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Research Article

New Nonlinear Systems Admitting Virasoro-Type Symmetry Algebra and Group-Invariant Solutions

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With the aid of symbolic computation by Maple, we extend the application of Virasoro-type symmetry prolongation method to coupled systems with two-component nonlinear equations. New nonlinear systems admitting infinitely dimensional centerless Virasoro-type symmetry algebra are constructed. Taking one of them as an example, we present some group-invariant solutions to one of the new model systems.

1. Introduction

Integrable models such as the KdV equation, the KP equation, the nonlinear Schrödinger equation, the NNV equation, the sine-Gordon equation, and the Toda lattice have played more and more important roles in almost all natural sciences. It becomes one of the most fundamental problems to seek for as much as possible new nonlinear equations and systems with some nice properties including Lax pair, Painlevé property, infinite number of conservation laws, and bi-Hamiltonian structures.

There exist many powerful methods to construct nonlinear equations and systems like the multiscale method, symmetry constraint method, and conformal invariant method [1–4]. Among these methods developed recently, the Virasoro-type symmetry prolongation (VSP) method is found to be very effective. Based on the fact that all the known (2 + 1)-dimensional integrable models possess the following centerless Virasoro-type subalgebra:

$$[\sigma(f_1), \sigma(f_2)] = \sigma(\dot{f}_1 f_2 - \dot{f}_2 f_1), \qquad (1)$$

where f_1 and f_2 are arbitrary functions of the same argument and there are no known nonintegrable models owning such type symmetry algebra, Lou and Hu introduced an idea that if an f-independent model possesses the Virasoro-type symmetry algebra (1), the model is Virasoro integrable [5]. By using this theory and selecting the special realizations, some new (2 + 1)-dimensional and (3 + 1)-dimensional Virasoro integrable models have been derived [6–8].

However, the VSP method and concrete realizations discussed above all belong to single equations. To our knowledge, there are few results concerning the construction of coupled systems with two-component nonlinear equations [9]. Therefore we extend the applications of this method to construct several (2 + 1)-dimensional Virasoro integrable systems by selecting special realization of algebra (1).

The remainder of this paper is organized as follows. The general theory of the VSP method for nonlinear systems is presented in Section 2. In Section 3, some (2+1)-dimensional Virasoro integrable systems are constructed by choosing appropriate realizations. For a concrete example, the one-dimensional optimal system and group-invariant solutions to system (21) are given in Section 4. The last section contains some concluding remarks.

2. The Generalized VSP Method

Firstly, let us give a brief account of the generalized VSP method for nonlinear systems. We consider the vector field with the following form:

$$\vec{V} = X(x,y,t,u,v)\partial_x + Y(x,y,t,u,v)\partial_y + T(x,y,t,u,v)\partial_t + U(x,y,t,u,v)\partial_u + V(x,y,t,u,v)\partial_v. \tag{2}$$

We define the functions T, X, Y, U, and V as follows:

$$T = f(t), \quad \{X,Y,U,V\} = \left\{\sum_{i=1}^{n} f^{(i)}X_i, \sum_{i=1}^{n} f^{(i)}Y_i, \sum_{i=1}^{n} f^{(i)}U_i, \sum_{i=1}^{n} f^{(i)}V_i\right\}, \quad n = 1,2,3,\ldots, \tag{3}$$

where $f^{(i)} = \mathrm{d}^i f/\mathrm{d}t^i$, X_i, Y_i, U_i, V_i, $i = 1,2,3,\ldots$, are functions of the variables $\{x,y,t,u,v\}$ and should be selected to satisfy the commutation relation (1). In order to construct invariant kth-order partial differential equations, we should calculate the kth prolongation of the vector field $\vec{V}$ firstly. The general formula for the kth prolongation of a vector field $\vec{V}$ is given by

$$pr^{(k)}\vec{V} = \vec{V} + U^x\partial_{u_x} + U^y\partial_{u_y} + U^t\partial_{u_t} + V^x\partial_{v_x} + V^y\partial_{v_y} + V^t\partial_{v_t} + \cdots + \sum_{1\le i+j+l\le k} U^{x^iy^jt^l}\partial_{u_{x^iy^jt^l}} + \sum_{1\le i+j+l\le k} V^{x^iy^jt^l}\partial_{v_{x^iy^jt^l}}, \tag{4}$$

with

$$\begin{aligned} U^{x^iy^jt^l} &= D_xU^{x^{i-1}y^jt^l} - (D_xX)u_{x^iy^jt^l} - (D_xY)u_{x^{i-1}y^{j+1}t^l} - (D_xT)u_{x^{i-1}y^jt^{l+1}} \\ &= D_yU^{x^iy^{j-1}t^l} - (D_yX)u_{x^{i+1}y^{j-1}t^l} - (D_yY)u_{x^iy^jt^l} - (D_yT)u_{x^iy^{j-1}t^{l+1}} \\ &= D_tU^{x^iy^jt^{l-1}} - (D_tX)u_{x^{i+1}y^jt^{l-1}} - (D_tY)u_{x^iy^{j+1}t^{l-1}} - (D_tT)u_{x^iy^jt^l}, \end{aligned}$$

