EXISTENCE OF SOLUTIONS FOR ELLIPTIC EQUATIONS HAVING NATURAL GROWTH TERMS IN ORLICZ SPACES

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Existence result for strongly nonlinear elliptic equation with a natural growth condition on the nonlinearity is proved.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with the segment property.

Consider the nonlinear Dirichlet problem

$$A(u) + g(x,u,\nabla u) = f, \tag{1.1}$$

where $A(u) = -\operatorname{div} a(x,u,\nabla u)$ is a Leray-Lions operator defined on $D(A) \subset W_0^1 L_M(\Omega) \to W^{-1}L_{\overline{M}}(\Omega)$ with M an N-function and where g is a nonlinearity with the "natural" growth condition

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + M(|\xi|)) \tag{1.2}$$

and which satisfies the classical sign condition $g(x,s,\xi)s \geq 0$. The right-hand side f is assumed to belong to $W^{-1}E_{\overline{M}}(\Omega)$.

It is well known that Gossez [12] solved (1.1) in the case where g depends only on x and u. If g depends also on ∇u, existence theorems have recently been proved by Benkirane and Elmahi in [3, 4] by making some restrictions.

In [3], g is supposed to satisfy a "nonnatural" growth condition of the form

$$|g(x,s,\xi)| \leq b(|s|)(c(x) + P(|\xi|)) \quad \text{with } P \ll M, \tag{1.3}$$

and in [4], g is supposed to satisfy a natural growth of the form (1.2) but the result is restricted to N-functions M satisfying a Δ_2-condition.

It is our purpose in this paper to extend the result of [4] to general N-functions (i.e., without assuming a Δ_2-condition on M) and hence generalize the results of [3, 4, 7].

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As an example of equations to which the present result can be applied, we give

(1)

$$-\operatorname{div}\left(\exp(m|u|)\frac{\exp(|\nabla u|)-1}{|\nabla u|^2}\nabla u\right)+u\sin^2 u\exp(|\nabla u|)=f,\quad m\geq 0,$$
$$\text{with } f=f_0+\sum_{i=1}^{N}\frac{\partial f_i}{\partial x_i},\ \int_\Omega f_i\log|f_i|\,dx<\infty, \tag{1.4}$$

(2)

$$-\operatorname{div}\left(\frac{p(|\nabla u|)}{|\nabla u|}\nabla u\right)+ug(u)p(|\nabla u|)=f, \tag{1.5}$$

with suitable data f, where p is a given positive and continuous function which increases from 0 to $+\infty$ and where g is a positive function on $\mathbb{R}$.

For classical existence results for nonlinear elliptic equations in Orlicz-Sobolev spaces, see, for example, [2, 3, 4, 6, 8, 9, 10].

2. Preliminaries

2.1. Let $M:\mathbb{R}^+\to\mathbb{R}^+$ be an N-function, that is, M is continuous and convex, with $M(t)>0$ for $t>0$, $M(t)/t\to 0$ as $t\to 0$, and $M(t)/t\to\infty$ as $t\to\infty$.

Equivalently, M admits the following representation: $M(t)=\int_0^t m(\tau)d\tau$, where $m:\mathbb{R}^+\to\mathbb{R}^+$ is nondecreasing and right continuous, with $m(0)=0$, $m(t)>0$ for $t>0$, and $m(t)\to\infty$ as $t\to\infty$.

The N-function $\overline{M}$, conjugate to M, is defined by $\overline{M}(t)=\int_0^t\overline{m}(\tau)d\tau$, where $\overline{m}:\mathbb{R}^+\to\mathbb{R}^+$ is given by $\overline{m}(t)=\sup\{s:m(s)\leq t\}$ (see [1, 14, 15]).

The N-function M is said to satisfy the Δ_2-condition if, for some $k>0$,

$$M(2t)\leq kM(t)\quad\forall t\geq 0. \tag{2.1}$$

When (2.1) holds only for $t\geq$ some $t_0>0$, then M is said to satisfy the Δ_2-condition near infinity.

