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The reciprocal sums of the Fibonacci 3-subsequences

Andrew YZ Wang[*] and Fan Zhang

[*]Correspondence:
yzwang@uestc.edu.cn
School of Mathematical Sciences,
University of Electronic Science and
Technology of China, Chengdu,
611731, P.R. China

Abstract

A Fibonacci 3-subsequence is a subsequence of the type $F_n, F_{n+3}, F_{n+6}, \ldots$, where F_k denotes the kth Fibonacci number. In this article, we investigate the reciprocal sums of the Fibonacci 3-subsequences and obtain several interesting families of identities involving the Fibonacci numbers.

MSC: 11B39

Keywords: Fibonacci 3-subsequence; reciprocal sum; floor function

1 Introduction

The *Fibonacci sequence* is defined by the linear recurrence relation

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2,$$

where F_n is called the nth *Fibonacci number* with $F_0 = 0$ and $F_1 = 1$. There exists a simple and nonobvious formula for the Fibonacci numbers,

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

The Fibonacci sequence plays an important role in the theory and applications of mathematics, and its various properties have been investigated by many authors: see [1–5].

In recent years, there has been an increasing interest in studying the reciprocal sums of the Fibonacci numbers. For example, Elsner, Shimomura, and Shiokawa [6–9] investigated the algebraic relations for reciprocal sums of the Fibonacci numbers. Ohtsuka and Nakamura [10] studied the partial infinite sums of the reciprocal Fibonacci numbers. They established the following results, where $\lfloor \cdot \rfloor$ denotes the floor function.

Theorem 1.1 *For all $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{1.1}$$

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Theorem 1.2 *For all $n \geq 1$,*

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1 & \text{if } n \text{ is even,} \\ F_n F_{n-1} & \text{if } n \text{ is odd.} \end{cases} \tag{1.2}$$

Recently, Wang and Wen [11] considered the partial finite sum of the reciprocal Fibonacci numbers and strengthened Theorem 1.1 and Theorem 1.2 to the finite sum case.

Theorem 1.3

(i) *For all $n \geq 4$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_k} \right)^{-1} \right\rfloor = F_{n-2}. \tag{1.3}$$

(ii) *If $m \geq 3$ and $n \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2} & \text{if } n \text{ is even,} \\ F_{n-2} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{1.4}$$

Theorem 1.4 *For all $m \geq 2$ and $n \geq 1$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1 & \text{if } n \text{ is even,} \\ F_n F_{n-1} & \text{if } n \text{ is odd.} \end{cases} \tag{1.5}$$

Furthermore, the present authors [12] studied the reciprocal sums of even and odd terms in the Fibonacci sequence and obtained the following main results.

Theorem 1.5 *We have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}} \right)^{-1} \right\rfloor = \begin{cases} F_{2n-1} & \text{if } m = 2 \text{ and } n \geq 3, \\ F_{2n-1} - 1 & \text{if } m \geq 3 \text{ and } n \geq 1. \end{cases} \tag{1.6}$$

Theorem 1.6 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}} \right)^{-1} \right\rfloor = F_{2n-2}. \tag{1.7}$$

Theorem 1.7 *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k}^2} \right)^{-1} \right\rfloor = F_{4n-2} - 1. \tag{1.8}$$

Theorem 1.8 *For all $n \geq 1$ and $m \geq 2$, we have*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{2k-1}^2} \right)^{-1} \right\rfloor = F_{4n-4}. \tag{1.9}$$

In this article, applying elementary methods, we investigate the reciprocal sums of the Fibonacci 3-subsequences, by which we mean the subsequences of the type F_n, F_{n+3}, $F_{n+6}, \ldots$ and obtain several interesting families of identities involving the Fibonacci numbers.

2 Main results I: the reciprocal sums

We first present several well-known results on Fibonacci numbers, which will be used throughout the article. The detailed proofs can be found in [5].

Lemma 2.1 *For any positive integers a and b, we have*

$$F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1}. \tag{2.1}$$

As a consequence of (2.1), we have the following conclusion.

Corollary 2.2 *If $n \geq 1$, then*

$$F_{2n+1} = F_{n-1} F_{n+1} + F_n F_{n+2}, \tag{2.2}$$

$$F_{2n+1} = F_{n+1} F_{n+2} - F_{n-1} F_n. \tag{2.3}$$

The following interesting identity concerning the Fibonacci numbers plays a central role in the proofs of our main results.

Lemma 2.3 *Assume that a, b, c are given nonnegative integers with $a \geq b$. For $n \geq a + c$, we have*

$$F_{n+a} F_{n-a-c} - F_{n+b} F_{n-b-c} = (-1)^{n-a-c+1} F_{a+b+c} F_{a-b}. \tag{2.4}$$

Proof We proceed by induction on n. It is clearly true for $n = a + c$. Assuming that the result holds for any integer $n \geq a + c$, we show that the same is true for $n + 1$.

