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Some new identities of Frobenius-Euler numbers and polynomials

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Abstract
In this paper, we give some new and interesting identities which are derived from the basis of Frobenius-Euler. Recently, several authors have studied some identities of Frobenius-Euler polynomials. From the methods of our paper, we can also derive many interesting identities of Frobenius-Euler numbers and polynomials.

1 Introduction

Let $\lambda(\neq 1) \in \mathbf{C}$. As is well known, the Frobienius-Euler polynomials are defined by the generating function to be

$$\frac{1-\lambda}{e^t-\lambda}e^{xt} = e^{H(x|\lambda)t} = \sum_{n=0}^{\infty} H_n(x|\lambda)\frac{t^n}{n!}, \tag{1}$$

with the usual convention about replacing $H^n(x|\lambda)$ by $H_n(x|\lambda)$ (see [1–6]).

In the special case, $x = 0$, $H_n(0|\lambda) = H_n(\lambda)$ are called the nth Frobenius-Euler numbers.

Thus, by (1), we get

$$\bigl(H(\lambda)+1\bigr)^n - \lambda H_n(\lambda) = H_n(1|\lambda) - \lambda H_n(\lambda) = (1-\lambda)\delta_{0,n}, \tag{2}$$

where $\delta_{0,n}$ is the Kronecker symbol.

From (1), we can derive the following equation:

$$H_n(x|\lambda) = \bigl(H(\lambda)+x\bigr)^n = \sum_{0\le l\le n}\binom{n}{l}H_{n-l}(\lambda)x^l \quad \text{(see [6–16])}. \tag{3}$$

Thus, by (3), we easily see that the leading coefficient of $H_n(x|\lambda)$ is $H_0(\lambda) = 1$. So, $H_n(x|\lambda)$ are monic polynomials of degree n with coefficients in $\mathbf{Q}(\lambda)$.

From (1), we have

$$\sum_{n=0}^{\infty}\bigl(H_n(x+1|\lambda) - \lambda H_n(x|\lambda)\bigr)\frac{t^n}{n!} = \frac{(1-\lambda)e^{(x+1)t}}{e^t-\lambda} - \lambda\frac{1-\lambda}{e^t-\lambda}e^{xt}. \tag{4}$$

Thus, by (4), we get

$$H_n(x+1|\lambda) - \lambda H_n(x|\lambda) = (1-\lambda)x^n, \quad \text{for } n \in \mathbf{Z}_+. \tag{5}$$

It is easy to show that

$$\frac{d}{dx}H_n(x|\lambda) = \frac{d}{dx}\big(H(\lambda)+x\big)^n = nH_{n-1}(x|\lambda) \quad (n \in \mathbf{N}). \tag{6}$$

From (6), we have

$$\int_0^1 H_n(x|\lambda)\,dx = \frac{1}{n+1}\big(H_{n+1}(1|\lambda) - H_{n+1}(\lambda)\big) = \frac{\lambda-1}{n+1}H_{n+1}(\lambda). \tag{7}$$

Let $\mathbb{P}_n(\lambda) = \{p(x) \in \mathbf{Q}(\lambda)[x] \mid \deg p(x) \leq n\}$ be a vector space over $\mathbf{Q}(\lambda)$. Then we note that $\{H_0(x|\lambda), H_1(x|\lambda), \ldots, H_n(x|\lambda)\}$ is a good basis for $\mathbb{P}_n(\lambda)$.

In this paper, we develop some new methods to obtain some new identities and properties of Frobenius-Euler polynomials which are derived from the basis of Frobenius-Euler polynomials. Those methods are useful in studying the identities of Frobenius-Euler polynomials.