We will extend these N-functions into even functions on all $\mathbb{R}$.

Let P and Q be two N-functions. $P\ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\varepsilon>0$,

$$\frac{P(t)}{Q(\varepsilon t)}\longrightarrow 0\quad\text{as } t\longrightarrow\infty. \tag{2.2}$$

This is the case if and only if

$$\lim_{t\to\infty}\frac{Q^{-1}(t)}{P^{-1}(t)}=0. \tag{2.3}$$

2.2. Let Ω be an open subset of $\mathbb{R}^N$. The Orlicz class $\mathcal{L}_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions

u on Ω such that

$$\int_\Omega M(u(x))\,dx < +\infty \qquad \left(\text{resp., } \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx < +\infty \text{ for some } \lambda > 0\right). \tag{2.4}$$

$L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{u(x)}{\lambda}\right)dx \le 1\right\} \tag{2.5}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$.

The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2-condition for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M}}$.

The space $L_M(\Omega)$ is reflexive if and only if M and $\overline{M}$ satisfy the Δ_2-condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.3. We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp., $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\le 1} \|D^\alpha u\|_M, \tag{2.6}$$

thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N+1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M, we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$.

We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$,

$$\int_\Omega M\left(\frac{D^\alpha u_n - D^\alpha u}{\lambda}\right)dx \longrightarrow 0 \quad \forall |\alpha| \le 1; \tag{2.7}$$

this implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$.

If M satisfies the Δ_2-condition on $\mathbb{R}^+$ (near infinity only if Ω has finite measure), then modular convergence coincides with norm convergence.

2.4. Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp., $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order less than or equal to 1 of functions in $L_{\overline{M}}(\Omega)$ (resp., $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and thus for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [9, 11]). Consequently, the action of a distribution S in $W^{-1}L_{\overline{M}}(\Omega)$ on an element u of $W_0^1L_M(\Omega)$ is well defined. It will be denoted by $\langle S, u\rangle$.

3. The main result

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$ with the segment property. Let M and P be two N-functions such that $P \ll M$.

Let $A : D(A) \subset W_0^1 L_M(\Omega) \to W^{-1} L_{\overline{M}}(\Omega)$ be a mapping (not everywhere defined) given by

$$A(u) = -\operatorname{div} a(x, u, \nabla u), \tag{3.1}$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$ and all $\xi, \xi^* \in \mathbb{R}^N$, $\xi \neq \xi^*$,

$$|a(x,s,\xi)| \leq \beta\left[c(x) + \overline{P}^{-1} M(\gamma|s|) + \overline{M}^{-1} M(\gamma|\xi|)\right], \tag{3.2}$$

$$\left[a(x,s,\xi) - a(x,s,\xi^*)\right][\xi - \xi^*] > 0, \tag{3.3}$$

$$\alpha M(|\xi|) \leq a(x,s,\xi)\xi, \tag{3.4}$$

where $c(x)$ belongs to $E_{\overline{M}}(\Omega)$, $c \geq 0$, and $\alpha, \beta, \gamma > 0$.

Furthermore, let $g(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$g(x,s,\xi)s \geq 0, \tag{3.5}$$

$$|g(x,s,\xi)| \leq b(|s|)\left(c'(x) + M(|\xi|)\right), \tag{3.6}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous and non decreasing function and $c'(x)$ is a given non-negative function in $L^1(\Omega)$. Finally, we assume that

$$f \in W^{-1} E_{\overline{M}}(\Omega). \tag{3.7}$$

Consider the following elliptic problem with Dirichlet boundary condition:

$$\begin{gathered} u \in W_0^1 L_M(\Omega), \quad g(x,u,\nabla u) \in L^1(\Omega), \quad g(x,u,\nabla u)u \in L^1(\Omega), \\ \langle A(u), v\rangle + \int_\Omega g(x,u,\nabla u) v\, dx = \langle f, v\rangle \\ \text{for all } v \in W_0^1 L_M(\Omega) \cap L^\infty(\Omega) \text{ and for } v = u. \end{gathered} \tag{3.8}$$

We will prove the following existence theorem.