$$\begin{aligned} V^{x^iy^jt^l} &= D_xV^{x^{i-1}y^jt^l} - (D_xX)v_{x^iy^jt^l} - (D_xY)v_{x^{i-1}y^{j+1}t^l} - (D_xT)v_{x^{i-1}y^jt^{l+1}} \\ &= D_yV^{x^iy^{j-1}t^l} - (D_yX)v_{x^{i+1}y^{j-1}t^l} - (D_yY)v_{x^iy^jt^l} - (D_yT)v_{x^iy^{j-1}t^{l+1}} \\ &= D_tV^{x^iy^jt^{l-1}} - (D_tX)v_{x^{i+1}y^jt^{l-1}} - (D_tY)v_{x^iy^{j+1}t^{l-1}} - (D_tT)v_{x^iy^jt^l}, \end{aligned} \tag{5}$$

where D_x, D_y, and D_t are total derivatives with respect to x, y, t, respectively. Thus we can calculate the kth prolongation of a concrete vector $\vec{V}$. It is well known that the invariant system should have the following form:

$$\Delta\left(x,y,t,u,v,u_x,v_x,u_y,v_y,u_t,v_t,\ldots,u_{x^iy^jt^l},v_{x^iy^jt^l},\ldots\right) = 0, \tag{6}$$

where Δ satisfies

$$pr^{(k)}\vec{V}(\Delta)\big|_{\Delta=0} = 0. \tag{7}$$

In order to construct group invariant equations, we should solve the corresponding characteristic equations

$$\frac{\mathrm{d}t}{T} = \frac{\mathrm{d}x}{X} = \frac{\mathrm{d}y}{Y} = \frac{\mathrm{d}u}{U} = \frac{\mathrm{d}v}{V} = \cdots = \frac{\mathrm{d}u_{x^iy^jt^l}}{U^{x^iy^jt^l}} = \frac{\mathrm{d}v_{x^iy^jt^l}}{V^{x^iy^jt^l}} = \cdots. \tag{8}$$

After solving the above system, we can obtain a set of elementary invariants

$$I_m\left(x,y,t,u,v,\ldots,u_{x^iy^jt^l},v_{x^iy^jt^l}\right) \equiv I_m \quad (1 \le i+j+l \le k,\ m = 1,2,3,\ldots). \tag{9}$$

The general $\vec{V}$ invariant system has the following form:

$$\begin{aligned} H_1(I_1,I_2,I_3,\ldots,I_r,\ldots) &= 0, \\ H_2(I_1,I_2,I_3,\ldots,I_r,\ldots) &= 0. \end{aligned} \tag{10}$$

According to the definition of the Virasoro integrability, the model should be f-independent. Therefore, when we find out the f-independent group invariants, we can construct the new Virasoro integral models from (10). Compared with the VSP method (see [7]), this method can be used to deal with coupled systems with two-component nonlinear equations.

3. Applications

In this section, we will construct several coupled systems admitting Virasoro-type symmetry algebra by selecting concrete realization of (1). The realization we consider is

$$\vec{V} = f\partial_t + C_1\dot{f}x\partial_x + C_2\dot{f}y\partial_y + C_3\dot{f}p\partial_p + \left(C_4\dot{f}r + C_5\ddot{f}x\right)\partial_r, \quad (11)$$

where $\dot{f}$, $\ddot{f}$, and $\dddot{f}$ denote the first, second, and third order derivatives of function $f = f(t)$ with respect to t, respectively, and C_i, $i = 1, \ldots, 5$, are arbitrary constants. It is easy to verify that $\vec{V}$ is a Virasoro type symmetry when $C_1 - C_4 = 1$. According to the prolongation formula (4), one can obtain the corresponding kth prolongation of $\vec{V}$ with the aid of symbolic computation by Maple:

