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Research Article

Chaos in a Discrete Delay Population Model

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This paper is concerned with chaos in a discrete delay population model. The map of the model is proved to be chaotic in the sense of both Devaney and Li-Yorke under some conditions, by employing the snap-back repeller theory. Some computer simulations are provided to visualize the theoretical result.

1. Introduction

Delay differential equations have been largely used to model phenomena in economics, biology, medicine, ecology, and other sciences. The studies on delay differential equations in population dynamics not only focus on the discussion of stability, attractivity, and persistence, but also involve many other dynamical behaviors such as periodic phenomenon, bifurcation, and chaos, see [1–5], and many references therein.

As we well know, the discrete time population models governed by difference equations are more appropriate than the continuous time population models governed by differential equations when the populations have nonoverlapping generations or the size of the population is rarely small. Moreover, some qualitative properties of the difference equations can also provide a lot of useful information for analyzing the properties of the original differential equations. In addition, discrete time models can also provide efficient computational models of continuous time models for numerical simulations. Therefore, many researchers studied the complex behaviors of the discrete population model, see, for example, [6–11].

Recently, some researchers used the Euler discretization to explore the complex dynamical behaviors of nonlinear differential systems, such as determining the bifurcation diagrams with Hopf bifurcation, observing stable or unstable orbits, and chaotic behavior, see [8–16], and so forth. However, some complicated behaviors such as chaos they observed were obtained only by numerical simulations, and have not been proved rigorously. It is noted that there could exist some false phenomena only by virtue of numerical simulations. Therefore, the existence of chaotic behavior of these systems needs to be studied rigorously.

In this paper, we study the chaotic behavior of the following discrete delay population model

$$x(n+1) = x(n) + \gamma x(n)\Big(a + bx(n-k) - cx^2(n-k)\Big), \tag{1.1}$$

where $a > 0$, $c > 0$, and $b \in R$ are constants, k is a positive integer, and $\gamma > 0$ is a parameter.

Equation (1.1) can be viewed as a discrete analogue of the following delay differential equation by using the forward Euler scheme when $r(t) = r$, which is the model of single species population with a quadratic per capita growth rate

$$\dot{x}(t) = r(t)x(t)\Big(a + bx(t-\tau) - cx^2(x-\tau)\Big), \tag{1.2}$$

where $r(t) \in C([0,+\infty),(0,+\infty))$, $a > 0$, $c > 0$, and $b \in R$ are constants, and $\tau > 0$ is the delay. In [1], Gopalsamy studied the global attractivity of the equilibrium of (1.2). As $r(t) = r$ in (1.2), Gopalsamy [1] discussed the existence conditions of Hopf bifurcation, and gave an approximate expression of the bifurcation periodic solution.

When $r(t) = 1$ in (1.2), it becomes

$$\dot{x}(t) = x(t)\Big(a + bx(t-\tau) - cx^2(x-\tau)\Big), \tag{1.3}$$

which was studied by Gopalsamy and Ladas for the oscillation and asymptotic behavior in [2]; Rodrigues [8], Huang and Peng [9] studied the discretization of (1.3) by using the forward and backward Euler scheme, respectively. They obtained some results about oscillation and stability of the solutions. Peng [11] also studied the backward difference form of (1.3), and observed much rich dynamical behaviors, such as Neimark-Sacker bifurcation and chaotic behavior by using the computer-assisted method and computer simulations.

To the best of our knowledge, the research works on the chaotic behavior of (1.2) or its discrete analogue (1.1) with rigorously mathematical proof up to now are still few. The main purpose of this paper is to study the chaotic behavior of (1.1) by using the snap-back repeller theory.

The rest of the paper is organized as follows. In Section 2, some basic concepts and lemmas are introduced. The transformation of the chaos problem is given in Section 3. In Section 4, it is rigorously proved that there exists chaotic behavior in the delay population model by using the snap-back repeller theory. Finally, an illustrative example is provided with computer simulations.

2. Preliminaries

In this section, some basic concepts and lemmas are introduced.

Since Li and Yorke [17] first introduced a precise mathematical definition of chaos, there appeared several different definitions of chaos, some are stronger and some are weaker, depending on the requirements in different problems. We refer to [17–23] for the some definitions of chaos and discussions of their relationships. For convenience, we present two definitions of chaos in the sense of Li-Yorke and Devaney.

Definition 2.1. Let (X, d) be a metric space, $f : X \to X$ be a map, and S be a set of X with at least two distinct points. Then S is called a scrambled set of f if for any two distinct points $x, y \in S$,

(i) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$;

(ii) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$.