First, it is easy to check that

$$
\begin{aligned}
F_{n-1+a} F_{n-a-c} + F_{n+a} F_{n-1-a-c} &= F_{n-1+a} F_{n-a-c} + F_{n+a} (F_{n+1-a-c} - F_{n-a-c}) \\
&= F_{n-1+a} F_{n-a-c} + F_{n+a} F_{n+1-a-c} - F_{n+a} F_{n-a-c} \\
&= F_{2n-c} - F_{n+a} F_{n-a-c},
\end{aligned}
$$

where the last equality follows from (2.1).

Now we have

$$
\begin{aligned}
F_{n+1+a} F_{n+1-a-c} &= (F_{n-1+a} + F_{n+a})(F_{n-1-a-c} + F_{n-a-c}) \\
&= F_{n-1+a} F_{n-a-c} + F_{n+a} F_{n-1-a-c} + F_{n+a} F_{n-a-c} + F_{n-1+a} F_{n-1-a-c} \\
&= F_{2n-c} + F_{n-1+a} F_{n-1-a-c}.
\end{aligned}
$$

Similarly, we get $F_{n+1+b} F_{n+1-b-c} = F_{2n-c} + F_{n-1+b} F_{n-1-b-c}$.

Therefore, by the induction hypothesis, we arrive at

$$F_{n+1+a}F_{n+1-a-c} - F_{n+1+b}F_{n+1-b-c} = F_{(n-1)+a}F_{(n-1)-a-c} - F_{(n-1)+b}F_{(n-1)-b-c}$$
$$= (-1)^{n-a-c}F_{a+b+c}F_{a-b},$$

which completes the induction on n. □

Before introducing our main results, we first establish an inequality.

Lemma 2.4 *If $n \geq 8$, then*

$$F_{n-2}F_nF_{n+1} > 2F_{2n+1}. \tag{2.5}$$

Proof It follows from (2.3) that

$$F_{n+1}F_{n+2} > F_{2n+1}.$$

Since $n \geq 8$, we have

$$F_{n-2}F_n - 2F_{n+2} = F_{n-2}F_n - 2(F_n + F_{n+1})$$
$$= F_{n-2}F_n - 2(F_{n-1} + 2F_n)$$
$$= F_n(F_{n-2} - 4) - 2F_{n-1}$$
$$> 2F_n - 2F_{n-1}$$
$$> 0.$$

Thus, $F_{n-2}F_nF_{n+1} > 2F_{n+2}F_{n+1} > 2F_{2n+1}$. □

Theorem 2.5 *For all $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_{3k}} \right)^{-1} \right\rfloor = 2F_{3n-2}. \tag{2.6}$$

Proof Setting $n = 3k$, $a = 1$, $b = 0$, and $c = 2$ in (2.4), we obtain

$$F_{3k+1}F_{3k-3} - F_{3k}F_{3k-2} = (-1)^{3k-2}2 = (-1)^k 2. \tag{2.7}$$

It is straightforward to check that, for $k \geq 1$,

$$\frac{1}{2F_{3k-2}} - \frac{1}{F_{3k}} - \frac{1}{2F_{3k+1}} = \frac{F_{3k}F_{3k+1} - 2F_{3k-2}F_{3k+1} - F_{3k}F_{3k-2}}{2F_{3k-2}F_{3k}F_{3k+1}}$$
$$= \frac{F_{3k+1}(F_{3k} - 2F_{3k-2}) - F_{3k}F_{3k-2}}{2F_{3k-2}F_{3k}F_{3k+1}}$$
$$= \frac{F_{3k+1}F_{3k-3} - F_{3k}F_{3k-2}}{2F_{3k-2}F_{3k}F_{3k+1}} = \frac{(-1)^k}{F_{3k-2}F_{3k}F_{3k+1}}, \tag{2.8}$$

where the last equality follows from (2.7).

Therefore, we get that

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} = \frac{1}{2F_{3n-2}} - \frac{1}{2F_{6n+1}} + \sum_{k=n}^{2n} \frac{(-1)^{k-1}}{F_{3k-2}F_{3k}F_{3k+1}}.$$

If n is even, it is clear that

$$\sum_{k=n}^{2n} \frac{(-1)^{k-1}}{F_{3k-2}F_{3k}F_{3k+1}} < 0;$$

thus,

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2}}.$$

Now we consider the case where n is odd. It follows from (2.5) and the condition $n \geq 3$ that

$$F_{3n-2}F_{3n}F_{3n+1} > 2F_{6n+1},$$

from which we derive

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} = \frac{1}{2F_{3n-2}} - \frac{1}{2F_{6n+1}} + \frac{1}{F_{3n-2}F_{3n}F_{3n+1}} + \sum_{k=n+1}^{2n} \frac{(-1)^{k-1}}{F_{3k-2}F_{3k}F_{3k+1}}$$

$$< \frac{1}{2F_{3n-2}} - \frac{1}{2F_{6n+1}} + \frac{1}{F_{3n-2}F_{3n}F_{3n+1}}$$

$$< \frac{1}{2F_{3n-2}}.$$

Hence, we always have

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2}}, \tag{2.9}$$

irrespective of the parity of n.