$$\begin{aligned}
pr^{(k)}\vec{V} &= \vec{V} + (C_3 - C_1)\dot{f}p_x\partial_{p_x} - \left(\dot{f}r_x - C_5\ddot{f}\right)\partial_{r_x} \\
&+ (C_3 - C_2)\dot{f}p_y\partial_{p_y} + (C_4 - C_2)\dot{f}r_y\partial_{r_y} \\
&+ \left[(C_3 - 1)\dot{f}p_t + C_3p\ddot{f} - C_1\ddot{f}xp_x - C_2\ddot{f}yp_y\right]\partial_{p_t} \\
&+ \left[(C_4 - 1)\dot{f}r_t + C_4\ddot{f}r + C_5\dddot{f}x - C_1\ddot{f}xr_x - C_2\ddot{f}yr_y\right] \\
&\times \partial_{r_t} + (C_3 - 2C_1)\dot{f}p_{xx}\partial_{p_{xx}} - (1 + C_1)\dot{f}r_{xx}\partial_{r_{xx}} \\
&+ (C_3 - C_2 - C_1)\dot{f}p_{xy}\partial_{p_{xy}} - (1 + C_2)\dot{f}r_{xy}\partial_{r_{xy}} \\
&+ \left[(C_3 - C_1 - 1)\dot{f}p_{xt} + (C_3 - C_1)\ddot{f}p_x \right. \\
&\quad \left. - C_1\ddot{f}xp_{xx} - C_2\ddot{f}yp_{xy}\right]\partial_{p_{xt}} \\
&+ \left[-2\dot{f}r_{xt} - \ddot{f}r_x + C_5\dddot{f} - C_1\ddot{f}xr_{xx} - C_2\ddot{f}yr_{xy}\right]\partial_{r_{xt}} \\
&+ (C_3 - 2C_2)p_{yy}\dot{f}\partial_{p_{yy}} + (C_4 - 2C_2)r_{yy}\dot{f}\partial_{r_{yy}} \\
&+ \left[(C_3 - C_2 - 1)\dot{f}p_{yt} + (C_3 - C_2)\ddot{f}p_y \right. \\
&\quad \left. - C_1\ddot{f}xp_{xy} - C_2\ddot{f}yp_{yy}\right]\partial_{p_{yt}} \\
&+ \left[(C_4 - C_2 - 1)\dot{f}r_{yt} + (C_4 - C_2)\ddot{f}r_y \right. \\
&\quad \left. - C_1x\ddot{f}r_{xy} - C_2\ddot{f}yr_{yy}\right]\partial_{r_{yt}} \\
&+ \left[(C_3 - 2)\dot{f}p_{tt} + (2C_3 - 1)\ddot{f}p_t + C_3p\dddot{f} - C_1x\dddot{f}p_x \right. \\
&\quad \left. - 2C_1x\ddot{f}p_{xt} - C_2y\dddot{f}p_y - 2C_2y\ddot{f}p_{yt}\right]\partial_{p_{tt}} \\
&+ \left[(C_4 - 2)\dot{f}r_{tt} + (2C_4 - 1)\ddot{f}r_t + C_4r\dddot{f} + C_5xf^{(4)} \right. \\
&\quad \left. - C_1x\dddot{f}r_x - 2C_1x\ddot{f}r_{xt} - C_2y\dddot{f}r_y - 2C_2y\ddot{f}r_{yt}\right]\partial_{r_{tt}} \\
&+ (C_3 - 3C_1)\dot{f}p_{xxx}\partial p_{xxx} + (C_4 - 3C_1)\dot{f}r_{xxx}\partial_{r_{xxx}} \\
&+ (C_3 - 2C_1 - C_2)\dot{f}p_{xxy}\partial_{p_{xxy}} \\
&+ (C_4 - 2C_1 - C_2)\dot{f}r_{xxy}\partial_{r_{xxy}} \\
&+ \left[(C_3 - 2C_1)\ddot{f}p_{xx} + (C_3 - 2C_1 - 1)\dot{f}p_{xxt} \right. \\
&\quad \left. - C_1x\ddot{f}p_{xxx} - C_2y\ddot{f}p_{xxy}\right]\partial_{p_{xxt}} \\
&+ \left[(C_4 - 2C_1)\ddot{f}r_{xx} + (C_4 - 2C_1 - 1)\dot{f}r_{xxt} \right. \\
&\quad \left. - C_1x\ddot{f}r_{xxx} - C_2y\ddot{f}r_{xxy}\right]\partial_{r_{xxt}} + \cdots .
\end{aligned} \quad (12)$$

The corresponding characteristic equations of $pr^{(k)}\vec{V}$ are

$$\begin{aligned}
\frac{\mathrm{d}t}{f} &= \frac{\mathrm{d}x}{C_1\dot{f}x} = \frac{\mathrm{d}y}{C_2\dot{f}y} = \frac{\mathrm{d}p}{C_3p\dot{f}} = \frac{\mathrm{d}r}{C_4r\dot{f} + C_5x\ddot{f}} \\
&= \cdots = \frac{\mathrm{d}u_{x^iy^jt^r}}{U^{x^iy^jt^r}} = \frac{\mathrm{d}v_{x^iy^jt^r}}{V^{x^iy^jt^r}}.
\end{aligned} \quad (13)$$