The map f is said to be chaotic in the sense of Li-Yorke if there exists an uncountable scrambled set S of f.

There are three conditions in the original characterization of chaos in Li-Yorke's theorem [17]. Besides the previous conditions (i) and (ii) mentioned in Definition 2.1, the third one is that for all $x \in S$ and for all periodic point p of f,

$$\limsup_{n\to\infty} d\left(f^n(x), f^n(p)\right) > 0. \tag{2.1}$$

But conditions (i) and (ii) together (Definition 2.1) imply that the scrambled set S contains at most one point x that does not satisfy the above condition. So the third condition is not essential and can be removed.

Definition 2.2 (see [19]). Let (X, d) be a metric space. A map $f : V \subset X \to V$ is said to be chaotic on V in the sense of Devaney if

(i) the set of the periodic points of f is dense in V;

(ii) f is topologically transitive in V;

(iii) f has sensitive dependence on initial conditions in V.

In Definition 2.2, condition (i) implies that all systems with no periodic points are not chaotic; condition (ii) means that a chaotic system is indecomposable, that is, the system cannot be decomposed into the sum of two subsystems; condition (iii) says that the system is unpredictable, which means that a small change of initial conditions can cause an unavoidable error after many iterations. In 1992, Banks et al. [18] proved that conditions (i) and (ii) together imply condition (iii) if f is continuous in V. So, condition (iii) is redundant in the above definition. It has been proved that under some conditions, chaos in the sense of Devaney is stronger than that in the sense of Li-Yorke [24].

Remark 2.3. Some researchers consider that condition (i) in Definition 2.2 is not essential in the chaotic behavior. In 1990, Wiggins [23] gave another definition of chaos, that is, f is said to be chaotic on V in the sense of Wiggins if it satisfies conditions (ii) and (iii) in Definition 2.2. It is evident that chaos in the sense of Devaney is stronger than that in the sense of Wiggins.

For convenience, we present some definitions in [25].

Definition 2.4 ([25, Definitions 2.1–2.4]). Let (X, d) be a metric space and $f : X \to X$ be a map.

(i) A point $z \in X$ is called an expanding fixed point (or a repeller) of f in $\overline{B}_r(z)$ for some constant $r > 0$, if $f(z) = z$ and there exists a constant $\lambda > 1$ such that

$$d(f(x), f(y)) \geq \lambda d(x, y) \quad \forall x, y \in \overline{B}_r(z), \tag{2.2}$$

where $\overline{B}_r(z) := \{x \in X : d(x, z) \leq r\}$ is the closed ball centered at z. The constant λ is called an expanding coefficient of f in $\overline{B}_r(z)$. Furthermore, z is called a regular expanding fixed point of f in $\overline{B}_r(z)$ if z is an interior point of $f(B_r(z))$, where $B_r(z) := \{x \in X : d(x, z) < r\}$ is the open ball centered at z. Otherwise, z is called a singular expanding fixed point of f in $\overline{B}_r(z)$.

(ii) Assume that z is an expanding fixed point of f in $\overline{B}_r(z)$ for some $r > 0$. Then z is said to be a snap-back repeller of f if there exists a point $x_0 \in B_r(z)$ with $x_0 \neq z$ and $f^m(x_0) = z$ for some positive integer m. Furthermore, z is said to be a nondegenerate snap-back repeller of f if there exist positive constants μ and $r_0 < r$ such that $B_{r_0}(x_0) \subset B_r(z)$ and

$$d(f^m(x), f^m(y)) \geq \mu d(x, y) \quad \forall x, y \in B_{r_0}(x_0). \tag{2.3}$$

z is called a regular snap-back repeller of f if $f(B_r(z))$ is open and there exists a positive constant δ_0 such that $B_{\delta_0}(x_0) \subset B_r(z)$ and for each positive constant $\delta \leq \delta_0$, z is an interior point of $f^m(B_\delta(x_0))$. Otherwise, z is called a singular snap-back repeller of f.

Remark 2.5. In 1978, Marotto [26] introduced the concept of snap-back repeller for maps in the Euclidean space $\mathbf{R}^n$. It is obvious that Definition 2.4 extended the concept of snap-back repeller to maps in metric spaces. According to the above classifications of snap-back repellers for maps in metric spaces, the snap-back repeller in the Marotto paper [26] is regular and nondegenerate.

We now present two lemmas which will be used to study chaos in the delay population model.

