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Research Article

A Matrix Approach for Divisibility Properties of the Generalized Fibonacci Sequence

Aynur Yalçiner

Department of Mathematics, Science Faculty, Selcuk University, 42075 Konya, Turkey

Correspondence should be addressed to Aynur Yalçiner; aynuryalciner@gmail.com

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We give divisibility properties of the generalized Fibonacci sequence by matrix methods. We also present new recursive identities for the generalized Fibonacci and Lucas sequences.

1. Introduction

The generalized Fibonacci sequence $\{U_n\}$ and the generalized Lucas sequence $\{V_n\}$ are defined for $n > 1$, by,

$$U_n = pU_{n-1} + U_{n-2},$$
$$V_n = pV_{n-1} + V_{n-2}, \quad (1)$$

where $U_0 = 0, U_1 = 1$ and $V_0 = 2$, $V_1 = p$, respectively.

Let α and β be the roots of the equation $x^2 - px - 1 = 0$. Then the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$ are given by

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad V_n = \alpha^n + \beta^n. \quad (2)$$

If $p = 1$, then $U_n = F_n$ (nth Fibonacci number) and $V_n = L_n$ (nth Lucas number).

It is a well-known fact that

$$\gcd(F_n, F_m) = F_{\gcd(n,m)}. \quad (3)$$

It is also known that F_{kn} is a multiple of F_n, for all integers k and n. In [1], the author showed that, for $n > 2$, the Fibonacci number F_m is a multiplication of F_n^2 if and only if m is multiplication of nF_n (for more details see [2]). Also, in [3], the author obtained the following divisibility properties:

(i) $F_{kn-1} - F_{n-1}^k$ is divisible by F_n^2;

(ii) $F_{kn-2} - (-1)^{k+1}F_{n-2}^k$ is divisible by F_n^2,

where $n, k \geq 1$. Kiliç [4] generalized these results for a general second-order linear recursion $\{U_n\}$ as follows:

$$U_r^{k-1}U_{kn-r} - (-1)^{(r-1)(k+1)}U_{n-r}^k \text{ is divisible by } U_n^2. \quad (4)$$

In this paper, we investigate divisibility properties of the generalized Fibonacci numbers by U_n^l, where $l \geq 3$. For $l = 3$, we show that

$$U_2^{k-2}U_{kn-3} - U_2^{k-2}U_{n-1}^k + (-1)^{k-1}U_{n-2}^k \text{ is divisible by } U_n^3. \quad (5)$$

We use matrix methods to prove the claim. We recall that matrix methods are useful tools for deriving some properties of linear recurrences (see [4–9]). We consider the quotient

$$\frac{U_2^{k-2}U_{kn-3} - U_2^{k-2}U_{n-1}^k + (-1)^{k-1}U_{n-2}^k}{U_n^3} \quad (6)$$

for all positive integers n and k. We define a generating matrix for this quotient for fixed n and increasing values of k. Then we give an explicit statement for the quotient. Also, by considering this explicit statement, we find new recursive identities for the general second-order linear recurrences. Finally, we give divisibility properties of the generalized Fibonacci numbers in the case $l > 3$. Thus we obtain a generalization of the results given in [4].

2. Main Results

We denote the quotient $(U_2^{k-2}U_{kn-3} - U_2^{k-2}U_{n-1}^k + (-1)^{k-1}U_{n-2}^k)/U_n^3$ by $s(n,k)$.

Define a second-order linear sequence $\{A_n\}$, for $n > 1$,

$$A_n = pA_{n-1} + A_{n-2} \tag{7}$$

with initial conditions $A_0 = p^2 - 1$ and $A_1 = p^3 + 3p$.