After solving the above characteristic equations, we can obtain the explicit elementary invariants of $\vec{V}$ and some of them are listed as follows:

$$I_1 = xf^{-C_1}, \qquad I_2 = yf^{-C_2}, \qquad I_3 = pf^{-C_3},$$
$$I_4 = rf^{-C_4} - C_5I_1\dot{f}, \qquad I_5 = p_xf^{-(C_3-C_1)},$$
$$I_6 = r_xf - C_5\dot{f}, \qquad I_7 = p_yf^{-(C_3-C_2)},$$
$$I_8 = r_yf^{-(C_4-C_2)},$$
$$I_9 = p_tf^{1-C_3} - C_3I_3\dot{f} + C_1I_1I_5\dot{f} + C_2I_2I_7\dot{f},$$
$$\begin{aligned}
I_{10} = r_tf^{1-C_4} &- C_5I_4\dot{f} - \frac{1}{2}C_5^2I_1\left(\dot{f}\right)^2 \\
&+ C_5I_1\left(f\ddot{f} - \frac{1}{2}\left(\dot{f}\right)^2\right) - C_1I_1I_6\dot{f} \\
&- C_2I_2I_8\dot{f} - \frac{1}{2}C_1I_1C_5\dot{f}^2,
\end{aligned}$$
$$I_{11} = p_{xx}f^{-(C_3-2C_1)}, \qquad I_{12} = r_{xx}f^{1+C_1},$$
$$I_{13} = p_{xy}f^{-(C_3-C_2-C_1)}, \qquad I_{14} = r_{xy}f^{1+C_2},$$
$$I_{15} = p_{xt}f^{1+C_1-C_3} - \left(\left(C_3 - C_1\right)I_5 - C_1I_1I_{11} - C_2I_2I_{13}\right)\dot{f},$$
$$\begin{aligned}
I_{16} = r_{xt}f^2 &+ I_6\dot{f} + \frac{1}{2}C_5\left(\dot{f}\right)^2 \\
&+ C_5\left(f\ddot{f} - \frac{1}{2}\left(\dot{f}\right)^2\right) - C_1I_1I_{12}\dot{f} - C_2I_2I_{14}\dot{f},
\end{aligned}$$
$$I_{17} = f^{2C_2-C_3}p_{yy}, \qquad I_{18} = f^{2C_2-C_4}r_{yy},$$

$$
\begin{aligned}
I_{19} &= f^{1+C_2-C_3} p_{yt} - (C_3 - C_2) I_7 \dot{f} + C_1 I_1 I_{13} \dot{f} + C_2 I_2 I_{17} \dot{f}, \\
I_{20} &= r_{yt} f^{1+C_2-C_4} - [(C_4 - C_2) I_8 - C_1 I_1 I_{14} - C_2 I_2 I_{18}] \dot{f}, \\
I_{21} &= p_{xxx} f^{3C_1-C_3}, \qquad I_{22} = r_{xxx} f^{3C_1-C_4}, \\
I_{23} &= p_{xxy} f^{2C_1+C_2-C_3}, \qquad I_{24} = r_{xxy} f^{2C_1+C_2-C_4}.
\end{aligned} \tag{14}
$$

Substituting the above invariants into (10), one can establish various (2 + 1)-dimensional nonlinear systems. Generally speaking, it is difficult to find out all of the f-independent invariant systems. Here we only list some concrete examples.

Case 1. When selecting $C_1 = 1/3$, $C_2 = 0$, $C_3 = -1/3$, $C_4 = -2/3$, and $C_5 = -1/9$, we obtain the following group invariant system:

$$
\begin{aligned}
H_1 &\equiv I_9 + I_{21} - 3(I_4 I_5 + I_3 I_6) + k_1 I_{12} + k_2 I_{13}^2 = 0, \\
H_2 &\equiv I_8 + k_3 I_5 + k_4 I_{17}^2 + k_5 I_{18} = 0.
\end{aligned} \tag{15}
$$

Here and hereafter k_i, $i = 1, \ldots, 5$, are arbitrary constants. From the above invariant system, we deduce the corresponding Virasoro f-independent integrable system:

$$
\begin{aligned}
p_t + p_{xxx} - 3rp_x - 3pr_x + k_1 r_{xx} + k_2 p_{xy}^2 &= 0, \\
r_y + k_3 p_x + k_4 p_{yy}^2 + k_5 r_{yy} &= 0.
\end{aligned} \tag{16}
$$

Taking $k_i = 0$, $i = 1, 2, 4, 5$, and $k_3 = -1$, the above system is changed to be the asymmetry NNV equation which is considered as a model for an incompressible fluid and where p and r are the components of the velocity.