Lemma 2.6. *Let $h : [-r, r] \subset \mathbf{R} \to \mathbf{R}$ be a continuously differentiable map. Assume that $h(0) = 0$, $h'(0) \neq 0$, then for a sufficiently small neighborhood N of $0 \in \mathbf{R}$ and any bounded interval I of $\mathbf{R}$, there exists a positive constant $\gamma^* := \gamma^*(N, I)$ such that the equation $\gamma h(x) = y$ has a solution $x \in N$ for any $|\gamma| > \gamma^*$ and $y \in I$.*

Proof. Since h is continuously differentiable on $[-r, r]$, and $h(0) = 0$, $h'(0) \neq 0$, for any sufficiently small neighborhood $N \subset [-r, r]$ of 0, there exist two neighborhoods U and V of 0 such that $U, V \subset N$, and $h : U \to V$ is a homeomorphism by [27, Theorem 10.39]. In addition, for any bounded interval $I \subset \mathbf{R}$, there exists a positive constant γ_0 such that

$$\frac{1}{\gamma_0} I := \left\{ \frac{1}{\gamma_0} y : y \in I \right\} \subset V. \tag{2.4}$$

Take $\gamma^* := \gamma^*(N, I) = \gamma_0$, then $(1/\gamma)I \subset V$ for any $|\gamma| > \gamma^*$. Hence, for any $y \in I$ and $|\gamma| > \gamma^*$, the equation $h(x) = (1/\gamma)y$, that is, $\gamma h(x) = y$ has a solution $x \in U \subset N$. This completes the proof. □

Lemma 2.7 ([28, Theorem 4.4]). *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a map with a fixed point $z \in \mathbf{R}^n$. Assume that*

(1) *f is continuously differentiable in a neighborhood of z and all the eigenvalues of $Df(z)$ have absolute values larger than 1, which implies that there exist a positive constant r and a norm $\|\cdot\|$ in $\mathbf{R}^n$ such that f is expanding in $\overline{B}_r(z)$ in $\|\cdot\|$, where $\overline{B}_r(z)$ is the closed ball of radius r centered at z in $(\mathbf{R}^n, \|\cdot\|)$;*

(2) *z is a snap-back repeller of f with $f^m(x_0) = z$, $x_0 \neq z$, for some $x_0 \in B_r(z)$ and some positive integer m, where $B_r(z)$ is the open ball of radius r centered at z in $(\mathbf{R}^n, \|\cdot\|)$. Furthermore, f is continuously differentiable in some neighborhoods of $x_0, x_1, \ldots, x_{m-1}$, respectively, and $Df(x_j) \neq 0$ for $0 \le j \le m-1$, where $x_j = f(x_{j-1})$ for $1 \le j \le m-1$.*

Then for each neighborhood U of z, there exist a positive integer $k > m$ and a Cantor set $\Lambda \subset U$ such that $f^k : \Lambda \to \Lambda$ is topologically conjugate to the symbolic system $\sigma : \sum_2^+ \to \sum_2^+$. Consequently, f^k is chaotic on Λ in the sense of Devaney and f is chaotic in the sense of Li-Yorke. Further, there exists a compact and perfect invariant set $V \subset \mathbf{R}^n$, containing the Cantor set Λ, such that f is chaotic on V in the sense of Devaney.

Remark 2.8. The conclusions of Lemma 2.7 is slightly different from the original Theorem 4.4 in [28]. From [28, Theorem 4.4], we get that f^k is chaotic on Λ in the sense of Devaney. By [29, Lemma 2.4], we obtain that f^k is chaotic in the sense of Li-Yorke. Consequently f is chaotic in the sense of Li-Yorke. The last conclusion of Lemma 2.7 can be conferred to [30, Theorem 4.2]. Under the conditions of Lemma 2.7, z is a regular and nondegenerate snap-back repeller. Therefore, Lemma 2.7 can be briefly stated as the following: "a regular and nondegenerate snap-back repeller in $\mathbf{R}^n$ implies chaos in the sense of both Devaney and Li-York." We refer to [25, 31] for details.

3. Transformation of the Chaos Problem

In this section, we will transform the delay population model (1.1) into a $(k+1)$-dimensional discrete dynamical system, and give some definitions about chaos of the two systems.

Let $u_j(n) := x(n+j-k-1)$ for $1 \le j \le k+1$, then system (1.1) can be transformed into the following $(k+1)$-dimensional discrete system

$$u(n+1) = \begin{pmatrix} u_2(n) \\ u_3(n) \\ \vdots \\ u_{k+1}(n) \\ u_{k+1}(n) + \gamma u_{k+1}(n)\left(a + bu_1(n) - cu_1^2(n)\right) \end{pmatrix} := F(u(n)), \tag{3.1}$$

where $u = (u_1, u_2, \ldots, u_{k+1})^T \in \mathbf{R}^{k+1}$.