Theorem 3.1. *Assume that (3.2), (3.3), (3.4), (3.5), (3.6), and (3.7) hold true. Then there exists at least one solution u of (3.8).*

Remark 3.2. Note that conditions (3.4) and (3.6) can be replaced by the following ones:

$$\alpha M\left(\frac{|\xi|}{\lambda}\right) \le a(x,s,\xi)\xi,$$
$$|g(x,s,\xi)| \le b(|s|)\left(c'(x)+M\left(\frac{|\xi|}{\lambda'}\right)\right), \tag{3.9}$$

with $\lambda' \ge \lambda > 0$.

Remark 3.3. The Euler equation of the integral

$$\int_\Omega \left(a(u)\int_0^{|\nabla u|}\frac{M(t)}{t}dt\right)dx - \langle f,u\rangle \tag{3.10}$$

is

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(a(u)\frac{M(|\nabla u|)}{|\nabla u|^2}\frac{\partial u}{\partial x_i}\right)+a'(u)\int_0^{|\nabla u|}\frac{M(t)}{t}dt = f, \tag{3.11}$$

where $a(s)$ is a smooth function satisfying $a'(s)s \ge 0$. Note that

$$a'(u)\int_0^{|\nabla u|}\frac{M(t)}{t}dt \tag{3.12}$$

satisfies the growth condition (3.6) and then Theorem 3.1 can be applied to Dirichlet problems related to (3.11).

Proof of Theorem 3.1

Step 1 (a priori estimates). Consider the sequence of approximate problems

$$u_n \in W_0^1L_M(\Omega),$$
$$\langle A(u_n),v\rangle + \int_\Omega g_n(x,u_n,\nabla u_n)v\,dx = \langle f,v\rangle \quad \forall v \in W_0^1L_M(\Omega), \tag{3.13}$$

where

$$g_n(x,s,\xi) = T_n(g(x,s,\xi)) \tag{3.14}$$

and where for $k>0$, T_k is the usual truncation at height k defined by $T_k(s) = \max(-k,\min(k,s))$ for all $s \in \mathbb{R}$.

Note that $g_n(x,s,\xi)s \ge 0$, $|g_n(x,s,\xi)| \le |g(x,s,\xi)|$, and $|g_n(x,s,\xi)| \le n$. Since g_n is bounded for any fixed $n>0$, there exists at least one solution u_n of (3.13) (see [13, Propositions 1 and 5]).

Using in (3.13) the test function u_n, we get

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla u_n\,dx \le \langle f,u_n\rangle. \tag{3.15}$$

Consequently, one has that (u_n) is bounded in $W_0^1L_M(\Omega)$. By [13, Proposition 5] (see [13, Remark 8]), $(a(x,u_n,\nabla u_n))_n$ is bounded in $(L_{\overline{M}}(\Omega))^N$,

$$\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx \le C, \tag{3.16}$$

where C is a real constant which does not depend on n.

Passing to a subsequence, if necessary, we can assume that

$$\begin{gathered} u_n \rightharpoonup u \text{ weakly in } W_0^1L_M(\Omega) \text{ for } \sigma(\Pi L_M,\Pi E_{\overline{M}}), \text{ strongly in } E_M(\Omega), \text{ and a.e. in } \Omega; \\ a(x,u_n,\nabla u_n) \rightharpoonup h \text{ and } a(x,T_k(u_n),\nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } (L_{\overline{M}}(\Omega))^N \\ \text{for } \sigma(\Pi L_{\overline{M}},\Pi E_M) \text{ for some } h \text{ and } h_k \in (L_{\overline{M}}(\Omega))^N. \end{gathered} \tag{3.17}$$