If we let $n = 3k$, $a = 1$, $b = 0$, and $c = 1$ in (2.4), then

$$F_{3k+1}F_{3k-2} - F_{3k}F_{3k-1} = (-1)^{k-1}. \tag{2.10}$$

By elementary manipulations and (2.10) we deduce that, for $k \geq 1$,

$$\frac{1}{2F_{3k-2}+1} - \frac{1}{F_{3k}} - \frac{1}{2F_{3k+1}+1} = \frac{(-1)^k 4 - 2F_{3k-2} - (2F_{3k+1}+1)}{(2F_{3k-2}+1)F_{3k}(2F_{3k+1}+1)}$$

$$= \frac{(-1)^k 4 - 2F_{3k-2}}{(2F_{3k-2}+1)F_{3k}(2F_{3k+1}+1)} - \frac{1}{(2F_{3k-2}+1)F_{3k}}$$

$$< -\frac{1}{(2F_{3k-2}+1)F_{3k}},$$

which implies that

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} > \frac{1}{2F_{3n-2}+1} - \frac{1}{2F_{6n+1}+1} + \sum_{k=n}^{2n} \frac{1}{(2F_{3k-2}+1)F_{3k}}$$

$$> \frac{1}{2F_{3n-2}+1} - \frac{1}{2F_{6n+1}+1} + \frac{1}{(2F_{3n-2}+1)F_{3n}}.$$

It follows from (2.2) that

$$2F_{6n+1} = 2F_{3n-1}F_{3n+1} + 2F_{3n}F_{3n+2}$$

$$> 2F_{3n-2}F_{3n} + F_{3n}$$

$$= (2F_{3n-2}+1)F_{3n},$$

from which we get that

$$\sum_{k=n}^{2n} \frac{1}{F_{3k}} > \frac{1}{2F_{3n-2}+1}. \tag{2.11}$$

Combining (2.9) and (2.11), we have

$$\frac{1}{2F_{3n-2}+1} < \sum_{k=n}^{2n} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2}},$$

which yields the desired identity. □

We now study a generalization of Theorem 2.5 and start with an inequality.

Lemma 2.6 *If $m \geq 3$ and $n \geq 3$, then*

$$\frac{1}{F_{n-2}F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+3}F_{n+4}} > \frac{1}{2F_{mn+1}}. \tag{2.12}$$

Proof Letting $a = b = n-1$ in (2.1), we obtain $F_nF_n < F_{2n-1}$. If we set $a = 2n-1$ and $b = n+1$ in (2.1), then $F_{2n-1}F_{n+1} < F_{3n+1}$. Therefore,

$$F_nF_nF_{n+1} < F_{3n+1},$$

from which we derive that

$$\frac{1}{F_{n-2}F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+3}F_{n+4}} - \frac{1}{2F_{mn+1}} \geq \frac{1}{F_{n-2}F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+3}F_{n+4}} - \frac{1}{2F_{3n+1}}$$

$$> \frac{1}{F_{n-2}F_nF_{n+1}} - \frac{1}{F_{n+1}F_{n+3}F_{n+4}} - \frac{1}{F_nF_nF_{n+1}}$$

$$= \frac{F_{n-1}}{F_{n-2}F_nF_nF_{n+1}} - \frac{F_{n-1}}{F_{n-1}F_{n+1}F_{n+3}F_{n+4}}$$

$$> 0,$$

which completes the proof. □

Theorem 2.7 *If $m \geq 3$ and $n \geq 1$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n-2} & \text{if } n \text{ is even,} \\ 2F_{3n-2} - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{2.13}$$

Proof Applying the same argument as in the proof of Theorem 2.5, it is easy to see that if n is even, then

$$\frac{1}{2F_{3n-2} + 1} < \sum_{k=n}^{mn} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2}},$$

and thus the statement is true when n is even. We now concentrate ourselves on the case where n is odd.

Invoking (2.10), a direct calculation shows that, for $k \geq 1$,

$$\begin{aligned} \frac{1}{2F_{3k-2} - 1} - \frac{1}{F_{3k}} - \frac{1}{2F_{3k+1} - 1} &= \frac{2F_{3k}(F_{3k+1} - F_{3k-2}) - (2F_{3k-2} - 1)(2F_{3k+1} - 1)}{(2F_{3k-2} - 1)F_{3k}(2F_{3k+1} - 1)} \\ &= \frac{4(F_{3k}F_{3k-1} - F_{3k+1}F_{3k-2}) + 2F_{3k-2} + 2F_{3k+1} - 1}{(2F_{3k-2} - 1)F_{3k}(2F_{3k+1} - 1)} \\ &= \frac{(-1)^k 4 + 2F_{3k-2} + 2F_{3k+1} - 1}{(2F_{3k-2} - 1)F_{3k}(2F_{3k+1} - 1)} \\ &> 0. \end{aligned}$$