The map F is said to be induced by f, and system (3.1) is said to be induced by system (1.1). It is evident that a solution $\{x(n-k), \ldots, x(n)\}_{n=1}^{\infty}$ of system (1.1) with an initial condition $\{x(-k),\ldots,x(0)\}$ corresponds to a solution $\{u(n)\}_{n=1}^{\infty}$ of system (3.1) with an initial condition $u(0) = (u_1(0),\ldots,u_{k+1}(0))^T \in \mathbf{R}^{k+1}$. We call the solution $\{u(n)\}_{n=1}^{\infty}$ of (3.1) is induced by the solution $\{x(n-k),\ldots,x(n)\}_{n=1}^{\infty}$ of (1.1). Therefore, the dynamical behavior of system (1.1) is the same as that of its induced system (3.1) in $\mathbf{R}^{k+1}$. So, we introduce some relative concepts for system (1.1), which are motivated from some works in [31, Definitions 5.1 and 5.2].

Definition 3.1. (i) A point $x \in \mathbf{R}^{k+1}$ is called an m-periodic point of system (1.1) if $x \in \mathbf{R}^{k+1}$ is an m-periodic point of its induced system (3.1), that is, $F^m(x) = x$ and $F^j(x) \neq x$ for $1 \le j \le m-1$. In the special case of $m = 1$, x is called a fixed point or steady state of system (1.1).

(ii) The concept of snap-back repeller and its classifications of system (1.1) are defined similarly to those for its induced system (3.1) in $\mathbf{R}^{k+1}$.

(iii) The concepts of density of periodic points, topological transitivity, sensitive dependence on initial conditions, and the invariant set for system (1.1) are defined similarly to those for its induced system (3.1) in $\mathbf{R}^{k+1}$.

Definition 3.2. System (1.1) is said to be chaotic in the sense of Devaney (or Li-Yorke) on $V \subset \mathbf{R}^{k+1}$ if its induced system (3.1) is chaotic in the sense of Devaney (or Li-Yorke) on $V \subset \mathbf{R}^{k+1}$.

4. Chaos in the Model

In this section, we will investigate the chaotic behavior of system (3.1), that is, system (1.1), by showing that there exists a regular and nondegenerate snap-back repeller under some conditions.

It is obvious that system (3.1) has three fixed points

$$O := (0,\ldots,0)^T, \qquad P := (p,\ldots,p)^T, \qquad Q := (q,\ldots,q)^T \in \mathbf{R}^{k+1}, \tag{4.1}$$

where $p := (b - \sqrt{b^2 + 4ac})/2c < 0$, $q := (b + \sqrt{b^2 + 4ac})/2c > 0$. It is noticed that the Jacobian matrix of F at the fixed-point O always has a k-multiple eigenvalue 0. So the fixed-point O cannot be a snap-back repeller. However, the fixed-points P and Q can be regular and nondegenerate snap-back repellers of system (3.1) when γ satisfies some conditions. We only show the fixed-point P can be a regular and nondegenerate snap-back repeller of system (3.1) under some conditions, since the situation for the fixed point Q is similar.

Theorem 4.1. *There exists a positive constant γ_0 such that for any $\gamma > \gamma_0$, the fixed-point P is a regular and nondegenerate snap-back repeller of system* (3.1)*. Then system* (3.1) *and consequently, system* (1.1)*, is chaotic in the sense of both Devaney and Li-Yorke.*

Proof. The idea in the proof is motivated by the proof of [32, Theorem 3.2]. We will apply Lemma 2.7 to prove this theorem. So, it suffices to show that all the assumptions in Lemma 2.7 are satisfied.

For convenience, we translate the fixed-point P to the origin O. Let $\overline{u} = u - P$. Then system (3.1) becomes the following

$$\overline{u}(n+1) = \begin{pmatrix} \overline{u}_2(n) \\ \overline{u}_3(n) \\ \vdots \\ \overline{u}_{k+1}(n) \\ \overline{u}_{k+1}(n) + \gamma(\overline{u}_{k+1}(n) + p)h(\overline{u}_1(n) + p) \end{pmatrix} := \overline{F}(u(n)), \tag{4.2}$$

where $h(x) := a + bx - cx^2$ satisfying $h(p) = 0$ and $h(q) = 0$. Therefore, $O = (0, \dots, 0)^T \in \mathbf{R}^{k+1}$ is a fixed point of the map $\overline{F}$, and we only need to prove that O is a regular and nondegenerate snap-back repeller of system (4.2) under some conditions.