2 Some identities of Frobenius-Euler polynomials

Let us take $p(x) \in \mathbb{P}_n(\lambda)$. Then $p(x)$ can be expressed as a $\mathbf{Q}(\lambda)$-linear combination of $H_0(x|\lambda), \ldots, H_n(x|\lambda)$ as follows:

$$p(x) = b_0H_0(x|\lambda) + b_1H_1(x|\lambda) + \cdots + b_nH_n(x|\lambda) = \sum_{0\leq k\leq n} b_kH_k(x|\lambda). \tag{8}$$

Let us define the operator $\triangle_\lambda$ by

$$g(x) = \triangle_\lambda p(x) = p(x+1) - \lambda p(x). \tag{9}$$

From (9), we can derive the following equation (10):

$$g(x) = \triangle_\lambda p(x) = \sum_{0\leq k\leq n} b_k\big(H_k(x+1|\lambda) - \lambda H_k(x|\lambda)\big) = (1-\lambda)\sum_{0\leq k\leq n} b_kx^k. \tag{10}$$

For $r \in \mathbf{Z}_+$, let us take the rth derivative of $g(x)$ in (10) as follows:

$$g^{(r)}(x) = (1-\lambda)\sum_{r\leq k\leq n} k(k-1)\cdots(k-r+1)b_kx^{k-r}, \quad \text{where } g^{(r)}(x) = \frac{d^rg(x)}{dx^r}. \tag{11}$$

Thus, by (11), we get

$$g^r(0) = \frac{d^rg(x)}{dx^r}\Big|_{x=0} = (1-\lambda)r!b_r. \tag{12}$$

From (12), we have

$$b_r = \frac{g^{(r)}(0)}{(1-\lambda)r!} = \frac{1}{(1-\lambda)r!}\big(p^{(r)}(1) - \lambda p^{(r)}(0)\big), \tag{13}$$

where $r \in \mathbf{Z}_+$ and $p^{(r)}(0) = \frac{d^rp(x)}{dx^r}|_{x=0}$. Therefore, by (13), we obtain the following theorem.

Theorem 1 *For $\lambda(\neq 1) \in \mathbf{C}$, $n \in \mathbf{Z}_+$, let $p(x) \in \mathbb{P}_n(\lambda)$ with $p(x) = \sum_{0 \le k \le n} b_k H_k(x|\lambda)$. Then we have*

$$b_k = \frac{1}{(1-\lambda)k!} g^{(k)}(0) = \frac{1}{(1-\lambda)k!}\big(p^{(k)}(1) - \lambda p^{(k)}(0)\big).$$

Let us take $p(x) = H_n(x|\lambda^{-1})$. Then, by Theorem 1, we get

$$H_n\big(x|\lambda^{-1}\big) = \sum_{0 \le k \le n} b_k H_k(x|\lambda), \tag{14}$$

where

$$\begin{aligned} b_k &= \frac{1}{(1-\lambda)k!}\frac{n!}{(n-k)!}\big\{H_{n-k}\big(1|\lambda^{-1}\big) - \lambda H_{n-k}\big(\lambda^{-1}\big)\big\} \\ &= \frac{1}{1-\lambda}\binom{n}{k}\big\{H_{n-k}\big(1|\lambda^{-1}\big) - \lambda H_{n-k}\big(\lambda^{-1}\big)\big\} \\ &= \frac{1}{1-\lambda}\binom{n}{k}\left\{\big(1-\lambda^{-1}\big)0^{n-k} + \frac{1}{\lambda}H_{n-k}\big(\lambda^{-1}\big) - \lambda H_{n-k}\big(\lambda^{-1}\big)\right\}. \end{aligned} \tag{15}$$

By (14) and (15), we get

$$\begin{aligned} &H_n\big(x|\lambda^{-1}\big) \\ &\quad= -\frac{1}{\lambda}H_n(x|\lambda) + \sum_{k=0}^{n}\left\{\frac{\binom{n}{k}}{\lambda(1-\lambda)}H_{n-k}\big(\lambda^{-1}\big) - \frac{\lambda\binom{n}{k}}{1-\lambda}H_{n-k}\big(\lambda^{-1}\big)\right\}H_k(x|\lambda) \\ &\quad= -\frac{1}{\lambda}H_n(x|\lambda) + \sum_{k=0}^{n}\binom{n}{k}\frac{1+\lambda}{\lambda}H_{n-k}\big(\lambda^{-1}\big)H_k(x|\lambda). \end{aligned} \tag{16}$$

Therefore, by (16), we obtain the following theorem.