Hence, we arrive at

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2} - 1} - \frac{1}{2F_{3mn+1} - 1} < \frac{1}{2F_{3n-2} - 1}. \tag{2.14}$$

Employing (2.8) and the fact that n is odd, we conclude that

$$\begin{aligned} \sum_{k=n}^{mn} \frac{1}{F_{3k}} &= \frac{1}{2F_{3n-2}} - \frac{1}{2F_{3mn+1}} + \sum_{k=n}^{mn} \frac{(-1)^{k-1}}{F_{3k-2}F_{3k}F_{3k+1}} \\ &> \frac{1}{2F_{3n-2}} - \frac{1}{2F_{3mn+1}} + \frac{1}{F_{3n-2}F_{3n}F_{3n+1}} - \frac{1}{F_{3n+1}F_{3n+3}F_{3n+4}} \\ &> \frac{1}{2F_{3n-2}}, \end{aligned} \tag{2.15}$$

where the last inequality follows from Lemma 2.6.

It follows from (2.14) and (2.15) that if $m \geq 3$ and n is odd, then

$$\frac{1}{2F_{3n-2}} < \sum_{k=n}^{mn} \frac{1}{F_{3k}} < \frac{1}{2F_{3n-2} - 1},$$

from which the desired identity follows immediately. □

Applying a similar analysis to the subsequences $\{F_{3k+1}\}$ and $\{F_{3k+2}\}$, we obtain the following results, whose proofs are omitted here.

Theorem 2.8 *For all $n \geq 1$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_{3k+1}} \right)^{-1} \right\rfloor = 2F_{3n-1}. \tag{2.16}$$

Theorem 2.9 *If $m \geq 3$ and $n \geq 1$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k+1}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n-1} - 1 & \text{if n is even,} \\ 2F_{3n-1} & \text{if n is odd.} \end{cases} \tag{2.17}$$

Theorem 2.10 *For all $n \geq 1$,*

$$\left\lfloor \left(\sum_{k=n}^{2n} \frac{1}{F_{3k+2}} \right)^{-1} \right\rfloor = 2F_{3n}. \tag{2.18}$$

Theorem 2.11 *If $m \geq 3$ and $n \geq 1$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k+2}} \right)^{-1} \right\rfloor = \begin{cases} 2F_{3n} & \text{if n is even,} \\ 2F_{3n} - 1 & \text{if n is odd.} \end{cases} \tag{2.19}$$

3 Main results II: the reciprocal square sums

In the rest of the article, we study the reciprocal square sums of the Fibonacci 3-subsequences.

Lemma 3.1 *Let a and b be two given integers with $a \geq b \geq 0$. For all $n \geq a$, we have*

$$F_{n+a}^2 F_{n-a}^2 - F_{n+b}^2 F_{n-b}^2 = F_{a+b}F_{a-b}\left(F_a^2 + (-1)^{a-b}F_b^2 + (-1)^{n-a+1}2F_n^2\right). \tag{3.1}$$

Proof Employing (2.4), we derive that

$$\begin{aligned} F_{n+a}^2 F_{n-a}^2 - F_{n+b}^2 F_{n-b}^2 &= (F_{n+a}F_{n-a} - F_{n+b}F_{n-b})(F_{n+a}F_{n-a} + F_{n+b}F_{n-b}) \\ &= (-1)^{n-a+1}F_{a+b}F_{a-b}\left(2F_n^2 + F_{n+a}F_{n-a} - F_n^2 + F_{n+b}F_{n-b} - F_n^2\right) \\ &= (-1)^{n-a+1}F_{a+b}F_{a-b}\left(2F_n^2 + (-1)^{n-a+1}F_a^2 + (-1)^{n-b+1}F_b^2\right) \\ &= F_{a+b}F_{a-b}\left(F_a^2 + (-1)^{a-b}F_b^2 + (-1)^{n-a+1}2F_n^2\right). \end{aligned}$$

The proof is complete. $\square$

Lemma 3.2 *If $n \geq 1$, we have*

$$F_{6n+3}^2 - F_{6n}^2 > F_{3n}^2\left(F_{3n+3}^2 - F_{3n}^2 + 1\right). \tag{3.2}$$

Proof First, it is easy to see that

$$F_{6n+3}^2 - F_{6n}^2 = (F_{6n+3} + F_{6n})(F_{6n+3} - F_{6n}) = 4F_{6n+1}F_{6n+2}. \tag{3.3}$$