Step 2 (almost everywhere convergence of the gradients). Fix $k>0$ and let $\varphi(t)=te^{\sigma t^2}$, $\sigma>0$. It is well known that when $\sigma \ge (b(k)/2\alpha)^2$, one has

$$\varphi'(t)-\frac{b(k)}{\alpha}|\varphi(t)| \ge \frac{1}{2} \quad \forall t\in\mathbb{R}. \tag{3.18}$$

Take a sequence $(v_j)\subset\mathcal{D}(\Omega)$ which converges to u for the modular convergence in $W_0^1L_M(\Omega)$ (cf. [11]) and set $\theta_n^j=T_k(u_n)-T_k(v_j)$, $\theta^j=T_k(u)-T_k(v_j)$, and $z_n^j=\varphi(\theta_n^j)$.

Using in (3.13) the test function z_n^j, we get

$$\langle A(u_n),z_n^j\rangle+\int_\Omega g_n(x,u_n,\nabla u_n)z_n^j\,dx=\langle f,z_n^j\rangle. \tag{3.19}$$

Denote by $\varepsilon_i(n,j)$ $(i=0,1,2,\dots)$ various sequences of real numbers which tend to 0 when n and $j\to\infty$, that is,

$$\lim_{j\to\infty}\lim_{n\to\infty}\varepsilon_i(n,j)=0. \tag{3.20}$$

In view of (3.17), we have $z_n^j\to\varphi(\theta^j)$ weakly in $W_0^1L_M(\Omega)$ for $\sigma(\Pi L_M,\Pi E_{\overline{M}})$ as $n\to\infty$ and then $\langle f,z_n^j\rangle\to\langle f,\varphi(\theta^j)\rangle$ as $n\to\infty$. Using, now, the modular convergence of (v_j), we get $\langle f,\varphi(\theta^j)\rangle\to 0$ as $j\to\infty$ so that

$$\langle f,z_n^j\rangle=\varepsilon_0(n,j). \tag{3.21}$$

Since $g_n(x,u_n,\nabla u_n)z_n^j\ge 0$ on the subset $\{x\in\Omega:|u_n|>k\}$, we have

$$\langle A(u_n),z_n^j\rangle+\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n)z_n^j\,dx \le \varepsilon_0(n,j). \tag{3.22}$$

The first term on the left-hand side of (3.22) reads as

$$\begin{aligned}\langle A(u_n), z_n^j\rangle &= \int_{\{|u_n|\le k\}} a(x,u_n,\nabla u_n)[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi'(\theta_n^j)\,dx\\ &\quad-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx\\ &=\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(v_j)]\varphi'(\theta_n^j)\,dx\\ &\quad-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx\end{aligned} \tag{3.23}$$

and then

$$\begin{aligned}\langle A(u_n), z_n^j\rangle &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)]\\ &\quad\times[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\varphi'(\theta_n^j)\,dx\\ &\quad+\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n)-\nabla T_k(v_j)\chi_j^s]\varphi'(\theta_n^j)\,dx\\ &\quad-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx\\ &\quad-\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx,\end{aligned} \tag{3.24}$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_j^s=\{x\in\Omega: |\nabla T_k(v_j)|\le s\}. \tag{3.25}$$

We will pass to the limit in n and in j for s fixed in the last three terms of the right-hand side of (3.24).

Starting with the fourth term, observe that, since

$$|\nabla T_k(v_j)\chi_{\{|u_n|>k\}}\varphi'(\theta_n^j)|\le\varphi'(2k)|\nabla T_k(v_j)|\le\varphi'(2k)\|\nabla v_j\|_\infty=a_j\in\mathbb{R}, \tag{3.26}$$

we have

$$\nabla T_k(v_j)\chi_{\{|u_n|>k\}}\varphi'(\theta_n^j)\longrightarrow\nabla T_k(v_j)\chi_{\{|u|\ge k\}}\varphi'(\theta^j)\text{ strongly in }(E_M(\Omega))^N\quad\text{as } n\longrightarrow\infty, \tag{3.27}$$

and hence

$$\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx\longrightarrow\int_{\{|u|\ge k\}} h\nabla T_k(v_j)\varphi'(\theta^j)\,dx\quad\text{as } n\longrightarrow\infty. \tag{3.28}$$