Theorem 2 *For $n \in \mathbf{Z}_+$, we have*

$$\lambda H_n\big(x|\lambda^{-1}\big) + H_n(x|\lambda) = (1+\lambda)\sum_{0 \le k \le n}\binom{n}{k}H_{n-k}\big(\lambda^{-1}\big)H_k(x|\lambda).$$

Let

$$p(x) = \sum_{0 \le k \le n} H_k(x|\lambda)H_{n-k}(x|\lambda) \in \mathbb{P}_n(\lambda). \tag{17}$$

From Theorem 2, we note that $p(x)$ can be generated by $\{H_0(x|\lambda), H_1(x|\lambda), \dots, H_n(x|\lambda)\}$ as follows:

$$p(x) = \sum_{0 \le k \le n} H_k(x|\lambda)H_{n-k}(x|\lambda) = \sum_{0 \le k \le n} b_k H_k(x|\lambda). \tag{18}$$

By (17), we get

$$p^{(k)}(x) = \frac{(n+1)!}{(n-k+1)!}\sum_{k \le l \le n} H_{l-k}(x|\lambda)H_{n-k}(x|\lambda), \tag{19}$$

and

$$\begin{aligned}
b_k &= \frac{1}{(1-\lambda)k!}\left\{p^{(k)}(1) - \lambda p^{(k)}(0)\right\} \\
&= \frac{(n+1)!}{(1-\lambda)k!(n-k+1)!}\sum_{l=k}^{n}\left\{H_{l-k}(1|\lambda)H_{n-l}(1|\lambda) - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\right\} \\
&= \frac{n+1}{(1-\lambda)(n-k+1)}\binom{n}{k}\sum_{l=k}^{n}\left\{\left(\lambda H_{l-k}(\lambda) + (1-\lambda)\delta_{0,l-k}\right)\left(\lambda H_{n-l} + (1-\lambda)\delta_{0,n-l}\right)\right. \\
&\quad \left. - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\right\} \\
&= \frac{n+1}{(1-\lambda)(n-k+1)}\binom{n}{k}\sum_{l=k}^{n}\left\{\lambda(1-\lambda)\delta_{0,l-k}H_{n-l}(\lambda) + \lambda(1-\lambda)\right. \\
&\quad \left. \times H_{l-k}(\lambda)\delta_{0,n-l} + (1-\lambda)^2\delta_{0,l-k}\delta_{0,n-l} + \lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)\right\} \\
&= \frac{n+1}{(1-\lambda)(n-k+1)}\binom{n}{k}\sum_{l=k}^{n}\left\{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda) + \lambda(1-\lambda)\right. \\
&\quad \left. \times H_{n-k}(\lambda) + \lambda(1-\lambda)H_{n-k}(\lambda) + (1-\lambda)^2\delta_{n,k}\right\} \\
&= \frac{n+1}{n-k+1}\binom{n}{k}\sum_{l=k}^{n}\left\{-\lambda H_{l-k}(\lambda)H_{n-l}(\lambda) + 2\lambda H_{n-k}(\lambda) + (1-\lambda)\delta_{n,k}\right\}. \qquad (20)
\end{aligned}$$

From (18) and (20), we have

$$\begin{aligned}
\sum_{0\le k\le n} H_k(x|\lambda)H_{n-k}(x|\lambda) &= (n+1)\sum_{0\le k\le n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{k\le l\le n}\left\{(-\lambda)H_{l-k}(\lambda)H_{n-l}(\lambda)\right. \\
&\quad \left. + 2\lambda H_{n-k}(\lambda)\right\}H_k(x|\lambda) + (n+1)H_n(x|\lambda). \qquad (21)
\end{aligned}$$

Therefore, by (21), we obtain the following theorem.