Case 2. Let $C_1 = 2$, $C_2 = 0$, $C_i = 1$, $i = 3, 4, 5$. We find the following group invariant system:

$$
\begin{aligned}
H_1 &\equiv I_{15} + I_5 I_6 + 2 I_{11} I_4 + k_1 I_5^2 + k_2 I_{13}^2 + k_3 I_{14}^2 = 0, \\
H_2 &\equiv I_9 + 2 I_1 I_5 I_6 - I_3 I_6 = 0,
\end{aligned} \tag{17}
$$

from which we construct the Virasoro f-independent integrable system as follows:

$$
\begin{aligned}
p_{xt} + p_x r_x + 2r p_{xx} + k_1 p_x^2 + k_2 p_{xy}^2 + k_3 r_{xy}^2 &= 0, \\
p_t + 2x r_x p_x - p r_x &= 0.
\end{aligned} \tag{18}
$$

Case 3. When choosing $C_1 = C_3 = 1/2$, $C_2 = 1$, $C_4 = C_5 = -1/2$, one can arrive at the following group invariant system:

$$
\begin{aligned}
H_1 &\equiv I_9 + I_3 I_6 - 2 I_2 I_6 I_7 - I_1 I_5 I_6 + k_1 I_7 + k_2 I_{11} = 0, \\
H_2 &\equiv I_{19} - I_6 I_7 - I_4 I_{13} - 2 I_2 I_6 I_{17} + k_3 I_8 + k_4 I_{17} + k_5 I_{23} = 0.
\end{aligned} \tag{19}
$$

Using the above system, we construct the corresponding Virasoro f-independent integrable system as follows:

$$
\begin{aligned}
p_t + pr_x - 2yp_y r_x - xp_x r_x + k_1 p_y + k_2 p_{xx} &= 0, \\
p_{yt} - p_y r_x - r p_{xy} - 2y p_{yy} r_x + k_3 r_y + k_4 p_{yy} + k_5 p_{xxy} &= 0.
\end{aligned} \tag{20}
$$

In the next section, we will find the group invariant solutions to the special case of the above system which reads

$$
\begin{aligned}
p_t + pr_x - 2yp_y r_x - xp_x r_x &= 0, \\
p_{yt} - p_y r_x - r p_{xy} - 2y p_{yy} r_x &= 0.
\end{aligned} \tag{21}
$$

Case 4. Taking $C_i = -1$, $i = 1, 2, 3, 5$, and $C_4 = -2$, we have the following group invariant system:

$$
\begin{aligned}
H_1 &\equiv I_9 - I_3 I_6 + I_4 I_5 + I_2 I_6 I_7 + k_1 I_8^2 + k_2 I_3^2 = 0, \\
H_2 &\equiv I_{20} - I_6 I_8 - I_4 I_{14} + I_2 I_6 I_{18} + k_3 I_3^2 I_8^2 + k_4 I_3^2 = 0,
\end{aligned} \tag{22}
$$

from which one can construct the Virasoro f-independent integrable system as follows:

$$
\begin{aligned}
p_t - pr_x + p_x r + yr_x + k_1 r_y^2 + k_2 p^2 &= 0, \\
r_{yt} + r_x r_y - r r_{xy} + y r_x + k_3 r_y^2 + k_4 p^2 &= 0.
\end{aligned} \tag{23}
$$

4. Group-Invariant Solutions of System (21)

Since group-invariant solutions of nonlinear models play an important role in simulation of natural phenomena [10–16], therefore we construct the group-invariant solutions to the system (21) as an example. We utilize the classical Lie symmetry group method to construct corresponding infinitesimals admitted by system (21) firstly.

Theorem 1. *The symmetries of system* (21) *form a Lie algebra h_1 generated by the following vector fields:*

$$
\begin{aligned}
&V_1 = p\partial_p, \qquad V_2 = y\partial_y, \qquad V_3 = x\partial_p, \\
&V_4 = \sqrt{y}\partial_p, \qquad V_5 = f(t)\partial_t - \dot{f}(t) r\partial_r, \\
&V_6 = g(t) x\partial_x + 2y g(t)\partial_y + g(t) p\partial_p + (g(t) r - \dot{g}(t) x)\partial_r,
\end{aligned} \tag{24}
$$

where $f(t)$ and $g(t)$ are arbitrary functions of t.

We consider three special cases of functions $f(t)$ and $g(t)$.