Observe that

$$|\nabla T_k(v_j)\chi_{\{|u|\ge k\}}\varphi'(\theta^j)|\le\varphi'(2k)|\nabla T_k(v_j)|\le\varphi'(2k)|\nabla v_j|; \tag{3.29}$$

then, by using the modular convergence of $|\nabla v_j|$ in $L_M(\Omega)$ and Vitali's theorem, we get

$$\nabla T_k(v_j)\chi_{\{|u|\geq k\}}\varphi'(\theta^j) \longrightarrow 0 \tag{3.30}$$

for the modular convergence in $(L_M(\Omega))^N$, and thus

$$\int_{\{|u|\geq k\}} h\nabla T_k(v_j)\varphi'(\theta^j)\,dx \longrightarrow 0 \quad \text{as } j \longrightarrow \infty. \tag{3.31}$$

We have then proved that

$$\int_{\{|u_n|>k\}} a(x,u_n,\nabla u_n)\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx = \varepsilon_1(n,j). \tag{3.32}$$

The second term on the right-hand side of (3.24) tends to (by letting $n \to \infty$)

$$\int_{\Omega} a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta^j)\,dx \tag{3.33}$$

since $a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\varphi'(\theta_n^j) \to a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)\varphi'(\theta^j)$ strongly in $(E_{\overline{M}}(\Omega))^N$ as $n \to \infty$ by [3, Lemma 2.3], while $\nabla T_k(u_n) \rightharpoonup \nabla T_k(u)$ weakly in $(L_M(\Omega))^N$ by (3.17).

Since $\nabla T_k(v_j)\chi_j^s \to \nabla T_k(u)\chi^s$ strongly in $(E_M(\Omega))^N$ as $j \to \infty$, where χ^s denotes the characteristic function of $\Omega_s = \{x \in \Omega : |\nabla T_k(u)| \leq s\}$, it is easy to see that

$$\int_{\Omega} a(x,T_k(u),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta^j)\,dx \longrightarrow 0 \quad \text{as } j \longrightarrow \infty, \tag{3.34}$$

and thus

$$\int_{\Omega} a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta_n^j)\,dx = \varepsilon_2(n,j). \tag{3.35}$$

Concerning the third term on the right-hand side of (3.24), we have

$$-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx \longrightarrow -\int_{\Omega\setminus\Omega_j^s} h_k\nabla T_k(v_j)\varphi'(\theta^j)\,dx \tag{3.36}$$

as $n \to \infty$ by using the fact that $\nabla T_k(v_j)$ belongs to $(E_M(\Omega))^N$.

In view of the modular convergence of (∇v_j) in $(L_M(\Omega))^N$, we have

$$-\int_{\Omega\setminus\Omega_j^s} h_k\nabla T_k(v_j)\varphi'(\theta^j)\,dx \longrightarrow -\int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\,dx \quad \text{as } j \longrightarrow \infty \tag{3.37}$$

and thus

$$-\int_{\Omega\setminus\Omega_j^s} a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\varphi'(\theta_n^j)\,dx = \varepsilon_3(n,j) - \int_{\Omega\setminus\Omega_s} h_k\nabla T_k(u)\,dx. \tag{3.38}$$

Combining now (3.32), (3.35), and (3.38), we obtain

$$\langle A(u_n), z_n^j\rangle = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]\varphi'(\theta_n^j)\,dx - \int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)\,dx + \varepsilon_4(n,j). \tag{3.39}$$