Theorem 3 *For $n \in \mathbf{Z}_+$, we have*

$$\begin{aligned}
&\frac{1}{n+1}\sum_{0\le k\le n} H_k(x|\lambda)H_{n-k}(x|\lambda) \\
&\quad = \sum_{0\le k\le n-1}\frac{\binom{n}{k}}{n-k+1}\sum_{k\le l\le n}\left\{(-\lambda)H_{l-k}(\lambda)H_{n-l}(\lambda) + 2\lambda H_{n-k}(\lambda)\right\}H_k(x|\lambda) + H_n(x|\lambda).
\end{aligned}$$

Let us consider

$$p(x) = \sum_{k=0}^{n}\frac{1}{k!(n-k)!}H_k(x|\lambda)H_{n-k}(x|\lambda) \in \mathbb{P}_n(\lambda). \qquad (22)$$

By Theorem 1, $p(x)$ can be expressed by

$$p(x) = \sum_{k=0}^{n} b_k H_k(x|\lambda). \qquad (23)$$

From (22), we have

$$p^{(r)}(x) = 2^r \sum_{k=r}^{n} \frac{H_{k-r}(x|\lambda)H_{n-k}(x|\lambda)}{(k-r)!(n-k)!} \quad (r \in \mathbf{Z}_+). \tag{24}$$

By Theorem 1, we get

$$\begin{aligned}
b_k &= \frac{1}{2k!}\left\{p^{(k)}(1) - p^{(k)}(0)\right\} \\
&= \frac{2^{k-1}}{k!} \sum_{l=k}^{n} \frac{1}{(l-k)!(n-l)!}\left\{H_{l-k}(1|\lambda)H_{n-l}(1|\lambda) - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\right\} \\
&= \frac{2^{k-1}}{k!} \sum_{l=k}^{n} \frac{1}{(l-k)!(n-l)!}\Big\{\big(\lambda H_{l-k}(\lambda) + (1-\lambda)\delta_{0,l-k}\big)\big(\lambda H_{n-l}(\lambda) + (1-\lambda)\delta_{0,n-l}\big) \\
&\quad - \lambda H_{l-k}(\lambda)H_{n-l}(\lambda)\Big\} \\
&= \frac{2^{k-1}}{k!}\left\{\sum_{l=k}^{n} \frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!} + (1-\lambda)^2\delta_{n,k}\right\} \\
&= \begin{cases} \frac{2^{k-1}}{k!}\sum_{l=k}^{n}\{\frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!}\}, & \text{if } k \neq n, \\ \frac{2^{n-1}(1-\lambda)}{n!}, & \text{if } k = n. \end{cases}
\end{aligned} \tag{25}$$

Therefore, by (25), we obtain the following theorem.

Theorem 4 *For $n \in \mathbf{Z}_+$, we have*

$$\begin{aligned}
&\sum_{0\le k\le n} \frac{1}{k!(n-k)!} H_k(x|\lambda)H_{n-k}(x|\lambda) \\
&\quad = \sum_{0\le k\le n-1} \frac{2^{k-1}}{k!} \sum_{k\le l\le n}\left\{\frac{\lambda(\lambda-1)H_{l-k}(\lambda)H_{n-l}(\lambda)}{(l-k)!(n-l)!} + \frac{2\lambda(1-\lambda)H_{n-k}(\lambda)}{(n-k)!}\right\} H_k(x|\lambda) \\
&\qquad + \frac{2^{n-1}(1-\lambda)}{n!} H_n(x|\lambda).
\end{aligned}$$