By the definitions of $\{U_n\}$ and $\{A_n\}$, we have

$$A_n = U_2V_{n+1} + U_2U_n - U_{n-1}. \tag{8}$$

Define a matrix $H(n)$ by

$$H(n) = \begin{bmatrix} A_{n-1} & B_n & C_n & (-1)^n U_2^3 U_{n-1}U_{n-2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \tag{9}$$

where

$$B_n = \left(U_2U_{n-2} - U_2^2U_{n-1}\right)V_n + U_{n-1}^2 - U_{n-2}^2 + (-1)^{n+1}U_3,$$
$$C_n = -U_2^2U_{n-1}U_{n-2}V_n + (-1)^{n+1}U_2^2U_{n-2} + (-1)^nU_2^3U_{n-1}. \tag{10}$$

We next define a matrix $G(n,k)$ of order 4 as follows:

$$G(n,k) = \begin{bmatrix} s(n,k+3) & t(n,k+3) & y(n,k+3) & (-1)^n U_2^3U_{n-1}U_{n-2}s(n,k+2) \\ s(n,k+2) & t(n,k+2) & y(n,k+2) & (-1)^n U_2^3U_{n-1}U_{n-2}s(n,k+1) \\ s(n,k+1) & t(n,k+1) & y(n,k+1) & (-1)^n U_2^3U_{n-1}U_{n-2}s(n,k) \\ s(n,k) & t(n,k) & y(n,k) & (-1)^n U_2^3U_{n-1}U_{n-2}s(n,k-1) \end{bmatrix}. \tag{11}$$

$t(n,k)$ and $y(n,k)$ are given by

$$\begin{aligned} t(n,k) = \Big[&-U_2^{k-2}U_{n-1}U_{n-2}\delta_nU_{kn} \\ &+ (-1)^{n+1}U_2^{k-2}\left(U_2U_{n-1}\delta_n + U_{n-2}^2\right)U_{n(k-1)} \\ &+ (-1)^{3n}U_2^kU_{n(k-3)} \\ &+ U_2^{k-2}U_n^2\left(2(-1)^nU_{-n+2} + U_{n+2}\right)U_{n-1}^k \\ &+ (-1)^kU_2U_n^2\left(2(-1)^nU_{-n+1} + U_{n+1}\right)U_{n-2}^k\Big] \\ &\times\left(U_n^5\right)^{-1}, \\ y(n,k) = \Big[&-U_2^{k-1}U_{n-1}^2U_{n-2}^2U_{kn} \\ &+ U_2^{k-1}\left(U_2U_{n-1}\delta_n + U_{n-2}^2\right)U_{n(k-2)} \\ &+ (-1)^{3n+1}U_2^k\delta_nU_{n(k-3)} \\ &+ U_n^2\left(2(-1)^{n+1}U_2 + U_{2n-2}\right)U_{n-1}^k \\ &+ (-1)^kU_2^2U_n^2\left(2(-1)^{n+1} - U_{2n-1}\right)U_{n-2}^k\Big] \\ &\times\left(U_n^5\right)^{-1}, \end{aligned} \tag{12}$$

where

$$\delta_n = U_2U_{n-1} - U_{n-2}. \tag{13}$$

Thus we give our first main result.

Theorem 1. *For $n \geq 1$,*

$$H(n)^k = G(n,k). \tag{14}$$

Proof. We will use induction on k. The result is clear for $k = 1$. Now assume that $H(n)^{k-1} = G(n,k-1)$. Then, by the definitions of $s(n,k)$, $t(n,k)$, and $y(n,k)$, we have

$$\begin{aligned} A_{n-1}s(n,k+2) + t(n,k+2) &= s(n,k+3), \\ B_ns(n,k+2) + y(n,k+2) &= t(n,k+3), \\ C_ns(n,k+2) + (-1)^nU_2^3U_{n-1}U_{n-2}s(n,k+1) & \\ = y(n,k+3). & \end{aligned} \tag{15}$$

Thus the proof is complete. □

As a consequence of this theorem, we can see that matrix $H(n)$ generates $s(n,k)$. Since the elements of $H(n)$ are integers, the quotient $s(n,k)$ are integers for all positive integers n and k.