We now turn to the second term on the left-hand side of (3.22). We have

$$\begin{aligned} &\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n) z_n^j\,dx\right| \\ &\quad = \left|\int_{\{|u_n|\le k\}} g_n(x,T_k(u_n),\nabla T_k(u_n)) z_n^j\,dx\right| \\ &\quad \le \int_\Omega b(k)c'(x)|\varphi(\theta_n^j)|\,dx + b(k)\int_\Omega M(|\nabla T_k(u_n)|)|\varphi(\theta_n^j)|\,dx \\ &\quad \le \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\varphi(\theta_n^j)|\,dx + \varepsilon_5(n,j). \end{aligned} \tag{3.40}$$

The first term of the right-hand side of this inequality reads as

$$\begin{aligned} &\frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]|\varphi(\theta_n^j)|\,dx \\ &\qquad + \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]|\varphi(\theta_n^j)|\,dx \\ &\qquad - \frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\varphi(\theta_n^j)|\,dx \end{aligned} \tag{3.41}$$

and, as above, it is easy to see that

$$\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]|\varphi(\theta_n^j)|\,dx = \varepsilon_6(n,j) \tag{3.42}$$

and that

$$-\frac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s|\varphi(\theta_n^j)|\,dx = \varepsilon_7(n,j) \tag{3.43}$$

so that

$$\begin{aligned} &\left|\int_{\{|u_n|\le k\}} g_n(x,u_n,\nabla u_n) z_n^j\,dx\right| \\ &\quad \le \frac{b(k)}{\alpha}\int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)] \\ &\qquad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]|\varphi(\theta_n^j)|\,dx + \varepsilon_8(n,j). \end{aligned} \tag{3.44}$$

Combining this inequality with (3.22) and (3.39), we obtain

$$\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right] \times \left[\varphi'(\theta_n^j) - \frac{b(k)}{\alpha}|\varphi(\theta_n^j)|\right]dx \le \varepsilon_9(n,j) + \int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx. \tag{3.45}$$

Consequently,

$$\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right]dx \le 2\varepsilon_9(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx. \tag{3.46}$$

On the other hand,

$$\begin{aligned}
&\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\right]dx \\
&\quad = \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right]dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right]dx \\
&\qquad - \int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\left[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\right]dx \\
&\qquad + \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right]dx.
\end{aligned} \tag{3.47}$$

We will pass to the limit in n and in j in the last three terms on the right-hand side of the above equality. Similar tools as in (3.24) and (3.41) give

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\left[\nabla T_k(v_j)\chi_j^s - \nabla T_k(u)\chi^s\right]dx = \varepsilon_{10}(n,j), \tag{3.48}$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u)\chi^s)\left[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\right]dx = \varepsilon_{11}(n,j), \tag{3.49}$$

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right]dx = \varepsilon_{12}(n,j) \tag{3.50}$$

which imply that

$$\begin{aligned}
&\int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(u)\chi^s\right]dx \\
&\quad = \int_\Omega \left[a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)\right]\left[\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s\right]dx \\
&\qquad + \varepsilon_{13}(n,j).
\end{aligned} \tag{3.51}$$

For $r \le s$, one has

$$\begin{aligned}
0 &\le \int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \\
&\le \int_{\Omega_s} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \\
&= \int_{\Omega_s} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dx \\
&\le \int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u)\chi^s)][\nabla T_k(u_n) - \nabla T_k(u)\chi^s]dx \\
&= \int_{\Omega} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s)][\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]dx \\
&\quad + \varepsilon_{13}(n,j) \\
&\le \varepsilon_{14}(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx.
\end{aligned} \tag{3.52}$$

This implies that, by passing at first to the limit sup over n and next over j,

$$\begin{aligned}
0 &\le \limsup_{n\to\infty} \int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \\
&\le 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx.
\end{aligned} \tag{3.53}$$

Using the fact that $h_k \nabla T_k(u) \in L^1(\Omega)$ and letting $s \to \infty$, we get

$$\int_{\Omega_r} [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]dx \longrightarrow 0 \tag{3.54}$$

as $n \to \infty$.