Case 5. When $f(t) = 0$, the symmetry generators of system (21) are reduced to

$$
V_1 = p\partial_p, \qquad V_2 = y\partial_y, \qquad V_3 = x\partial_p, \qquad V_4 = \sqrt{y}\partial_p, \tag{25}
$$

$$
\begin{aligned}
V_5 = {} & g(t) x\partial_x + 2y g(t)\partial_y + g(t) p\partial_p \\
& + (g(t) r - \dot{g}(t) x)\partial_r.
\end{aligned} \tag{26}
$$

The nonzero commutators of V_1, V_2, V_3, and V_4 are

$$
\begin{aligned}
&[V_1, V_3] = -V_3, \qquad [V_1, V_4] = -V_4, \qquad [V_2, V_4] = \frac{1}{2} V_4, \\
&[V_3, V_1] = V_3, \qquad [V_4, V_1] = V_4, \qquad [V_4, V_2] = -\frac{1}{2} V_4.
\end{aligned} \tag{27}
$$

Table 1: The adjoint representation of H_1 on h_1.

$\mathrm{Ad}(\varepsilon\cdot)$	V_1	V_2	V_3	V_4	V_5
V_1	V_1	V_2	$e^{\varepsilon}V_3$	$e^{\varepsilon}V_4$	V_5
V_2	V_1	V_2	V_3	$\cos\frac{\varepsilon}{2}V_4$	V_5
V_3	$V_1-\varepsilon V_3$	V_2	V_3	V_4	V_5
V_4	$V_1-\varepsilon V_4$	$V_2+\frac{\varepsilon}{2}V_4$	V_3	V_4	V_5
V_5	V_1	V_2	V_3	V_4	V_5

Table 2: Solutions to system (21) of Case 5.

r_i	$g(t)$	$p(x,y,t)$	$r(x,y,t)$
r_3	1	$\frac{1}{\sqrt{y}}+\frac{2C_1+1}{2x}$	$h(t)\sqrt{y}$
r_6	1	$\pm(x\ln y-2x\ln x+C_1x)$	$k(y,t)x$
r_6	1	$x\ln y+2e^{\int -h(t)\mathrm{d}t}\int^t e^{h(a)x/\int -h(t)\mathrm{d}t}\mathrm{d}a+e^{-\int h(t)\mathrm{d}t}F\left(e^{x/e^{\int -h(t)\mathrm{d}t}}\right)$	$h(t)x$
r_7	1	$-x\ln y-2e^{\int -h(t)\mathrm{d}t}\int^t e^{h(a)x/\int -h(t)\mathrm{d}t}\mathrm{d}a+e^{-\int h(t)\mathrm{d}t}F\left(e^{x/e^{\int -h(t)\mathrm{d}t}}\right)$	$h(t)x$
r_{13}	1	$y^{1/\alpha}h(t)$	$\frac{x\dot{h}(t)}{(2/\alpha-1)h(t)}+k(y,t)$
r_3	t	$\frac{x\ln x}{t}+C_1x+C_2y^{1/2}+xh(t)$	$\frac{-x\ln x+x}{t}+xt\dot{h}(t)+F(t)\sqrt{y}$

With the help of the adjoint representation:

$$Ad(\exp(\beta V))W=W-\beta[V,W]+\frac{\beta^2}{2}[V,[V,W]]-\cdots, \tag{28}$$

the adjoint action of the Lie group H_1 on the Lie algebra h_1 is listed in Table 1.

Applying the method initiated by Ovsiannikov [17], we obtain the following theorem.

Theorem 2. *The one-dimensional optimal system θ_1 of h_1 is generated by*

$$\begin{gathered} r_1=V_3, \quad r_2=-V_3, \quad r_3=V_3+V_5, \\ r_4=-V_3+V_5, \quad r_5=V_2, \quad r_6=V_2+V_3, \\ r_7=V_2-V_3, \quad r_8=V_2+\alpha V_5, \\ r_9=V_2+V_3+\alpha V_5, \quad r_{10}=V_2-V_3+\alpha V_5, \\ r_{11}=V_1, \quad r_{12}=V_1+\alpha V_5, \\ r_{13}=V_1+\alpha V_2, \quad r_{14}=V_1+\alpha V_2+\beta V_5, \end{gathered} \tag{29}$$

where α, β are nonzero real constants.

Therefore, we obtain 14 nonequivalent one-dimensional subalgebras and classify the group-invariant solutions into 14 nonequivalent types. After solving the characteristic equations, we can obtain the invariants and invariant forms. Substituting the invariant forms into system (21), we can reduce the original (2 + 1)-dimensional system to (1 + 1)-dimensional system. Since it is a tough task to find all solutions out for the every 14 nonequivalent subalgebras, we just show the results for the cases that we can deal with and list the solutions to system (21) in Table 2. Here and hereafter $h(\cdot)$, $F(\cdot)$, and $k(\cdot,\cdot)$ are arbitrary functions with respect to their variables.