First, we show that there exists a positive constant γ_1 such that O is an expanding fixed point of the map $\overline{F}$ in some norm in $\mathbf{R}^{k+1}$ for any $\gamma > \gamma_1$. In fact, the map $\overline{F}$ is continuously differentiable in $\mathbf{R}^{k+1}$, and the Jacobian matrix of $\overline{F}$ at O is

$$D\overline{F}(O) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\gamma(cp^2 + a) & 0 & 0 & \cdots & 1 \end{pmatrix}_{(k+1)\times(k+1).} \tag{4.3}$$

Its eigenvalues are determined by the following:

$$\lambda^{k+1} - \lambda^k + \gamma\left(cp^2 + a\right) = 0. \tag{4.4}$$

Let $\gamma_1 := 2/(cp^2 + a)$. From (4.4), we get that all the eigenvalues of $D\overline{F}(O)$ have absolute values larger than 1 for any $\gamma > \gamma_1$. Otherwise, suppose that there exists an eigenvalue λ_0 of $D\overline{F}(O)$ with $|\lambda_0| \leq 1$, then we get the following inequality

$$2 \geq \left|\lambda_0^{k+1}\right| + \left|\lambda_0^k\right| \geq \left|\lambda_0^{k+1} - \lambda_0^k\right| = \gamma\left(cp^2 + a\right) > 2, \tag{4.5}$$

which is a contradiction. Hence, it follows from the first condition of Lemma 2.7, there exist a positive constant r and a norm $\|\cdot\|^*$ in $\mathbf{R}^{k+1}$ such that O is an expanding fixed point of $\overline{F}$ in $\overline{B}_r(O)$ in the norm $\|\cdot\|^*$, that is,

$$\|\overline{F}(x) - \overline{F}(y)\|^* \geq \mu\|x - y\|^*, \quad \forall x, y \in \overline{B}_r(O), \tag{4.6}$$

where $\mu > 1$ is an expanding coefficient of $\overline{F}$ in $\overline{B}_r(O)$, and $\overline{B}_r(O)$ is the closed ball centered at $O \in \mathbf{R}^{k+1}$ of radius r with respect to the norm $\|\cdot\|^*$.

Next, we show that O is a snap-back repeller of $\overline{F}$ in the norm $\|\cdot\|^*$. Suppose that $W \subset \overline{B}_r(O)$ is an arbitrary neighborhood of O in $\mathbf{R}^{k+1}$, then there exists a small interval $U \subset \mathbf{R}$

containing 0 such that $\underbrace{U \times U \times \cdots \times U}_{k+1} \subset W$. In the following, we will show that there exists a positive constant γ_2 such that for any $\gamma > \gamma_2$, there exists a point $O_0 \in W$ with $O_0 \neq O$ satisfying

$$\overline{F}^{k+2}(O_0) = O, \tag{4.7}$$

which implies that O is a snap-back repeller of $\overline{F}$.

For convenience, Let $u^* = q - p$, and $\overline{h}(x) := h(x+p) = a + b(x+p) - c(x+p)^2$. It is clear that $\overline{h}(x)$ is continuously differentiable on $\mathbf{R}$ and satisfies

$$\overline{h}(0) = h(p) = 0, \qquad \overline{h}(u^*) = h(q) = 0, \qquad \overline{h}'(0) = -p^{-1}\left(cp^2 + a\right) > 0. \tag{4.8}$$

For $k = 1$. From Lemma 2.6, it follows that there exists a positive constant γ_2' such that for any $\gamma > \gamma_2'$, there exist two points $x_1, x_2 \in U$ satisfying the following equations

$$\begin{gathered} \gamma\overline{h}(x_2) = pq^{-1} - 1, \\ \gamma\overline{h}(x_1) = q\left(x_2 + p\right)^{-1} - 1, \end{gathered} \tag{4.9}$$

which can be written as follows

$$\begin{gathered} u^* + \gamma q h\left(x_2 + p\right) = 0, \\ x_2 + \gamma\left(x_2 + p\right)h\left(x_1 + p\right) = u^*. \end{gathered} \tag{4.10}$$

Set $O_0 = (x_1, x_2)^T \in \mathbf{R}^2$, then we get that $O_0 \in U \times U \subset W$ with $O_0 \neq O$ for any $\gamma > \gamma_2'$. It follows from (4.10) that $\overline{F}(O_0) = (x_2, u^*)^T$, $\overline{F}^2(O_0) = (u^*, 0)^T$, $\overline{F}^3(O_0) = O$.