Similarly, we have

$$F_{3n+3}^2 - F_{3n}^2 = 4F_{3n+1}F_{3n+2}. \tag{3.4}$$

It follows from (2.1) that $F_{6n+1} > F_{3n}^2 + 1$ and $F_{6n+2} > F_{3n+1}F_{3n+2} + 1$. Therefore,

$$F_{6n+1}F_{6n+2} > F_{3n}^2 F_{3n+1}F_{3n+2} + F_{3n}^2. \tag{3.5}$$

The desired result follows immediately from (3.3), (3.4), and (3.5). □

Lemma 3.3 *For $n \geq 1$, we have*

$$\frac{8F_{3n}^2 - 16}{(F_{3n}^2 - F_{3n-3}^2)F_{3n}^2(F_{3n+3}^2 - F_{3n}^2)} - \frac{8F_{3n+3}^2 + 16}{(F_{3n+3}^2 - F_{3n}^2)F_{3n+3}^2(F_{3n+6}^2 - F_{3n+3}^2)} > \frac{1}{F_{3n}^2 F_{3n+3}^2}.$$

Proof It is easy to see that $8F_{3n}^2 - 16 > F_{3n}^2 - F_{3n-3}^2$ and $8F_{3n+3}^2 + 16 < F_{3n+6}^2 - F_{3n+3}^2$, so

$$\frac{8F_{3n}^2 - 16}{(F_{3n}^2 - F_{3n-3}^2)F_{3n}^2(F_{3n+3}^2 - F_{3n}^2)} > \frac{1}{F_{3n}^2(F_{3n+3}^2 - F_{3n}^2)}$$

and

$$\frac{1}{(F_{3n+3}^2 - F_{3n}^2)F_{3n+3}^2} > \frac{8F_{3n+3}^2 + 16}{(F_{3n+3}^2 - F_{3n}^2)F_{3n+3}^2(F_{3n+6}^2 - F_{3n+3}^2)}.$$

Hence,

$$\frac{8F_{3n}^2 - 16}{(F_{3n}^2 - F_{3n-3}^2)F_{3n}^2(F_{3n+3}^2 - F_{3n}^2)} - \frac{8F_{3n+3}^2 + 16}{(F_{3n+3}^2 - F_{3n}^2)F_{3n+3}^2(F_{3n+6}^2 - F_{3n+3}^2)}$$

$$> \frac{1}{F_{3n}^2(F_{3n+3}^2 - F_{3n}^2)} - \frac{1}{(F_{3n+3}^2 - F_{3n}^2)F_{3n+3}^2} = \frac{1}{F_{3n}^2 F_{3n+3}^2},$$

which completes the proof. □

We now present the first reciprocal square sum of the Fibonacci 3-subsequence.

Theorem 3.4 *If $n \geq 2$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{3n}^2 - F_{3n-3}^2 & \text{if } n \text{ is even,} \\ F_{3n}^2 - F_{3n-3}^2 - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{3.6}$$

Proof We first consider the case where n is even. Applying (3.1), by a direct calculation we get that, for $k \geq 1$,

$$\frac{1}{F_{3k}^2 - F_{3k-3}^2} - \frac{1}{F_{3k}^2} - \frac{1}{F_{3k+3}^2 - F_{3k}^2}$$

$$= \frac{F_{3k}^2(F_{3k+3}^2 - 2F_{3k}^2 + F_{3k-3}^2) - (F_{3k}^2 - F_{3k-3}^2)(F_{3k+3}^2 - F_{3k}^2)}{(F_{3k}^2 - F_{3k-3}^2)F_{3k}^2(F_{3k+3}^2 - F_{3k}^2)}$$

$$= \frac{F_{3k+3}^2 F_{3k-3}^2 - F_{3k}^4}{(F_{3k}^2 - F_{3k-3}^2) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2)}$$

$$= \frac{16 + (-1)^{3k} 8 F_{3k}^2}{(F_{3k}^2 - F_{3k-3}^2) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2)}, \tag{3.7}$$

from which we get

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} = \frac{1}{F_{3n}^2 - F_{3n-3}^2} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2} - 8 \sum_{k=n}^{mn} \frac{(-1)^{3k} F_{3k}^2 + 2}{(F_{3k}^2 - F_{3k-3}^2) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2)}.$$

Since n is even, it is easy to see that

$$\sum_{k=n}^{mn} \frac{(-1)^{3k} F_{3k}^2 + 2}{(F_{3k}^2 - F_{3k-3}^2) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2)} > 0,$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} < \frac{1}{F_{3n}^2 - F_{3n-3}^2} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2} < \frac{1}{F_{3n}^2 - F_{3n-3}^2}. \tag{3.8}$$