As in [3], we deduce that there exists a subsequence still denoted by u_n such that

$$\nabla u_n \longrightarrow \nabla u \quad \text{a.e. in } \Omega, \tag{3.55}$$

which implies that

$$a(x,u_n,\nabla u_n) \rightharpoonup a(x,u,\nabla u) \text{ weakly in } (L_{\overline{M}}(\Omega))^N \quad \text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{3.56}$$

Step 3 (modular convergence of the truncations). Going back to (3.46), we can write

$$\begin{aligned}
\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)dx \le{}& \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s\,dx \\
&+ \int_\Omega a(x,T_k(u_n),\nabla T_k(v_j)\chi_j^s) \\
&\qquad \times [\nabla T_k(u_n) - \nabla T_k(v_j)\chi_j^s]dx \\
&+ 2\varepsilon_9(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx,
\end{aligned} \tag{3.57}$$

which implies, by using (3.50),

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \\ \leq \int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(v_j)\chi_j^s\,dx + \varepsilon_{15}(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx. \tag{3.58}$$

Passing to the limit sup over n in both sides of this inequality yields

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \\ \leq \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(v_j)\chi_j^s\,dx + \lim_{n\to\infty}\varepsilon_{15}(n,j) + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx, \tag{3.59}$$

in which we can pass to the limit in j to obtain

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \\ \leq \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)\chi^s\,dx + 2\int_{\Omega\setminus\Omega_s} h_k \nabla T_k(u)dx \tag{3.60}$$

which gives, by letting $s\to\infty$,

$$\limsup_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \leq \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx. \tag{3.61}$$

On the other hand, we have, by using Fatou's lemma,

$$\int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx \leq \liminf_{n\to\infty}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx, \tag{3.62}$$

which implies that

$$\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)\,dx \longrightarrow \int_\Omega a(x,T_k(u),\nabla T_k(u))\nabla T_k(u)dx \quad \text{as } n\longrightarrow\infty, \tag{3.63}$$

and by using [4, Lemma 2.4], we conclude that

$$a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \longrightarrow a(x,T_k(u),\nabla T_k(u))\nabla T_k(u) \quad \text{in } L^1(\Omega). \tag{3.64}$$

This implies, by using (3.4), that

$$T_k(u_n) \longrightarrow T_k(u) \quad \text{in } W_0^1L_M(\Omega) \tag{3.65}$$

for the modular convergence.

Step 4 (equi-integrability of the nonlinearities and passage to the limit). We will prove that $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem.

Since $g_n(x,u_n,\nabla u_n) \to g(x,u,\nabla u)$ a.e. in Ω, thanks to (3.55), it suffices to prove that $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω. Let $E \subset \Omega$ be a measurable subset of Ω. We have, for any $m > 0$,

$$\begin{aligned}\int_E |g_n(x,u_n,\nabla u_n)|\,dx &= \int_{E\cap\{|u_n|\le m\}} |g_n(x,u_n,\nabla u_n)|\,dx + \int_{E\cap\{|u_n|>m\}} |g_n(x,u_n,\nabla u_n)|\,dx \\ &\le b(m)\int_E a(x,T_m(u_n),\nabla T_m(u_n))\nabla T_m(u_n)\,dx \\ &\quad + b(m)\int_E c'(x)\,dx + \frac{1}{m}\int_\Omega g_n(x,u_n,\nabla u_n)u_n\,dx.\end{aligned} \tag{3.66}$$

Standard arguments allow to deduce, using the strong convergence (3.64), that there exists $\mu > 0$ such that

$$|E| < \mu \Longrightarrow \int_E |g_n(x,u_n,\nabla u_n)|\,dx \le \varepsilon, \quad \forall n, \tag{3.67}$$

which shows that $g_n(x,u_n,\nabla u_n)$ are uniformly equi-integrable in Ω as required.