For $k > 1$. It also follows from Lemma 2.6 that there exists a positive constant γ_2^* such that for any $\gamma > \gamma_2^*$, there exist two points $x_1, x_2 \in U$ satisfying the following equations

$$\begin{gathered} \gamma\overline{h}(x_1) = qp^{-1} - 1, \\ \gamma\overline{h}(x_2) = pq^{-1} - 1, \end{gathered} \tag{4.11}$$

which can also be written as follows

$$\begin{gathered} \gamma p h\left(x_1 + p\right) = u^*, \\ u^* + \gamma q h\left(x_2 + p\right) = 0. \end{gathered} \tag{4.12}$$

Set $O_0 = (x_1, x_2, 0, \ldots, 0)^T \in \mathbf{R}^{k+1}$, then we get that $O_0 \in \underbrace{U \times U \times \cdots \times U}_{k+1} \subset W$ with $O_0 \neq O$ for any $\gamma > \gamma_2^*$. It follows from (4.12) that $\overline{F}(O_0) = (x_2, 0, \ldots, u^*)^T$, $\overline{F}^j(O_0) = (0, \ldots, 0, \underbrace{u^*, 0, \ldots, 0}_{j})^T$ for $2 \le j \le k+1$, and $\overline{F}^{k+2}(O_0) = O$.

Take $\gamma_2 := \max\{\gamma_2', \gamma_2^*\}$. Then for any $\gamma > \gamma_2$, there exists a point $O_0 \in W$ with $O_0 \neq O$ satisfying $\overline{F}^{k+2}(O_0) = O$, in the two cases. Let $\gamma_0 := \max\{\gamma_1, \gamma_2\}$. Then we get that O is a snap-back repeller of $\overline{F}$ for $\gamma > \gamma_0$.

Now, it is clear that $\overline{F}$ is continuously differentiable in $\mathbf{R}^{k+1}$, we shall show that for $\gamma > \gamma_0$,

$$\det D\overline{F}(O_j) \neq 0, \quad 0 \leq j \leq k+1, \tag{4.13}$$

where $O_j := \overline{F}(O_{j-1})$ for $1 \leq j \leq k+1$. It is obvious that $\overline{h}(x)$ is continuously differentiable on $\mathbf{R}$ and satisfies

$$\begin{gathered}\overline{h}(0) = \overline{h}(u^*) = 0,\\ \overline{h}'(0) = -p^{-1}\left(cp^2 + a\right) > 0,\\ \overline{h}'(u^*) = -q^{-1}\left(cq^2 + a\right) < 0.\end{gathered} \tag{4.14}$$

Hence, from the second conclusion of (4.14), it follows that there exists a sufficiently small neighborhood $U_1 \subset \mathbf{R}$ containing 0 such that $\overline{h}'(x) \neq 0$ for all $x \in U_1$. We can take sufficiently large γ_0 such that x_1, x_2 obtained in the above, also lie in U_1 for $\gamma > \gamma_0$ and satisfy (4.10) or (4.12). Consequently, we have

$$\overline{h}'(x_1) \neq 0, \qquad \overline{h}'(x_2) \neq 0. \tag{4.15}$$

A direct calculation shows that for any $u = (u_1, \ldots, u_{k+1})^T \in \mathbf{R}^{k+1}$,

$$\det D\overline{F}(u) = (-1)^k \gamma(u_{k+1} + p)\overline{h}'(u_1). \tag{4.16}$$

For $k = 1$, we get that $O_0 = (x_1, x_2)^T$, $O_1 = (x_2, u^*)^T$, $O_2 = (u^*, 0)^T \in \mathbf{R}^2$. It follows from the third conclusion of (4.14), (4.15), and (4.16) that for $\gamma > \gamma_0$,

$$\det D\overline{F}(O_j) \neq 0, \quad \text{for } 0 \leq j \leq 2. \tag{4.17}$$

For $k > 1$, we get that $O_0 = (x_1, x_2, 0, \ldots, 0)^T$, $O_1 = (x_2, 0, \ldots, 0, u^*)^T$, and $O_j = (0, \ldots, 0, \underbrace{u^*, 0, \ldots, 0}_{j})^T \in \mathbf{R}^{k+1}$ for $2 \leq j \leq k+1$. Hence, from the last two conclusions of (4.14), (4.15) and (4.16), we get that for $\gamma > \gamma_0$,

$$\det D\overline{F}(O_j) \neq 0, \quad \text{for } 0 \leq j \leq k+1. \tag{4.18}$$

Therefore, all the assumptions in Lemma 2.7 are satisfied and O is a regular and nondegenerate snap-back repeller of system (4.2). Consequently, P is a regular and