A similar manipulation yields that, for $k \geq 1$,

$$\frac{1}{F_{3k}^2 - F_{3k-3}^2 + 1} - \frac{1}{F_{3k}^2} - \frac{1}{F_{3k+3}^2 - F_{3k}^2 + 1} = \frac{F_{3k-3}^2 - F_{3k+3}^2 + 15 + (-1)^{3k} 8 F_{3k}^2}{(F_{3k}^2 - F_{3k-3}^2 + 1) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2 + 1)}$$

$$\leq -\frac{F_{3k+3}^2 - F_{3k-3}^2 - 15 - 8 F_{3k}^2}{(F_{3k}^2 - F_{3k-3}^2 + 1) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2 + 1)}$$

$$< -\frac{1}{F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2 + 1)},$$

where the last inequality follows from the easily checked fact

$$F_{3k+3}^2 - F_{3k-3}^2 - 15 - 8 F_{3k}^2 > F_{3k}^2 - F_{3k-3}^2 + 1.$$

Now we have

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} > \frac{1}{F_{3n}^2 - F_{3n-3}^2 + 1} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2 + 1} + \sum_{k=n}^{mn} \frac{1}{F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2 + 1)}$$

$$> \frac{1}{F_{3n}^2 - F_{3n-3}^2 + 1} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2 + 1} + \frac{1}{F_{3n}^2 (F_{3n+3}^2 - F_{3n}^2 + 1)}$$

$$\geq \frac{1}{F_{3n}^2 - F_{3n-3}^2 + 1} + \frac{1}{F_{3n}^2 (F_{3n+3}^2 - F_{3n}^2 + 1)} - \frac{1}{F_{6n+3}^2 - F_{6n}^2 + 1}$$

$$> \frac{1}{F_{3n}^2 - F_{3n-3}^2 + 1}, \tag{3.9}$$

where the last inequality follows from (3.2).

Combining (3.8) and (3.9), we get that

$$\frac{1}{F_{3n}^2 - F_{3n-3}^2 + 1} < \sum_{k=n}^{mn} \frac{1}{F_{3k}^2} < \frac{1}{F_{3n}^2 - F_{3n-3}^2},$$

which means that the statement is true when n is even.

We now consider the case where n is odd. It is not hard to derive that

$$\frac{1}{F_{3k}^2 - F_{3k-3}^2 - 1} - \frac{1}{F_{3k}^2} - \frac{1}{F_{3k+3}^2 - F_{3k}^2 - 1}$$

$$= \frac{(-1)^k 8 F_{3k}^2 + F_{3k+3}^2 - F_{3k-3}^2 + 15}{(F_{3k}^2 - F_{3k-3}^2 - 1) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2 - 1)}$$

$$> 0,$$

where the last inequality follows from the fact that $F_{3k+3} = 3F_{3k} + 2F_{3k-1}$.

Therefore, we arrive at

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} < \frac{1}{F_{3n}^2 - F_{3n-3}^2 - 1} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2 - 1} < \frac{1}{F_{3n}^2 - F_{3n-3}^2 - 1}. \tag{3.10}$$

Invoking (3.7) and Lemma 3.3, we derive that if n is odd, then

$$\sum_{k=n}^{mn} \frac{1}{F_{3k}^2} = \frac{1}{F_{3n}^2 - F_{3n-3}^2} - \frac{1}{F_{3mn+3}^2 - F_{3mn}^2}$$

$$\qquad + \sum_{k=n}^{mn} \frac{(-1)^{k-1} 8 F_{3k}^2 - 16}{(F_{3k}^2 - F_{3k-3}^2) F_{3k}^2 (F_{3k+3}^2 - F_{3k}^2)}$$

$$> \frac{1}{F_{3n}^2 - F_{3n-3}^2} + \frac{8 F_{3n}^2 - 16}{(F_{3n}^2 - F_{3n-3}^2) F_{3n}^2 (F_{3n+3}^2 - F_{3n}^2)}$$

$$\qquad - \frac{8 F_{3n+3}^2 + 16}{(F_{3n+3}^2 - F_{3n}^2) F_{3n+3}^2 (F_{3n+6}^2 - F_{3n+3}^2)} - \frac{1}{F_{6n+3}^2 - F_{6n}^2}$$

$$> \frac{1}{F_{3n}^2 - F_{3n-3}^2} + \frac{1}{F_{3n}^2 F_{3n+3}^2} - \frac{1}{F_{6n+3}^2 - F_{6n}^2}$$

$$= \frac{1}{F_{3n}^2 - F_{3n-3}^2} + \frac{1}{F_{3n}^2 F_{3n+3}^2} - \frac{1}{4 F_{6n+1} F_{6n+2}}$$

$$> \frac{1}{F_{3n}^2 - F_{3n-3}^2}, \tag{3.11}$$

where the last inequality follows from the fact that $F_{6n+2} > F_{3n} F_{3n+3}$ and $2F_{6n+1} > F_{6n+2}$.

It follows from (3.10) and (3.11) that

$$\frac{1}{F_{3n}^2 - F_{3n-3}^2} < \sum_{k=n}^{mn} \frac{1}{F_{3k}^2} < \frac{1}{F_{3n}^2 - F_{3n-3}^2 - 1},$$

which yields the desired result. $\qquad\qquad\qquad \square$

For the subsequences $\{F_{3k+1}\}$ and $\{F_{3k+2}\}$, we have similar results.

