha \in \mathbf{R}_+$, we have from Lemma 5 that

$$x^{(\alpha)_\diamond} \circ_t 1 = (x \circ_t 1)^{(\alpha)_*} = (x \circ_t 1)^\alpha = \left(x^t + 1\right)^{\alpha/t}. \tag{15}$$

Then we have from (11) and (15) that

$$\begin{aligned} x_1^{(\alpha_1)_\diamond} \diamond \cdots \diamond x_n^{(\alpha_n)_\diamond} \circ_t 1 &= \left(x_1^{(\alpha_1)_\diamond} \circ_t 1\right) * \cdots * \left(x_n^{(\alpha_n)_\diamond} \circ_t 1\right) \\ &= \left(x_1^{(\alpha_1)_\diamond} \circ_t 1\right) \cdots \left(x_n^{(\alpha_n)_\diamond} \circ_t 1\right) \\ &= \left(x_1^t + 1\right)^{\alpha_1/t} \cdots \left(x_n^t + 1\right)^{\alpha_n/t}, \end{aligned}$$

and hence

$$x_1^{(\alpha_1)_\diamond} \diamond \cdots \diamond x_n^{(\alpha_n)_\diamond} = \left(\left(x_1^t + 1\right)^{\alpha_1} \cdots \left(x_n^t + 1\right)^{\alpha_n} - 1\right)^{1/t}. \tag{16}$$

Then we obtain from (13), (14), (16) and Theorem 2 that

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \leq \left(\left(x_1^t + 1\right)^{\alpha_1} \cdots \left(x_n^t + 1\right)^{\alpha_n} - 1\right)^{1/t} \leq \left(\alpha_1 x_1^t + \cdots + \alpha_n x_n^t\right)^{1/t}.$$

This is a refinement of the geometric-arithmetic mean inequality.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
YN carried out the design of the study and performed the analysis. ST participated in the sequence alignment and drafted the manuscript. All authors read and approved the final manuscript.

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