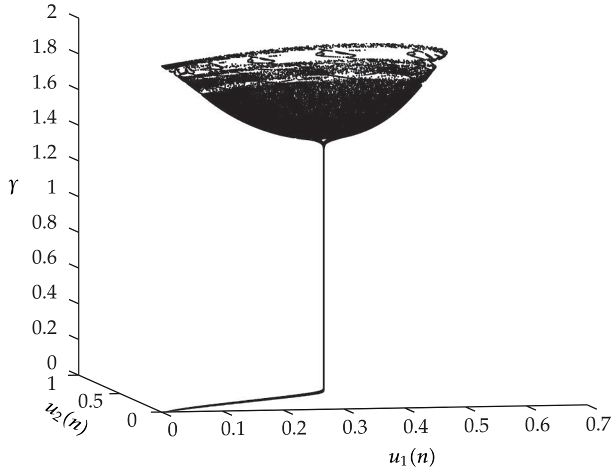

Figure 1: Bifurcation diagram of system (3.1) in the $(\gamma, u_1(n), u_2(n))$ space for γ from 0 to 2, $k = 1$, and $u(0) = (0.01, 0.01)^T$.

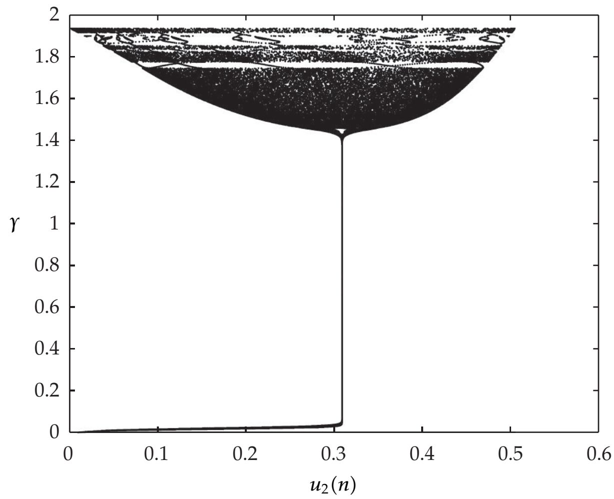

Figure 2: Bifurcation diagram of system (3.1) in the $(\gamma, u_2(n))$ plane for γ from 0 to 2, $k = 1$, and $u(0) = (0.01, 0.01)^T$.

nondegenerate snap-back repeller of system (3.1). Hence, system (3.1), that is, system (1.1), is chaotic in the sense of both Devaney and Li-Yorke. The proof is complete. □

Remark 4.2. From the proof of Theorem 4.1, we see there exists some positive constant γ_0 such that for any $\gamma > \gamma_0$, system (1.1) is chaotic in the sense of both Devaney and Li-Yorke. However, it is very difficult to determine the concrete value γ_0 since the concrete expanding area of a fixed point is not easy to obtain. This will be left for our further research.

In order to help better visualize the theoretical result, six computer simulations are done, which exhibit complicated dynamical behaviors of the induced system (3.1), that is, system (1.1). We take $a = 0.5$, $b = -1$, $c = 2$, $k = 1, 2$, and γ as a bifurcation parameter

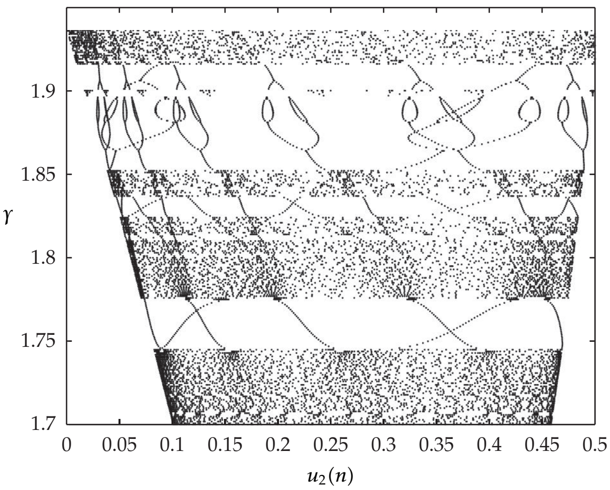

Figure 3: Zoom area of the rectangular box $[0, 0.5] \times [1.7, 1.95]$ in Figure 2.