Theorem 3.5 *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k+1}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{3n+1}^2 - F_{3n-2}^2 - 1 & \text{if } n \text{ is even,} \\ F_{3n+1}^2 - F_{3n-2}^2 & \text{if } n \text{ is odd.} \end{cases} \tag{3.12}$$

Theorem 3.6 *If $n \geq 1$ and $m \geq 2$, then*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} \right)^{-1} \right\rfloor = \begin{cases} F_{3n+2}^2 - F_{3n-1}^2 & \text{if } n \text{ is even,} \\ F_{3n+2}^2 - F_{3n-1}^2 - 1 & \text{if } n \text{ is odd.} \end{cases} \tag{3.13}$$

Remark The proof of Theorem 3.5 is similar to that of Theorem 3.4 and is omitted here. Since the telescoping technique for the proof of Theorem 3.6 is very different from that for Theorem 3.4, we give a detailed proof of Theorem 3.6 in the next section.

4 Proof of Theorem 3.6

We first present a preliminary result, which plays a central role in the later proof.

Lemma 4.1 *For all $n \geq 1$, we have*

$$2\left(F_{6n+5}^2 - F_{6n+2}^2 \right) > F_{3n+2}^2 F_{3n+5}^2$$
$$> \left(F_{3n+2}^2 - F_{3n-1}^2 + 1 \right)\left(F_{3n+5}^2 - F_{3n+2}^2 + 1 \right).$$

Proof The second inequality is obvious, so we concentrate ourselves on the first one. It is easy to see that

$$2\left(F_{6n+5}^2 - F_{6n+2}^2 \right) = 2 \cdot 4 F_{6n+3} F_{6n+4} > 4 F_{6n+4}^2. \tag{4.1}$$

Employing (2.1), we get $F_{6n+4} = F_{3n+2}(F_{3n+1} + F_{3n+3})$; therefore,

$$F_{6n+4}^2 = F_{3n+2}^2 (F_{3n+1} + F_{3n+3})^2. \tag{4.2}$$

It follows from $2F_{3n+1} > F_{3n+2} = F_{3n+5} - 2F_{3n+3}$ that $2(F_{3n+1} + F_{3n+3}) > F_{3n+5}$; thus,

$$4(F_{3n+1} + F_{3n+3})^2 > F_{3n+5}^2. \tag{4.3}$$

Combining (4.1), (4.2), and (4.3) yields the desired result. □

Proof of Theorem 3.6 We first consider the case where n is even. Employing (3.1) again, by a direct calculation we deduce that, for $k \geq 1$,

$$\frac{1}{F_{3k+2}^2 - F_{3k-1}^2} - \frac{1}{F_{3k+2}^2} - \frac{1}{F_{3k+5}^2 - F_{3k+2}^2}$$

$$= \frac{F_{3k+2}^2(F_{3k+5}^2 - 2F_{3k+2}^2 + F_{3k-1}^2) - (F_{3k+2}^2 - F_{3k-1}^2)(F_{3k+5}^2 - F_{3k+2}^2)}{(F_{3k+2}^2 - F_{3k-1}^2)F_{3k+2}^2(F_{3k+5}^2 - F_{3k+2}^2)}$$

$$= \frac{F_{3k-1}^2 F_{3k+5}^2 - F_{3k+2}^4}{(F_{3k+2}^2 - F_{3k-1}^2)F_{3k+2}^2(F_{3k+5}^2 - F_{3k+2}^2)}$$

$$= \frac{16 + (-1)^k 8 F_{3k+2}^2}{(F_{3k+2}^2 - F_{3k-1}^2)F_{3k+2}^2(F_{3k+5}^2 - F_{3k+2}^2)}, \tag{4.4}$$

from which we obtain

$$\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} = \frac{1}{F_{3n+2}^2 - F_{3n-1}^2} - \frac{1}{F_{3mn+5}^2 - F_{3mn+2}^2}$$
$$- 8 \sum_{k=n}^{mn} \frac{(-1)^k F_{3k+2}^2 + 2}{(F_{3k+2}^2 - F_{3k-1}^2) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2)}.$$

Since n is even, it is easy to see that

$$\sum_{k=n}^{mn} \frac{(-1)^k F_{3k+2}^2 + 2}{(F_{3k+2}^2 - F_{3k-1}^2) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2)} > 0,$$

which implies that

$$\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} < \frac{1}{F_{3n+2}^2 - F_{3n-1}^2}. \tag{4.5}$$

Similarly, we can derive that, for $k \geq 1$,

$$\frac{1}{F_{3k+2}^2 - F_{3k-1}^2 + 1} - \frac{1}{F_{3k+2}^2} - \frac{1}{F_{3k+5}^2 - F_{3k+2}^2 + 1}$$
$$= \frac{F_{3k-1}^2 - F_{3k+5}^2 + 15 + (-1)^k 8 F_{3k+2}^2}{(F_{3k+2}^2 - F_{3k-1}^2 + 1) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2 + 1)}$$
$$< 0.$$