Case 6. When $g(t)=0$, the symmetry generators of system (21) are reduced to

$$\begin{gathered} V_1=p\partial_p, \quad V_2=y\partial_y, \quad V_3=x\partial_p, \\ V_4=\sqrt{y}\partial_p, \quad V_5=f(t)\partial_t-\dot{f}(t)r\partial_r. \end{gathered} \tag{30}$$

In this case, the optimal system is the same as that in Theorem 2 and we can find some new solutions which are listed in Table 3.

Case 7. When $f(t)=1$, $g(t)=e^t$, the symmetry generators of system (21) are

$$\begin{gathered} V_1=p\partial_p, \quad V_2=y\partial_y, \quad V_3=x\partial_p, \\ V_4=\sqrt{y}\partial_p, \quad V_5=\partial_t, \\ V_6=e^t\left(x\partial_x+2y\partial_y+p\partial_p+(r-x)\partial_r\right). \end{gathered} \tag{31}$$

Table 3: Solutions to system (21) of Case 6.

r_i	$f(t)$	$p(x,y,t)$	$r(x,y,t)$
r_3	1	$xt+F(x)+C_1\sqrt{y}+C_2$	$\int \frac{x}{x\dot{F}(x)-C_1-F(x)}\mathrm{d}x+K(y,t)$
r_3	t	$x\ln t+F(x)+C_1\sqrt{y}+C_2$	$\frac{\int(x/(x\dot{F}(x)-C_1-F(x)))\mathrm{d}x}{t}+\frac{F(y)}{t}$
r_3	t^2	$-\frac{x}{t}+F(x)+C_1\sqrt{y}+C_2$	$\frac{\int(x/(x\dot{F}(x)-C_1-F(x)))\mathrm{d}x}{t^2}+\frac{F(y)}{t^2}$
r_8	1	C_1x	$K(x,y^{\alpha}e^{-t})$
r_8	1	$F(x)$	$h(y^{\alpha}e^{-t})$
r_8	1	$C_1x+\frac{C_2x^{C_3}\sqrt{y}}{e^{t/2\alpha}}$	$\frac{-x}{2\alpha C_3}$
r_8	1	$C_1x+C_2(y^{\alpha}e^{-t})^{C_3}$	$\frac{C_3x}{1-2C_3\alpha}+h(y^{\alpha}e^{-t})$
r_8	t	C_1x	$\frac{K(x,y^{\alpha}e^{-t})}{t}$
r_8	t	$F(x)$	$\frac{h(y^{\alpha}e^{-t})}{t}$
r_8	t	$C_3x+\frac{C_2x^{C_1}\sqrt{y}}{t^{1/2a}}$	$\frac{-x}{2aC_1}$
r_8	t	$\frac{C_2C_3xy^{\alpha C_1}-t^{C_1}}{C_2y^{\alpha C_1}}$	$\frac{(1+2C_1\alpha)h(y^{\alpha}/t)-C_1x}{(1+2C_1\alpha)t}$
r_8	t^2	C_1x	$\frac{k(x,ye^{1/\alpha t})}{t^2}$
r_8	t^2	$F(x)$	$\frac{h(ye^{1/\alpha t})}{t^2}$
r_8	t^2	$C_1x+C_2x^{C_3}\sqrt{y}e^{1/2\alpha t}$	$\frac{-x}{2C_3\alpha t^2}$
r_8	t^2	$C_1x+C_2(ye^{1/\alpha t})^{C_3}$	$\frac{-C_3x}{(2C_3-1)\alpha t^2}+\frac{h(ye^{1/\alpha t})}{t^2}$
r_9	1	$\frac{xt}{\alpha}+C_3x^2(y^{\alpha}e^{-t})^{1/2\alpha}-2x(2\ln x-C_2)$	$\frac{-x}{4\alpha}$
r_9	1	$\frac{xt}{\alpha}+(y^{\alpha}e^{-t})^{1/(\alpha(2+C_4))}+C_3x+C_4x\ln x$	$\frac{x}{4\alpha}+h(y^{\alpha}e^{-t})$
r_9	1	$\frac{x(\alpha\ln y-2\alpha\ln x+C_1\alpha)}{\alpha}$	$xh(y^{\alpha}e^{-t})$
r_9	1	$\frac{xt}{\alpha}+(h(y^{\alpha}e^{-t})-2\ln x)x$	$\frac{-x}{2\alpha}$
r_{12}	1	$C_3e^tx^{C_2}(C_1x+\sqrt{y})$	$\frac{x}{C_2}$
r_{12}	1	$h(x)e^{\alpha/t}$	$\int\frac{h(x)}{\alpha(x\dot{h}(x)-h(x))}\mathrm{d}x+K(y,t)$
r_{12}	t	$C_3tx^{C_2}(C_1x+\sqrt{y})$	$\frac{x}{C_2t}$
r_{12}	t	$t^{1/\alpha}h(x)$	$\frac{\int h(x)/(\alpha(x\dot{h}(x)-h(x)))\mathrm{d}x+K(y,t)}{t}$