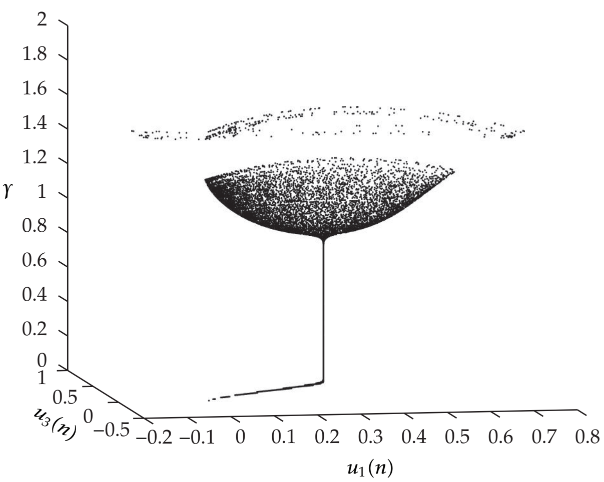

Figure 4: Bifurcation diagram of system (3.1) in the $(\gamma, u_1(n), u_3(n))$ space for γ from 0 to 2, for $k = 2$, and $u(0) = (0.01, 0.01, 0.01)^T$.

for computer simulations. It is clear that $p = (-1 - \sqrt{(-1)^2 + 4 \times 0.5 \times 2})/(2 \times 2) \approx -0.809$, and $\gamma_1 = 2/(2p^2 + 0.5) \approx 1.1056$. Then, it follows from the proof of Theorem 4.1 that O is an expanding fixed point of system (4.2) when $\gamma > \gamma_1$. Furthermore, there exists some positive constant $\gamma_0 > \gamma_1$, such that for $\gamma > \gamma_0$, O is a regular and nondegenerate snap-back repeller of system (4.2). Consequently, P is a regular and nondegenerate snap-back repeller of system (3.1). Some simulation results are shown in Figures 1, 2, 3, 4, 5, and 6 for $k = 1, 2$, which show the complicated dynamical behaviors of system (3.1), that is, system (1.1).

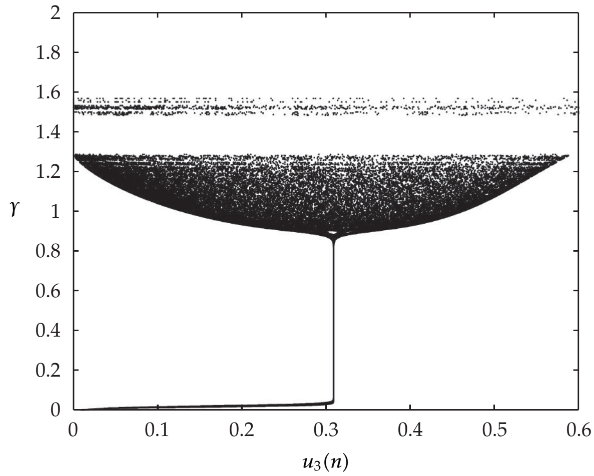

Figure 5: Bifurcation diagram of system (3.1) in the $(\gamma, u_3(n))$ plane for γ from 0 to 2, $k = 2$, and $u(0) = (0.01, 0.01, 0.01)^T$.

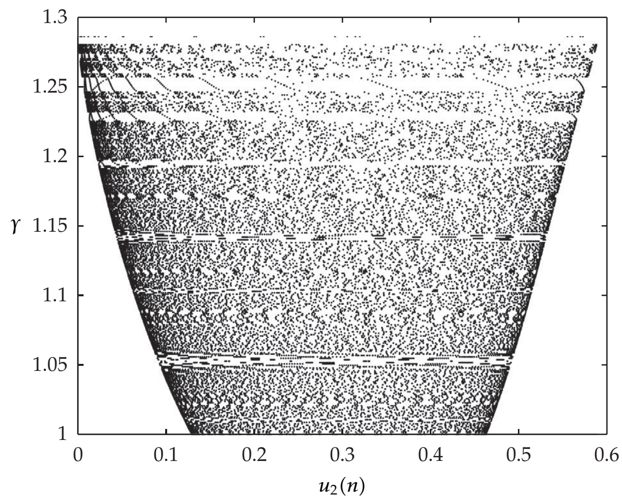

Figure 6: Zoom area of the rectangular box $[0, 0.6] \times [1, 1.3]$ in Figure 5.

5. Conclusion

In this paper, we rigorously prove the existence of chaos in a discrete delay population model. The map of the system is proved to be chaotic in the sense of both Devaney and Li-Yorke under some conditions, by employing the snap-back repeller theory. Computer simulations confirm the theoretical analysis. The system (3.1) consists of a k-dimensional linear subsystem and one-dimensional nonlinear subsystem. That is, the folding and stretching only occur in the variable u_{k+1}, and all the other variables u_j are taken placed by u_{j+1} for $1 \leq j \leq k$. So system (3.1) can be viewed as one of the simplest discrete systems that can show higher-dimensional chaos. Consequently, system (1.1) can be viewed as one of the simplest delay difference systems that show chaos.

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