In order to pass to the limit, we have, by going back to approximate equations (3.13),

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla w\,dx + \int_\Omega g_n(x,u_n,\nabla u_n)w\,dx = \langle f,w\rangle \tag{3.68}$$

for all $w \in \mathfrak{D}(\Omega)$, in which, we can easily pass to the limit as $n \to \infty$ to get

$$\int_\Omega a(x,u,\nabla u)\nabla w\,dx + \int_\Omega g(x,u,\nabla u)w\,dx = \langle f,w\rangle. \tag{3.69}$$

Let now $v \in W_0^1L_M(\Omega) \cap L^\infty(\Omega)$. There exists $(w_j) \subset \mathfrak{D}(\Omega)$ such that $\|w_j\|_{\infty,\Omega} \le (N+1)\|v\|_{\infty,\Omega}$ for all $j \in \mathbb{N}$ and

$$w_j \longrightarrow v \tag{3.70}$$

for the modular convergence in $W_0^1L_M(\Omega)$. Taking $w = w_j$ in (3.69) and letting $j \to \infty$ yields

$$\int_\Omega a(x,u,\nabla u)\nabla v\,dx + \int_\Omega g(x,u,\nabla u)v\,dx = \langle f,v\rangle. \tag{3.71}$$

By choosing $v = T_k(u)$ in the last equality, we get

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u)\,dx + \int_\Omega g(x,u,\nabla u)T_k(u)\,dx = \langle f,T_k(u)\rangle. \tag{3.72}$$

From (3.16), we deduce by Fatou's lemma that $g(x,u,\nabla u)u \in L^1(\Omega)$ and since $|g(x,u,\nabla u)T_k(u)| \le g(x,u,\nabla u)u$ and $T_k(u) \to u$ in $W_0^1L_M(\Omega)$ for the modular convergence and a.e. in Ω as $k \to \infty$, it is easy to pass to the limit in both sides of (3.72) (by using Lebesgue theorem) to obtain

$$\int_{\Omega} a(x,u,\nabla u)\nabla u\,dx + \int_{\Omega} g(x,u,\nabla u)u\,dx = \langle f,u\rangle. \tag{3.73}$$

This completes the proof of Theorem 3.1. □

Remark 3.4. If we replace, as in [5], (3.2) by the general growth condition

$$|a(x,s,\xi)| \leq \overline{b}(|s|)\left(c(x) + \overline{M}^{-1}M(\gamma|\xi|)\right), \tag{3.74}$$

where $\gamma > 0$, $c \in E_{\overline{M}}(\Omega)$, and $\overline{b} : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous nondecreasing function, we prove the existence of solutions for the following problem:

$$\begin{gathered} u \in W_0^1 L_M(\Omega), \qquad g(x,u,\nabla u) \in L^1(\Omega), \qquad g(x,u,\nabla u)u \in L^1(\Omega), \\ \langle A(u), T_k(u-v)\rangle + \int_{\Omega} g(x,u,\nabla u)T_k(u-v)\,dx \leq \langle f, T_k(u-v)\rangle \\ \forall v \in W_0^1 L_M(\Omega) \cap L^{\infty}(\Omega). \end{gathered} \tag{3.75}$$

Indeed, we consider the following approximate problems:

$$\begin{gathered} u_n \in W_0^1 L_M(\Omega), \\ -\operatorname{div} a(x,T_n(u_n),\nabla u_n) + g_n(x,u_n,\nabla u_n) = f \quad \text{in } \Omega, \end{gathered} \tag{3.76}$$

and we conclude by adapting the same steps.

As an application of this result, we can treat the following model equations:

$$-\operatorname{div}\left((1+|u|)^m \frac{\exp(|\nabla u|) - 1}{|\nabla u|^2}\nabla u\right) + u\cos^2 u \exp(|\nabla u|) = f, \quad m \geq 0. \tag{3.77}$$

Remark that the solutions of (3.77) belong to $L^{\infty}(\Omega)$ so that (3.77) holds in the distributional sense.

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