Table 4: Solutions to system (21) of Case 7.

r_i	$p(x,y,t)$	$r(x,y,t)$
r_3	$xt+F(x)+C_1+C_2\sqrt{y}$	$\int \frac{x}{-F(x)-C_1+xF'(x)}dx+h(y)$
r_4	$-xt+F(x)+C_1+C_2\sqrt{y}$	$\int \frac{-x}{-F(x)-C_1+xF'(x)}dx+h(y)$
r_8	C_1x	$K\left(x,\frac{y}{e^{t/\alpha}}\right)$
r_8	$F(x)$	$h\left(\frac{y}{e^{t/\alpha}}\right)$
r_8	$C_3x+C_2x_1^{C}\frac{\sqrt{y}}{e^{t/2\alpha}}$	$\frac{-x}{2\alpha C_1}$
r_8	$C_3x+C_2\frac{y^{C_1}}{e^{(C_1t)/\alpha}}$	$\frac{C_1x}{\alpha(1-2C_1)}+F\left(\frac{y}{e^{t/\alpha}}\right)$
r_9	$\frac{xt}{\alpha}+F(x)$	$\int \frac{x}{\alpha(-F(x)+xF'(x))}dx+h\left(\frac{y}{e^{t/\alpha}}\right)$
r_9	$\frac{xt}{\alpha}-2x\ln(C_3x)+C_1x+C_3x\sqrt{\frac{y}{e^{t/\alpha}}}$	$\frac{-x}{2\alpha}$
r_9	$\frac{xt}{\alpha}+\left(((C_2-2)\ln x+C_1)xe^{C_4/C_2}-\left(\frac{y}{e^{t/\alpha}}\right)^{1/C_2}\right)e^{-C_4/C_2}$	$\frac{x}{\alpha(C_2-2)}+F\left(\frac{y}{e^{t/\alpha}}\right)$
r_9	$\frac{xt}{\alpha}+x\left(\ln\left(\frac{y}{e^{t/\alpha}}\right)+C_1-2\ln x\right)$	$xF\left(\frac{y}{e^{t/\alpha}}\right)$
r_{10}	$\frac{-xt}{\alpha}+F(x)$	$\int \frac{-x}{\alpha(-F(x)+xF'(x))}dx+h\left(\frac{y}{e^{t/\alpha}}\right)$
r_{10}	$\frac{-xt}{\alpha}+2x\ln(C_3x)+C_1x+C_3x\sqrt{\frac{y}{e^{t/\alpha}}}$	$\frac{-x}{2\alpha}$
r_{10}	$\frac{-xt}{\alpha}+\left(((-C_2+2)\ln x+C_1)xe^{C_4/C_2}-\left(\frac{y}{e^{t/\alpha}}\right)^{1/C_2}\right)e^{-C_4/C_2}$	$\frac{x}{\alpha(C_2-2)}+F\left(\frac{y}{e^{t/\alpha}}\right)$
r_{10}	$\frac{-xt}{\alpha}+x\left(-\ln\left(\frac{y}{e^{t/\alpha}}\right)+C_1+2\ln x\right)$	$xF\left(\frac{y}{e^{t/\alpha}}\right)$
r_{12}	$e^{t/\alpha}+C_1x+C_2\sqrt{y}$	$h(y)+\frac{x}{\alpha}$
r_{14}	$y^{1/\alpha}F(x)\left(\frac{y}{e^{\alpha t/\beta}}\right)^{-1/\alpha}$	$\int \frac{F(x)}{\beta(xF'(x)-F(x))}dx+h\left(\frac{y}{e^{\alpha t/\beta}}\right)$
r_{14}	$y^{1/\alpha}\left(x^{(\alpha/2-1)}C_1\left(C_3x^{C_2}\right)^{(1-\alpha/2)}\left(\frac{y}{e^{\alpha t/\beta}}\right)^{\alpha-2/2\alpha}+C_3x^{C_2}\left(\frac{y}{e^{\alpha t/\beta}}\right)^{-1/\alpha}\right)$	$\frac{x}{\beta(C_2-1)}$
r_{14}	$C_1y^{1/\alpha}$	$F\left(\frac{y}{e^{\alpha t/\beta}}\right)$

By simple calculation, we obtain that the optimal system in this case is the same as that in Theorem 2. And we list the new solutions in Table 4.

5. Concluding Remarks

In this paper, we extend the Virasoro-type symmetry prolongation approach from single equations to coupled systems of two-component nonlinear equations. Four types of new nonlinear Virasoro integrable systems are constructed. Furthermore, we obtain the one-dimensional optimal system and group-invariant solutions to one of the model systems, namely, system (21).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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