Lemma 2. *For $n \geq 1$, the eigenvalues of $H(n)$ are $U_2\alpha^n$, $U_2\beta^n$, U_2U_{n-1}, and $-U_{n-2}$.*

Proof. The characteristic polynomial of $H(n)$ is

$$\begin{aligned} x^4 - A_{n-1}x^3 - B_nx^2 - C_nx & \\ + (-1)^{n+1}U_2^3U_{n-1}U_{n-2} &= 0, \end{aligned} \tag{16}$$

and it is factorized as

$$\left(x - U_2\alpha^n\right)\left(x - U_2\beta^n\right)\left(x - U_2U_{n-1}\right)\left(x + U_{n-2}\right) = 0, \tag{17}$$

which completes the proof. □

As another main result, we have the following theorem.

Theorem 3. *For $n, k \geq 1$,*

$$\begin{aligned}(G(n,k))_{4,1} &= s(n,k) \\ &= \left(U_2^{k-2}U_{n-1}U_{n-2}U_{kn} + (-1)^n U_2^{k-2}\delta_n U_{n(k-1)}\right. \\ &\quad - U_2^{k-1}U_{n(k-2)} - U_2^{k-2}U_n^2U_{n-1}^k \\ &\quad \left.+(-1)^{k-1}U_n^2U_{n-2}^k\right) \times \left(U_n^5\right)^{-1},\end{aligned} \tag{18}$$

where δ_n is defined as shown previously.

Proof. Since the eigenvalues of $H(n)$ are distinct, $H(n)$ is diagonalizable as

$$V^{-1}H(n)V = D, \tag{19}$$

where

$$V = \begin{bmatrix} U_2^3\alpha^{3n} & U_2^3\beta^{3n} & U_2^3U_{n-1}^3 & -U_{n-2}^3 \\ U_2^2\alpha^{2n} & U_2^2\beta^{2n} & U_2^2U_{n-1}^2 & U_{n-2}^2 \\ U_2\alpha^n & U_2\beta^n & U_2U_{n-1} & -U_{n-2} \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{20}$$

and $D = \operatorname{diag}(U_2\alpha^n, U_2\beta^n, U_2U_{n-1}, -U_{n-2})$. Therefore, we obtain $V^{-1}H(n)^kV = D^k$. By Theorem 1, we write $V^{-1}G(n,k)V = D^k$. Then we have the following linear equation system:

$$\begin{aligned} &g_{i1}U_2^3\alpha^{3n} + g_{i2}U_2^2\alpha^{2n} + g_{i3}U_2\alpha^n + g_{i4} \\ &\quad = U_2^{k-i+4}\alpha^{(k-i+4)n}, \\ &g_{i1}U_2^3\beta^{3n} + g_{i2}U_2^2\beta^{2n} + g_{i3}U_2\beta^n + g_{i4} \\ &\quad = U_2^{k-i+4}\beta^{(k-i+4)n}, \\ &g_{i1}U_2^3U_{n-1}^3 + g_{i2}U_2^2U_{n-1}^2 + g_{i3}U_2U_{n-1} + g_{i4} \\ &\quad = U_2^{k-i+4}U_{n-1}^{k-i+4}, \\ &-g_{i1}U_{n-2}^3 + g_{i2}U_{n-2}^2 - g_{i3}U_{n-2} + g_{i4} \\ &\quad = (-1)^{k-i+4}U_{n-2}^{k-i+4}. \end{aligned} \tag{21}$$

The solution of the above linear equation system gives the claimed result. □

By considering the definition of $s(n,k)$, we have the following consequence of Theorem 3.

Corollary 4. *For $n, k \geq 1$,*

$$\begin{aligned} &U_{n-1}U_{n-2}U_{kn} \\ &\quad = U_n^2U_{kn-3} + (-1)^{n+1}\delta_nU_{n(k-1)} + U_2U_{n(k-2)}. \end{aligned} \tag{22}$$

The next results generalize the result given by Corollary 4.

