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PROPERTY Q

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ABSTRACT. Some properties of property Q are stated, some new results are proved and implications to totally metacompact and totally paracompact are obtained.

KEY WORDS AND PHRASES. Property Q, metacompact, totally metacompact, totally paracompact.

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1. INTRODUCTION.

An open cover has **property Q** [1] if when $\{O_i : i \in N\}$ is a sequence of distinct members of the cover and p_i, q_i are points of O_i and $\{p_i\}$ has limit p, then $\{q_i\}$ has limit p. A topological space has **property Q** if each open cover has an open refinement having property Q. A topological space is **metacompact** if each open cover has a point finite refinement that covers the space. A topological space is **totally paracompact (totally metacompact)** if each open base contains a locally finite (point finite) subcover. A basis is a **uniform base** if each infinite collection from the basis containing a point is a basis at the point. All spaces are assumed to be Hausdorff topological spaces. Some previous results pertaining to property Q are:

THEOREM 1. [1] A complete Moore space that satisfies property Q is a metric space.

THEOREM 2. [2] A space that satisfies property Q is metacompact.

THEOREM 3. [2] A first countable space that satisfies property Q is paracompact.

THEOREM 4. [3] A developable space is metrizable if and only if it satisfies property Q.

It follows that a countably compact space satisfying property Q is compact and that an M–space satisfying property Q is metric.

2. RESULTS.

DEFINITION 1. A basis β is a **Q base** if β satisfies property Q.

LEMMA 1. If the space X has a Q base, then X has a uniform base.

PROOF. If X has the discrete topology, then the lemma is true. Therefore let Y be the set of nondiscrete points of X and B be a Q base for Y. For any infinite subcollection β of B containing a point p, we need to show that β is a basis at p. Suppose not, then there is an open set O containing p that contains no member of β. Select a countably infinite subcollection $\{B_i : i \in N\}$ of β containing p, and choose points $\{p_i\}$ from distinct members of $\{B_i\}$ but not in O. Then $\{p_i\}$ must have sequential limit p because β has property Q. This is a contradiction.

THEOREM 5. A space is metric if and only if it is a regular space with a Q base.

PROOF. Note that a regular space with a uniform base is developable [4]. And a regular developable space satisfying property Q is metric [3].

Conversely a metric space has a Q base. For each integer n, use locally finite refinements of balls with diameters less than $1/n$.

DEFINITION 2. A topological space is **totally** Q if each open base contains a subcover satisfying property Q.

THEOREM 6. If X is totally Q, then X is totally metacompact.

PROOF. Let B be a basis for X. Then there is a subcollection β of B covering X and having property Q. Well order β and let B_1 be the first member in this ordering. And let B_α be the first member of the well ordering that contains a point not in $\cup_{\beta<\alpha} B_\beta$. Claim $\{B_\alpha\}$ is point finite. Suppose that p is point in infinitely many members of $\{B_\alpha\}$; then we pick a countably infinite subsequence of sets $\{B_{\alpha_i}\}$ from $\{B_\alpha\}$ each containing p. Let p_1 be a point in B_{α_1}, then from each B_{α_i} we choose a point p_i not in $\cup_{j<i} B_{\alpha_j}$. Then $\{p_i\}$ has p as sequential limit by property Q but B_{α_1} is an open set containing p but no point of $\{p_i : i > 1\}$ a contradiction.

The converse of Theorem 6 is not true. Let X and Y be one-point compactifications of discrete spaces of size ω and ω_1, then the space $X \times Y - \{(\omega, \omega_1)\}$ with the product topology is totally metacompact but not totally Q.

THEOREM 7. A first countable, totally Q space X is totally paracompact.

PROOF. Let B be a basis for X. By Theorem 6 there is a subcollection β of B that is point finite and minimal (minimal in the sense that if b is in β then b is not a subset of any other member of β).

Claim β is locally finite. Suppose not, then there is a point p of X so that each open set containing p intersects infinitely many members of β. Let B_o be one of the finitely many members of β containing p. Let $\{O_i\}$ be a countable basis at p. Then for each natural number i, choose $B_i \in \beta$ such that $B_i \cap O_i$ in not empty, and the B_i's are distinct members of β which are also different from B_o. For each i, choose p_i in $B_i \cap O_i$ and $q_i \in (B_i - B_o)$. Since $\{p_i\}$ has sequential limit point p; therefore, $\{q_i\}$ must have sequential limit point p by property Q. This is a contradiction; hence, $\{B_\alpha\}$ is a locally finite subcollection of β.

Example 2.14 in [3] is an example of a totally Q space that is not totally paracompact. In [5] it is proved that a locally compact space is paracompact if and only if it is mesocompact. It is not true that a locally compact space is paracompact if and only if it satisfies property Q.

EXAMPLE. A locally compact property Q space that is not paracompact.

Let $\beta\omega$ and $\beta\omega_1$ be the Stone-Čech compactifications of discrete spaces of size ω and ω_1. Then the space

$$\beta\omega \times \beta\omega_1 - (\beta\omega - \omega) \times (\beta\omega_1 - \omega_1),$$

with the topology inherited as a subspace of the product space $\beta\omega \times \beta\omega_1$, has the desired property.

An open cover has **strong property Q** if it has property Q and when $\{p_i\}$ has cluster point p, then $\{q_i\}$ has cluster point p. A topological space is strong property Q if each open cover has a refinement satisfying strong property Q.

THEOREM 8. A regular, locally compact, strong property Q space is paracompact.

PROOF. First note that a regular, locally compact, strong property Q space is metacompact. And suppose we have a regular, locally compact, strong property Q space that is not paracompact. Then there is an open cover O and a point p so that every open refinement of O is not locally finite at p. Let R be a point finite minimal open refinement of O satisfying strong property Q. Let C be an open set containing p so that C is a subset of some member of R and the closure of C is compact. Let G be an open set containing p so that the closure of G is a subset of C. The set G must intersect infinitely many members of R and each member of R that intersects G must have a point in the complement of C (otherwise R would not be minimal). Hence, sequences $\{p_i\}$ and $\{q_i\}$ exist with p_i in G and q_i not in C and $\{p_i\}$ must have a cluster point that can not be a cluster point of $\{q_i\}$. This is a contradiction. Therefore, the space must be paracompact.

QUESTION When does totally Q imply totally paracompact?

REFERENCES

1. YOUNGLOVE, J. N. Concerning dense metric subspaces of certain nonmetric spaces, Fund. Math. 48(1960), 15-25.

2. BRIGGS, R. C. A Comparison of Covering Properties in T3 and T4 Spaces, Ph. D. University of Houston, 1968.

3. MANCUSO, V. J. Mesocompactness and related properties, Pacific J. Math. 33(1970), 345-355.

4. ALEKSANDROV, P. S. On the metrization of topological spaces, Bull. Acad. Polon. Sci. Ser. Sci Math. Astronom. Phys. 8(1960), 135-140.

5. BOONE, J. R. Some characterizations of paracompactness in K-spaces, Fund. Math. 72(1971), 145-153.