3 Higher-order Frobenius-Euler polynomials

For $n \in \mathbf{Z}_+$, the Frobenius-Euler polynomials of order r are defined by the generating function to be

$$\begin{aligned}
\left(\frac{1-\lambda}{e^t-\lambda}\right)^r e^{xt} &= e^{H^{(r)}(x|\lambda)t} \\
&= \sum_{n=0}^{\infty} H_n^{(r)}(x|\lambda)\frac{t^n}{n!},
\end{aligned} \tag{26}$$

with the usual convention about replacing $(H^{(r)}(x|\lambda))^n$ by $H_n^{(r)}(x|\lambda)$ (see [1–10]). In the special case, $x = 0$, $H_n^{(r)}(0|\lambda) = H_n^{(r)}(\lambda)$ are called the nth Frobenius-Euler numbers of order r (see [8, 9]).

From (26), we have

$$H_n^{(r)}(x|\lambda) = \left(H^{(r)}(\lambda) + x\right)^n = \sum_{l=0}^{n} \binom{n}{l} H_{n-l}^{(r)}(\lambda) x^l, \tag{27}$$

with the usual convention about replacing $(H^{(r)}(\lambda))^n$ by $H_n^{(r)}(\lambda)$.

By (26), we get

$$H_n^{(r)}(\lambda) = \sum_{n_1+\cdots+n_r=n} \binom{n}{n_1, n_2, \ldots, n_r} H_{n_1}(\lambda)\cdots H_{n_r}(\lambda), \tag{28}$$

where $\binom{n}{n_1,n_2,\ldots,n_r} = \frac{n!}{n_1!n_2!\cdots n_r!}$. From (27) and (28), we note that the leading coefficient of $H_n^{(r)}(x|\lambda)$ is given by

$$\begin{aligned} H_0^{(r)}(\lambda) &= \sum_{n_1+\cdots+n_r=0} \binom{n}{n_1, n_2, \ldots, n_r} H_{n_1}(\lambda)\cdots H_{n_r}(\lambda) \\ &= H_0(\lambda)\cdots H_0(\lambda) = 1. \end{aligned} \tag{29}$$

Thus, by (29), we see that $H_n^{(r)}$ is a monic polynomial of degree n with coefficients in $\mathbf{Q}(\lambda)$.
From (26), we have

$$H_n^{(0)}(x|\lambda) = x^n, \quad \text{for } n \in \mathbf{Z}_+, \tag{30}$$

and

$$\frac{\partial}{\partial x} H_n^{(r)}(x|\lambda) = \frac{\partial}{\partial x}\left(H^{(r)}(\lambda) + x\right)^n = nH_{n-1}^{(r)}(x|\lambda) \quad (r \geq 0). \tag{31}$$

It is not difficult to show that

$$H_n^{(r)}(x+1|\lambda) - \lambda H_n^{(r)}(x|\lambda) = (1-\lambda)H_n^{(r-1)}(x|\lambda). \tag{32}$$

Now, we note that $\{H_0^{(r)}(x|\lambda), H_1^{(r)}(x|\lambda), \ldots, H_n^{(r)}(x|\lambda)\}$ is also a good basis for $\mathbb{P}_n(\lambda)$.

Let us define the operator D as $Df(x) = \frac{df(x)}{dx}$ and let $p(x) \in \mathbb{P}_n(\lambda)$. Then $p(x)$ can be written as

$$p(x) = \sum_{k=0}^{n} C_k H_k^{(r)}(x|\lambda). \tag{33}$$

From (9) and (32), we have

$$\triangle_\lambda H_n^{(r)}(x|\lambda) = H_n^{(r)}(x+1|\lambda) - \lambda H_n^{(r)}(x|\lambda) = (1-\lambda)H_n^{(r-1)}(x|\lambda). \tag{34}$$

Thus, by (33) and (34), we get

$$\triangle_\lambda^r p(x) = (1-\lambda)^r \sum_{k=0}^{n} C_k H_k^{(0)}(x|\lambda) = (1-\lambda)^r \sum_{k=0}^{n} C_k x^k. \tag{35}$$

Let us take the kth derivative of $\triangle_\lambda^r p(x)$ in (35).