Theorem 5. *For all integers r,*

$$\begin{aligned} U_{n-r}U_{n-r-1}U_{kn} &= U_n^2U_{kn-2r-1} \\ &\quad + (-1)^{n+r}\left(U_{r+1}U_{n-r} - U_rU_{n-r-1}\right)U_{n(k-1)} \\ &\quad + U_rU_{r+1}U_{n(k-2)}, \\ U_{n-r}U_{n-r-1}V_{kn} &= U_n^2V_{kn-2r-1} \\ &\quad + (-1)^{n+r}\left(U_{r+1}U_{n-r} - U_rU_{n-r-1}\right)V_{n(k-1)} \\ &\quad + U_rU_{r+1}V_{n(k-2)}. \end{aligned} \tag{23}$$

Proof. The proof can be seen by the Binet formulas of the sequences $\{U_n\}$ and $\{V_n\}$. □

For $l = 3$, we give the general case of divisibility properties in the following result.

Corollary 6. *For all integers r,*

$$\begin{aligned} &U_r^{k-2}U_{r+1}^{k-2}U_{kn-2r-1} - (-1)^{k(r-1)}U_{r+1}^{k-2}U_{n-r}^k \\ &\quad + (-1)^{kr-1}U_r^{k-2}U_{n-r-1}^k \end{aligned} \tag{24}$$

is divisible by U_n^3.

3. Generalization of the Divisibility Properties

In this section, for a positive integer l, we generalize divisibility properties. For this purpose we introduce some new notations.

Let r_i be an integer for $i = 1, \ldots, l-1$. Let

$$\begin{aligned} \xi_1 &= \begin{cases} \sum_{1\le i\le m} r_i, & m \equiv 1 \pmod 2, \\ \sum_{1\le i\le m-1} r_i, & m \equiv 0 \pmod 2, \end{cases} \\ \xi_2 &= \begin{cases} \sum_{1\le i\le m-1} r_i, & m \equiv 1 \pmod 2, \\ \sum_{1\le i\le m} r_i, & m \equiv 0 \pmod 2, \end{cases} \\ \varphi &= \sum_{1\le i\le l-1} r_i, \\ \gamma &= \prod_{1\le j<i\le l-1} U_{r_i-r_j}. \end{aligned} \tag{25}$$

We denote the above product by γ_m for $i, j \neq m$.

Corollary 7. *(a) For an even positive integer l,*

$$\left(\prod_{1\le i\le l-1} U_{r_i}\right)^{k-l+1} \gamma U_{kn-\varphi} + \sum_{1\le m\le l-1} (-1)^{k(r_m-1)+m+\xi_1+1} \gamma_m U_{n-r_m}^{k} \left(\prod_{\substack{1\le i\le l-1\\ m\neq i}} U_{r_i}\right)^{k-l+1} \quad (26)$$

is divisible by γU_n^l.

(b) For an odd positive integer l,

$$\left(\prod_{1\le i\le l-1} U_{r_i}\right)^{k-l+1} \gamma U_{kn-\varphi} + \sum_{1\le m\le l-1} (-1)^{k(r_m-1)+m+\xi_2} \gamma_m U_{n-r_m}^{k} \left(\prod_{\substack{1\le i\le l-1\\ m\neq i}} U_{r_i}\right)^{k-l+1} \quad (27)$$

is divisible by γU_n^l.