We now have

$$\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} = \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 + 1} - \frac{1}{F_{3mn+5}^2 - F_{3mn+2}^2 + 1}$$
$$+ \sum_{k=n}^{mn} \frac{F_{3k+5}^2 - F_{3k-1}^2 - 15 - (-1)^k 8 F_{3k+2}^2}{(F_{3k+2}^2 - F_{3k-1}^2 + 1) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2 + 1)}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 + 1} + \frac{F_{3n+5}^2 - F_{3n-1}^2 - 15 - 8 F_{3n+2}^2}{(F_{3n+2}^2 - F_{3n-1}^2 + 1) F_{3n+2}^2 (F_{3n+5}^2 - F_{3n+2}^2 + 1)}$$
$$- \frac{1}{F_{6n+5}^2 - F_{6n+2}^2 + 1}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 + 1} + \frac{2}{(F_{3n+2}^2 - F_{3n-1}^2 + 1)(F_{3n+5}^2 - F_{3n+2}^2 + 1)}$$
$$- \frac{1}{F_{6n+5}^2 - F_{6n+2}^2 + 1}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 + 1}, \tag{4.6}$$

where the last inequality follows from Lemma 4.1.

Combining (4.5) and (4.6) yields that

$$\frac{1}{F_{3n+2}^2 - F_{3n-1}^2 + 1} < \sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} < \frac{1}{F_{3n+2}^2 - F_{3n-1}^2},$$

which shows that the statement is true when n is even.

Next we turn to consider the case where n is odd. It is not hard to derive that

$$\frac{1}{F_{3k+2}^2 - F_{3k-1}^2 - 1} - \frac{1}{F_{3k+2}^2} - \frac{1}{F_{3k+5}^2 - F_{3k+2}^2 - 1}$$
$$= \frac{(-1)^k 8 F_{3k+2}^2 + F_{3k+5}^2 - F_{3k-1}^2 + 15}{(F_{3k+2}^2 - F_{3k-1}^2 - 1) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2 - 1)}$$
$$> 0,$$

where the last inequality follows from the fact that $F_{3k+5} = 3F_{3k+2} + 2F_{3k+1}$.

Therefore, we obtain

$$\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} < \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 - 1} - \frac{1}{F_{3mn+5}^2 - F_{3mn+2}^2 - 1} < \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 - 1}. \tag{4.7}$$

Applying (4.4) and the fact that n is odd, we derive that

$$\sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} = \frac{1}{F_{3n+2}^2 - F_{3n-1}^2} - \frac{1}{F_{3mn+5}^2 - F_{3mn+2}^2}$$
$$+ \sum_{k=n}^{mn} \frac{(-1)^{k-1} 8 F_{3k+2}^2 - 16}{(F_{3k+2}^2 - F_{3k-1}^2) F_{3k+2}^2 (F_{3k+5}^2 - F_{3k+2}^2)}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2} + \frac{8 F_{3n+2}^2 - 16}{(F_{3n+2}^2 - F_{3n-1}^2) F_{3n+2}^2 (F_{3n+5}^2 - F_{3n+2}^2)}$$
$$- \frac{8 F_{3n+5}^2 + 16}{(F_{3n+5}^2 - F_{3n+2}^2) F_{3n+5}^2 (F_{3n+8}^2 - F_{3n+5}^2)} - \frac{1}{F_{6n+5}^2 - F_{6n+2}^2}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2} + \frac{2}{F_{3n+2}^2 (F_{3n+5}^2 - F_{3n+2}^2)} - \frac{2}{F_{3n+5}^2 (F_{3n+5}^2 - F_{3n+2}^2)}$$
$$- \frac{1}{F_{6n+5}^2 - F_{6n+2}^2}$$
$$= \frac{1}{F_{3n+2}^2 - F_{3n-1}^2} + \frac{2}{F_{3n+2}^2 F_{3n+5}^2} - \frac{1}{F_{6n+5}^2 - F_{6n+2}^2}$$
$$> \frac{1}{F_{3n+2}^2 - F_{3n-1}^2}, \tag{4.8}$$

where the last inequality follows from Lemma 4.1.

It follows from (4.7) and (4.8) that

$$\frac{1}{F_{3n+2}^2 - F_{3n-1}^2} < \sum_{k=n}^{mn} \frac{1}{F_{3k+2}^2} < \frac{1}{F_{3n+2}^2 - F_{3n-1}^2 - 1},$$

which yields the desired result.   □

5 Conclusions

In this paper, we investigate in two ways the reciprocal sums of the Fibonacci 3-subsequences, where a Fibonacci 3-subsequence is a subsequence of the type $F_n, F_{n+3}, F_{n+6}, \ldots$. One is focused on the ordinary sums, and the other is concerned with the square sums. By evaluating the integer parts of the reciprocals of these sums we get several interesting families of identities. The results are new and important to those with closely related research interests.

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