Then we have

$$D^k\bigl(\triangle_\lambda^r p(x)\bigr) = (1-\lambda)^r \sum_{l=k}^{n} \frac{l!}{(l-k)!} C_l x^{l-k}. \tag{36}$$

Thus, from (36), we have

$$D^k\bigl(\triangle_\lambda^r p(0)\bigr) = (1-\lambda)^r \sum_{l=k}^{n} \frac{l! C_l}{(l-k)!} 0^{l-k} = (1-\lambda)^r k! C_k. \tag{37}$$

Thus, by (37), we get

$$\begin{aligned} C_k &= \frac{D^k(\triangle_\lambda^r p(0))}{(1-\lambda)^r k!} \\ &= \frac{\triangle_\lambda^r (D^k p(0))}{(1-\lambda)^r k!} = \frac{1}{(1-\lambda)^r k!} \sum_{j=0}^{r} \binom{r}{j} (-\lambda)^{(r-j)} D^k p(j). \end{aligned} \tag{38}$$

Therefore, by (33) and (38), we obtain the following theorem.

Theorem 5 *For $r \in \mathbf{Z}_+$, let $p(x) \in \mathbb{P}_n(\lambda)$ with*

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{0 \le k \le n} C_k H_k^{(r)}(x|\lambda) \quad \bigl(C_k \in \mathbf{Q}(\lambda)\bigr).$$

Then we have

$$C_k = \frac{1}{(1-\lambda)^r k!} \sum_{0 \le j \le r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j).$$

That is,

$$p(x) = \frac{1}{(1-\lambda)^r} \sum_{0 \le k \le n} \left(\sum_{0 \le j \le r} \frac{1}{k!} \binom{r}{j} (-\lambda)^{r-j} D^k p(j) \right) H_k^{(r)}(x|\lambda).$$

Let us take $p(x) = H_n(x|\lambda) \in \mathbf{P}_n(\lambda)$. Then, by Theorem 5, $p(x) = H_n(x|\lambda)$ can be generated by $\{H_0^{(r)}(x|\lambda), H_1^{(r)}(\lambda), \ldots, H_n^{(r)}(x|\lambda)\}$ as follows:

$$H_n(x|\lambda) = \sum_{0 \le k \le n} C_k H_k^{(r)}(x|\lambda), \tag{39}$$

where

$$C_k = \frac{1}{(1-\lambda)^r} \frac{1}{k!} \sum_{0 \le j \le r} \binom{r}{j} (-\lambda)^{r-j} D^k p(j), \tag{40}$$

and

$$p^{(k)}(x) = D^k p(x) = n(n-1)\cdots(n-k+1) H_{n-k}(x|\lambda) = \frac{n!}{(n-k)!} H_{n-k}(x|\lambda). \tag{41}$$

By (40) and (41), we get

$$C_k = \frac{1}{(1-\lambda)^r}\binom{n}{k}\sum_{0\le j\le r}\binom{r}{j}(-\lambda)^{r-j}H_{n-k}(j|\lambda). \tag{42}$$

Therefore, by (39) and (42), we obtain the following theorem.