As an example, if we take $l = 4$, $r_1 = 1$, $r_2 = 2$, $r_3 = 3$, and $p = 1$, then

$$F_3^{k-3} F_{kn-6} - F_3^{k-3} F_{n-1}^{k} + (-1)^k F_3^{k-3} F_{n-2}^{k} + F_{n-3}^{k} \text{ is divisible by } F_n^4. \quad (28)$$

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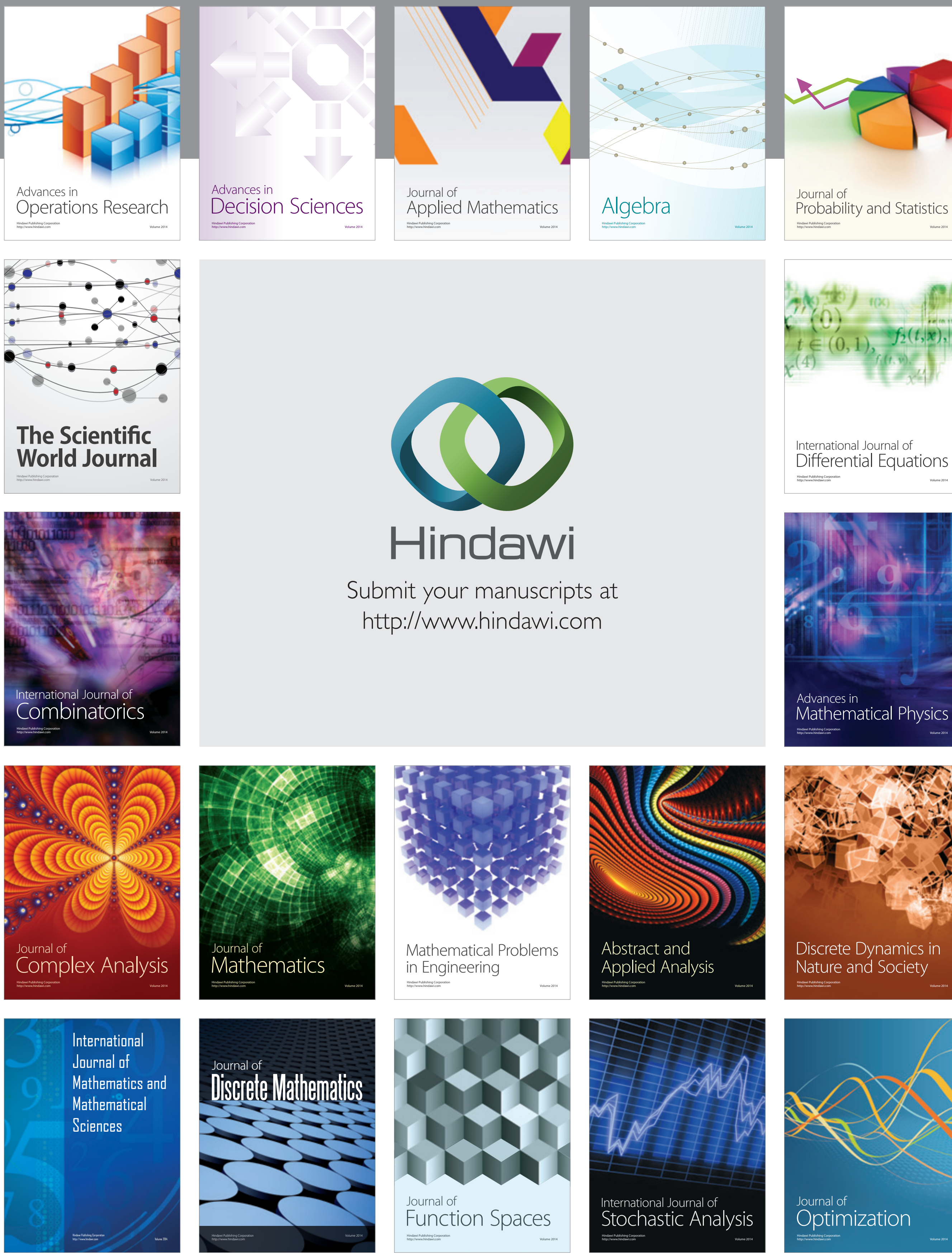
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