Theorem 6 *For $n \in \mathbf{Z}_+$, we have*

$$H_n(x|\lambda) = \frac{1}{(1-\lambda)^r}\sum_{0\le k\le n}\binom{n}{k}\left(\sum_{0\le j\le r}\binom{r}{j}(-\lambda)^{r-j}H_{n-k}(j|\lambda)\right)H_k^{(r)}(x|\lambda).$$

Let us assume that $p(x) = H_n^{(r)}(x|\lambda)$.
Then we have

$$\begin{aligned} p^k(x) &= n(n-1)\cdots(n-k+1)H_{n-k}^{(r)}(x|\lambda) \\ &= \frac{n!}{(n-k)!}H_{n-k}^{(r)}(x|\lambda). \end{aligned} \tag{43}$$

From Theorem 1, we note that $p(x) = H_n^{(r)}(x|\lambda)$ can be expressed as a linear combination of $H_0(x|\lambda), H_1(x|\lambda), \ldots, H_n(x|\lambda)$

$$H_n^{(r)}(x|\lambda) = \sum_{0\le k\le n} b_k H_k(x|\lambda), \tag{44}$$

where

$$\begin{aligned} b_k &= \frac{1}{(1-\lambda)k!}\left\{p^k(1) - \lambda p^{(k)}(0)\right\} \\ &= \frac{n!}{(1-\lambda)k!(n-k)!}\left\{H_{n-k}^{(r)}(1|\lambda) - \lambda H_{n-k}^{(r)}(\lambda)\right\}. \end{aligned} \tag{45}$$

By (34) and (45), we get

$$b_k = \binom{n}{k}H_{n-k}^{(r-1)}(\lambda). \tag{46}$$

Therefore, by (44) and (46), we obtain the following theorem.

Theorem 7 *For $n \in \mathbf{Z}_+$, we have*

$$H_n^{(r)}(x|\lambda) = \sum_{0\le k\le n}\binom{n}{k}H_{n-k}^{(r-1)}(\lambda)H_k(x|\lambda).$$

Remark From (2) and (37), we note that

$$\begin{aligned} \frac{d}{d\lambda}\left(\frac{1-\lambda}{e^t-\lambda}\right) = \frac{1-e^t}{(e^t-\lambda)^2} &= \frac{1}{(1-\lambda)^2}\left(\frac{(1-\lambda)^2}{(e^t-\lambda)^2} - \frac{(1-\lambda)^2}{(e^t-\lambda)^2}e^t\right) \\ &= \frac{1}{(1-\lambda)^2}\left(\frac{(1-\lambda)^2}{(e^t-\lambda)^2} - \frac{(1-\lambda)^2}{(e^t-\lambda)^2}\left(e^t-\lambda+\lambda\right)\right) \end{aligned}$$

$$= \frac{1}{1-\lambda}\left(\frac{(1-\lambda)^2}{(e^t-\lambda)^2} - \frac{1-\lambda}{e^t-\lambda}\right)$$

$$= \frac{1}{1-\lambda}\sum_{n=0}^{\infty}\left(H_n^{(2)}(\lambda) - H_n(\lambda)\right)\frac{t^n}{n!}, \tag{47}$$

and

$$\frac{d^2}{d\lambda^2}\left(\frac{1-\lambda}{e^t-\lambda}\right) = 2!\frac{1-e^t}{(e^t-\lambda)^3} = \frac{2!}{(1-\lambda)^3}\left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^3}{(e^t-\lambda)^3}e^t\right)$$

$$= \frac{2!}{(1-\lambda)^3}\left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^3}{(e^t-\lambda)^3}\left(e^t-\lambda+\lambda\right)\right)$$

$$= \frac{2!}{(1-\lambda)^2}\left(\frac{(1-\lambda)^3}{(e^t-\lambda)^3} - \frac{(1-\lambda)^2}{(e^t-\lambda)^2}\right)$$

$$= \frac{2!}{(1-\lambda)^2}\sum_{n=0}^{\infty}\left(H_n^{(3)}(\lambda) - H_n^{(2)}(\lambda)\right)\frac{t^n}{n!}. \tag{48}$$

Continuing this process, we obtain the following equation:

$$\frac{d^k}{d\lambda^k}\left(\frac{1-\lambda}{e^t-\lambda}\right) = \frac{k!}{(1-\lambda)^k}\left(\frac{(1-\lambda)^{k+1}}{(e^t-\lambda)^{k+1}} - \frac{(1-\lambda)^k}{(e^t-\lambda)^k}\right)$$

$$= \frac{k!}{(1-\lambda)^k}\sum_{n=0}^{\infty}\left(H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)\right)\frac{t^n}{n!} \quad \text{(see [8])}. \tag{49}$$

By (1), (2) and (49), we get

$$\frac{d^k}{d\lambda^k}H_n(\lambda) = \frac{k!}{(1-\lambda)^k}\left(H_n^{(k+1)}(\lambda) - H_n^{(k)}(\lambda)\right),$$

where k is a positive integer (see [7, 8]).

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

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References
1. Araci, S, Acikgoz, M: A note on the Frobenius-Euler numbers and polynomials associated with Bernstein polynomials. Adv. Stud. Contemp. Math. **22**(3), 399-406 (2012)
2. Can, M, Cenkci, M, Kurt, V, Simsek, Y: Twisted Dedekind type sums associated with Barnes' type multiple Frobenius-Euler *L*-functions. Adv. Stud. Contemp. Math. **18**(2), 135-160 (2009)
3. Cangul, IN, Kurt, V, Ozden, H, Simsek, Y: On the higher-order *w*-*q*-Genocchi numbers. Adv. Stud. Contemp. Math. **19**(1), 39-57 (2009)

4. Carlitz, L: A note on q-Eulerian numbers. J. Comb. Theory, Ser. A **25**(1), 90-94 (1978)
5. Kim, DS, Dolgy, DV, Kim, T, Rim, S-H: Some formulae for the product of two Bernoulli and Euler polynomials. Abstr. Appl. Anal. **2012**, Article ID 784307 (2012)
6. Kim, T, Choi, J: A note on the product of Frobenius-Euler polynomials arising from the p-adic integral on $\mathbb{Z}_p$. Adv. Stud. Contemp. Math. **22**(2), 215-223 (2012)
7. Kim, T: An identity of the symmetry for the Frobenius-Euler polynomials associated with the fermionic p-adic invariant q-integrals on $\mathbb{Z}_p$. Rocky Mt. J. Math. **41**(1), 239-247 (2011)
8. Kim, T: Identities involving Frobenius-Euler polynomials arising from non-linear differential equations. J. Number Theory **132**(12), 2854-2865 (2012)
9. Kim, T: Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbf{Z}_p$. Russ. J. Math. Phys. **16**(1), 93-96 (2009)
10. Rim, S-H, Jeong, J: On the modified q-Euler numbers of higher order with weight. Adv. Stud. Contemp. Math. **22**(1), 93-98 (2012)
11. Rim, S-H, Lee, J: Some identities on the twisted (h, q)-Genocchi numbers and polynomials associated with q-Bernstein polynomials. Int. J. Math. Math. Sci. **2011**, Article ID 482840 (2011)
12. Ryoo, CS: A note on the Frobenius-Euler polynomials. Proc. Jangjeon Math. Soc. **14**(4), 495-501 (2011)
13. Simsek, Y, Bayad, A, Lokesha, V: q-Bernstein polynomials related to q-Frobenius-Euler polynomials, l-functions, and q-Stirling numbers. Math. Methods Appl. Sci. **35**(8), 877-884 (2012)
14. Simsek, Y, Yurekli, O, Kurt, V: On interpolation functions of the twisted generalized Frobenius-Euler numbers. Adv. Stud. Contemp. Math - Jang'jun Math. Soc. **15**(2), 187-194 (2007)
15. Shiratani, K: On the Euler numbers. Mem. Fac. Sci., Kyushu Univ., Ser. A, Math. **27**, 1-5 (1973)
16. Shiratani, K, Yamamoto, S: On a p-adic interpolation function for the Euler numbers and its derivatives. Mem. Fac. Sci., Kyushu Univ., Ser. A, Math. **39**(1), 113-125 (1985)

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